STABILITY OF PERIODIC SOLUTION FOR QUASI-LINEAR CONTROLLED SYSTEMS

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The present study regards the class of controlled quasi-linear systems. It is presumed that the use of controls as linear combination of state variables will determine zero eigenvalues for the characteristic equation of the linearized system. An analysis method for the periodica solution stability valid for the autonomous system obtained by control substitution.

1. INTRODUCTION

The analysis of a class of dynamic phenomena leads to mathematical models represented by controlled differential systems. The major interest in the command systems is for the controllable ones. The construction of a control, given by a linear from of state variables, might modify the eigenvalues of the linearized system [2]. When all the eingenvalues have negative real part, the controlled system has an asymptotic stable trivial solution. A critical stability case occurs when some of the eigenvalues are zero and the other have the negative real part. An equivalent system, for which we have to determine the stability solution is obtained by a series of succesive transformations. The transformed system is a quasi-linear system with non-homogeneous state variables. Its stability conditions of the periodical solutions, or the orbital stability, will be analized.

2. QUASI-LINEAR CONTROLLED SYSTEM

We will consider the controlled quasi-linear system:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + bu + \sum_{m=2}^{oo} z^{(m)} , \tag{1}$$

where A, b are constant matrices, z is the state n-dimension vector, $z^{(m)}$ are homogenous froms of variable u is command r-dimension vector. The coefficient $z^{(m)}$ of are time-independent.

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Under the assumption that the linearized sistem (1) can be controlled, a control with the from:

$$u = c^{*}z \tag{2}$$

is constructed so that the matrix $\left(a + bc^{*}\right)$ has not eigenvalues with positive real part. This means that the instability occurs for the trivial solution $z = 0$ system (1).

Presuming that for a certain vector *c* the *k* eigenvalues are zero while the other *n-k* eigenvalues have negative real part. This represents one of the critical stability cases.

An answer regarding the trivial solution stability for such a system might be provided by the transformation of the non-linear system (1) for the control (2) in an autonomous system related to state variables.

The study of periodical stability of such a system requires the development of a mathematical model regarding the existence conditions of the orbital stability.

Emplo*yi*ng a control with the from, system (1) becomes

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = X_s\left(x, z\right) \quad s = 1, 2, ..., k \tag{3}$$

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = \sum_{j=1}^{N} p_{ij} + Z_i\left(x, z\right) \quad i = 1, 2, ..., N, \;\; N = n - k$$

where

$$X_s = \sum_{m=2}^{oo} X_s^{(m)}, \quad Z_i = \sum_{m=2}^{oo} Z_i^{(m)}, \tag{4}$$

where $X_s^{(m)}$ are homogeneous *m*-rank forms with the variables $x_1,..., x_k, z_1,..., z_N$ and coefficients time-independent. These series (4) converge for $|x_s|, s = 1,...,k$ and $|y_i|, i = 1,...,N$ sufficient small. The eigenvalues of the matrix $P = \{p_{ij}\}$ have negative real part.

We consider the second group of equations of the systems (3) written as:

$$Pz + Z(x, z) = 0 \tag{5}$$

for:

$$x_1 = x_2 = ... = x_k = 0, \qquad z_1 = z_2 = ... = z_N = 0 \tag{6}$$

so that the functional determinant related to $z_1,...., z_N$ does not vanish for x_s and $y_i = 0$.

Using the implicit functions theorem, it follows that the equations (5) can be solved with respect to $z_1,...,z_N$ and will admit a solution of the from:

$$z_i = u_i\left(x_1,...,x_k\right),\ i = 1,....,N, \tag{7}$$

with $u_i(x_1,...,x_k)$ holomorphic functions of variables $x_1,...,x_k$, vanishing for $x_s = 0$.

Taking into account (7), the functions $X_s(x,u(x))$ are holomorphic functions of the variables $x_1,...,x_k$.

We will assume in the X_s non-singular case defined for certain s_0.

Performing the exchange of function:

$$\begin{aligned} x_s &= \overline{x}_s \qquad (s = 1,...,k) \\ z_i &= \overline{z}_i + z_i(\overline{x}) \qquad (i = 1,..,N) \end{aligned} \tag{8}$$

and replacing into (3), yields

$$\frac{\mathrm{d}\overline{z}_i}{\mathrm{d}t} + \sum_{j=1}^{k} \frac{\partial u_i\left(\overline{x}\right)}{\partial \overline{x}_j}\frac{\mathrm{d}\overline{x}_j}{\mathrm{d}t} = \sum_{j=1}^{N} p_{ij}\left(z_j + u_j\left(\overline{x}\right) + Z_i\left(\overline{x}, \overline{z} + u\left(\overline{x}\right)\right)\right) \tag{9}$$

taking into account that $u_j(\overline{x})(j = 1,...,N)$ are roots of the equation (5), we have:

$$\sum_{j=1}^{N} p_{ij} u_j\left(\overline{x}\right) = -Z_i\left(\overline{x}, u\left(\overline{x}\right)\right), \tag{10}$$

so that the system (9) becomes

$$\begin{aligned} \frac{\mathrm{d}\overline{x}_s}{\mathrm{dt}} &= \overline{X}_s\left(\overline{x}, \overline{z}\right), \\ \frac{\mathrm{d}\overline{z}_i}{\mathrm{d}t} &= \sum_{j=1}^{N} p_{ij} z_j + \overline{Z}_i\left(\overline{x}, \overline{z}\right), \end{aligned} \tag{11}$$

where:

$$\begin{aligned} \overline{X}_s\left(\overline{x}, \overline{z}\right) &= X_s\left(\overline{x}, u\left(\overline{x}\right) + \overline{z}\right), \\ \overline{Z}_i\left(\overline{x}, \overline{z}\right) &= Z_i\left(\overline{x}, u\left(\overline{x}\right) + \overline{z}\right) - Z_i\left(\overline{x}, u\left(x\right)\right) - \sum_{j=1}^{N} \frac{\partial u_i\left(\overline{x}\right)}{\partial \overline{x}_j} X_j\left(\overline{x}, \overline{z}\right). \end{aligned} \tag{12}$$

Putting $\overline{z} = 0$ in $X_s(\overline{x},\overline{z})$ and $\overline{Z}_i(\overline{x},\overline{z})$ we get:

$$\overline{X}_s^{(0)}(x) = \overline{X}_s(x,0) = X_s(x, u(\overline{x})) = \sum_{l=\alpha}^{\infty} X_s^{(l)}$$
$$\overline{Z}_s^{(0)}(x) = \overline{Z}_i^{(0)}(\overline{x},0) = \sum_{l=\beta}^{\infty} Z_i^{(l)}. \tag{13}$$

It will noticed the condition $\beta \geq \alpha + 1$ must be satisfied. The solution stability study for the system (11) is equivalent with the solution stability analysis for the system:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = X_s(x) \quad s = 1,...,k, \tag{14}$$

where $X_s(x)$ is a homogeneous form.

3. ORBITAL STABILITY

We are considering the autonoumous system (14) written in vector from:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x). \tag{15}$$

We presume that this system admits the periodical solution $x = \overline{u}(t)$ with the period ω. Based on the unicity theorem we have $X(\overline{u}(t)) \neq 0$ for all t.

Denoting

$$\overline{X}(t) = \frac{X(\overline{u}(t))}{\left\| X(\overline{u}(t)) \right\|}, \tag{16}$$

if X is verifying the local Lipschitz condition, the curve $x = \overline{X}(t)$ does not cover the entire unit sphere, thus there exists a unit vector e_l so that $\overline{X}(t) + e_l \neq 0$ for each t.

An orthogonal and normalized system of vectors $e_1, e_2, ..., e_n$ will be constructed starting from the constant vector e_l.

Because:

$$\cos\theta_i = \frac{(\overline{X}(t), e_i)}{\|\overline{X}\| \|e_i\|} = (\overline{X}(t), e_i) \tag{17}$$

it follows that $\left|\cos\theta_i\right| \le 1$ using the equality of Cauchy-Buniakowschi and also that:

$$\cos\left(\overline{X}, e_i\right) = \cos\theta_1 \neq +\cos\pi \neq -1. \tag{18}$$

Under there the vectors

$$\xi_v = e_v - \frac{\cos\theta_v}{1+\cos\theta_1}\left(e_1 + \overline{X}\right) \tag{19}$$

can be constructed.

The vectors ξ_v are periodical functions with the period ω and have the same regularity property as $\overline{X}(t)$.

It is to easy demonstrat that the vectors $\left(\overline{X}(t), \xi_2, \xi_3, ..., \xi_n\right)$ represents an orthogonal normalizes system, thus:

$$\begin{aligned}&\left(\xi_v, \overline{X}\right) = 0 \quad v = 2,...,n,\\&\left(\xi_v, \xi_\mu\right) = \delta v\mu \quad v, \mu = 2,...,n.\end{aligned} \tag{20}$$

It follows that an orthogonal and normalizes vectors system on (the unitary vector of the tangent to the curve) consisting from period ω periodical vectors with the same regularity property as was attached to the periodical solution.

With this orthogonal and normalized system we perform the variable change given by the relation:

$$x = \overline{u}(\theta) + S(\theta)y, \tag{21}$$

where:

$$\begin{aligned}&S(\theta) = \left(\xi_2(\theta), \xi_3(\theta), ..., \xi_n(\theta)\right) = \left(s_{ij}(\theta)\right) \; i = 1,..,n; \; j = 1,...,n-1,\\&y = \left(y^1, y^2, ..., y^{n-1}\right)^{\mathrm{T}}\\&\overline{u}(\theta) = \left(\overline{u}^1(\theta), \overline{u}^2(\theta), ..., \overline{u}^n(\theta)\right)^{\mathrm{T}}.\end{aligned} \tag{22}$$

Developing the transformation (21), it follows that:

$$x^i = u^i(\theta) + \sum_{j=1}^{n-1} s_{ij}(\theta) y^j \quad i = 1,..,n, \tag{23}$$

thus:

$$\frac{\partial x^i}{\partial y^j} = s_{ij},$$
$$\left.\frac{\partial x^i}{\partial \theta}\right|_{y=0} = \frac{\partial \overline{u}^i}{\partial \theta} + \sum_{j=0}^{n-1} \frac{\partial s_{ij}}{\partial \theta} y^i \Bigg|_{y=0} = \frac{\mathrm{d}\overline{u}^i}{\mathrm{d}\theta} = X\left(\overline{u}^i(\theta)\right) = X^i(\theta). \tag{24}$$

Taking into account (24) follows:

$$\Delta = \left.\frac{\partial\left(X^1, X^2, ..., X^n\right)}{\partial\left(y^1, ..., y^{n-1}\theta\right)}\right|_{y=0} = \det\left(S(\theta), X\left[\overline{u}(\theta)\right]\right). \tag{25}$$

But we have:

$$\det\left(S(\theta), X\left[\overline{u}(\theta)\right]\right) = \left|X\left[\overline{u}(\theta)\right]\right| \det\left(S(\theta), \overline{X}\left[\overline{u}(\theta)\right]\right) = \left|X\left[\overline{u}(\theta)\right]\right| \neq 0. \tag{26}$$

In order to obtain the relation (26) we used $\left(S(\theta), \overline{X}\left[\overline{u}(\theta)\right]\right) = 1$ because system $\xi_1, ..., \xi_n, \overline{X}$ is orthogonal and normalized.

Thus $\Delta \neq 0$ requires that the transformation (21) unvariable.

By this change of variables the curve $x = \overline{u}(t)$ becames $y = 0,\ \theta = t$. With the new variables, system (15) will be written as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\overline{u}(\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} y + S(\theta)\frac{\mathrm{d}y}{\mathrm{d}t} = X\left[\overline{u}(\theta) + S(\theta)y\right] \tag{27}$$

or

$$X\left[\overline{u}(\theta)\right]\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta} y + S(\theta)\frac{\mathrm{d}y}{\mathrm{d}t} = X\left[\overline{u}(\theta) + S(\theta)y\right]. \tag{28}$$

Multiplying (28) with $X^*\left[\overline{u}(\theta)\right]$ and taking into account the othogonality of the columns of $S(\theta)$ and X^* it follows that:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{X^*\left[\overline{u}(\theta)\right]X\left[\overline{u}(\theta) + S(\theta)y\right]}{\left\|X\left[\overline{u}(\theta)\right]\right\|^2 + X^*\left[\overline{u}(\theta)\right]\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}y}. \tag{29}$$

Taking into account:

$$\left(\xi_\mu^*, X\left[\overline{u}(\theta)\right]\right) = 0$$
$$\left(\xi_\mu^*, S(\theta)\right) = \xi_{\mu v} \quad \mu, v = 2, ..., n, \tag{30}$$

by multiplying of relation (28) with ξ_μ^* we get

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}t}=\xi_\mu^*(\theta)X\left[\overline{u}(\theta)+S(\theta)y\right]-\frac{X^*\left[\overline{u}(\theta)\right]X\left[\overline{u}(\theta)+S(\theta)y\right]}{\left\|X\left[\overline{u}(\theta)\right]\right\|^2+X^*\left[\overline{u}(\theta)\right]\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}y}\xi_\mu^*(\theta)\frac{\mathrm{d}S}{\mathrm{d}\theta}y\,. \quad (31)$$

Thus, with the transformation (21), system (15) becames

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t}&=Y(\theta,y),\\ \frac{\mathrm{d}\theta}{\mathrm{d}t}&=\Theta(\theta,y),\end{aligned}\qquad (32)$$

where the expressions of $\Theta(\theta,y)$ and $Y(\theta,y)$ are given by (29) and, respectively, (31) with the property that:

$$\begin{aligned}Y(\theta,y)&=\xi_\mu^*(\theta)X\left[\overline{u}(\theta)\right]\equiv 0,\\ \Theta(\theta,y)&=\frac{X^*\left[\overline{u}(\theta)\right]X\left[\overline{u}(\theta)\right]}{\left\|X\left[\overline{u}(\theta)\right]\right\|^2}\equiv 1.\end{aligned}\qquad (33)$$

The system

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{\Theta(\theta,y)}Y(\theta,y) \qquad (34)$$

is formed next.

The function $\frac{Y(\theta,y)}{\Theta(\theta,y)}$ is periodical in θ with the period ω and, for a sufficient low *y,* has the same regularity properties as *X* because $\Theta(\theta,y)\neq 0$.

By Taylor – expanding $X\left[\overline{u}(\theta)+S(\theta)y\right]$ around $y=0$ we get

$$X\left[\overline{u}(\theta)+S(\theta)y\right]=X\left[\overline{u}(\theta)\right]+\frac{\mathrm{d}X\left[\overline{u}(\theta)\right]}{\mathrm{d}x}S(\theta)y+O\left(|y|^2\right). \qquad (35)$$

The variation of the matrix system is given by:

$$A(\theta)=\frac{\partial X\left[\overline{u}(\theta)\right]}{\partial x}. \qquad (36)$$

By multiplying the expression (35) with $X^*\left[\overline{u}(\theta)\right]$, it results

$$\frac{1}{X^*\left[\bar{u}(\theta)\right]X\left[\bar{u}(\theta)+S(\theta)y\right]}=\frac{1}{\left\|X\left[\bar{u}(\theta)\right]\right\|^2}\frac{1}{1+\dfrac{X^*\left[\bar{u}(\theta)\right]A(\theta)S(\theta)y}{\left\|X\left[\bar{u}(\theta)\right]\right\|^2}+O(|y|)} \tag{37}$$

where $O(|y|)$ represents the terms converging to zero with y.

Similarly with (21), by Taylor expression will get:

$$\frac{1}{1+\dfrac{X^*[\bar{u}(\theta)]A(\theta)S(\theta)y}{\|X[\bar{u}(\theta)]\|^2}}=1-\frac{y-0}{1!}\left[\frac{\dfrac{X^*[\bar{u}(\theta)]A(\theta)S(\theta)y}{\|X[\bar{u}(\theta)]\|^2}}{1+\dfrac{X^*[\bar{u}(\theta)]A(\theta)S(\theta)y}{\|X[\bar{u}(\theta)]\|^2}}\right]+O(|y|) \tag{38}$$

thus

$$\frac{1}{\Theta(\theta,y)}=1+O(|y|) \tag{39}$$

taking into account (30), (35), (36) and (39) the system (34) becomes

$$\frac{\mathrm{d}y^{\mu}}{\mathrm{d}\theta}=\xi_{\mu}^*(\theta)\left[A(\theta)S(\theta)-\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\right]y+\mathrm{O}(|y|) \tag{40}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=B(\theta)y+O(|y|), \tag{41}$$

where the elements $b_{\mu\nu}$ of the matrix B are given by:

$$b_{\mu\nu}(\theta)=\xi_{\mu}^*(\theta)\left[A(\theta)\xi_{\nu}-\frac{\mathrm{d}\xi_{\nu}}{\mathrm{d}\theta}\right]. \tag{42}$$

The elements $b_{\mu\nu}(\theta)$ are periodical functions of period ω.

Neglecting the term $O(|y|)$, which is approaching zero with y, the normal variations system can be written:

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=B(\theta)y. \tag{43}$$

The variations system corresponding to the periodical solution $u(\theta)$ is:

$$\frac{\mathrm{d}v}{\mathrm{d}\theta} = A(\theta)v. \tag{44}$$

Taking into account the vector space of the variation system (44), there exists a base in the orthogonal normalized system, so that each element of the solution space can be written as linear combination of the basis vectors.

Hence, it follows:

$$v(\theta) = \overline{p}(\theta)\overline{X}[\overline{u}(\theta)] + \sum_{\gamma=2}^{n} p^{v}(\theta)\xi_{v}(\theta) \tag{45}$$

or

$$v(\theta) = p(\theta)X[\overline{u}(\theta)] + \sum_{\gamma=2}^{n} p^{v}(\theta)\xi_{v}(\theta), \tag{46}$$

where

$$p(\theta) = \frac{\overline{p}(\theta)}{\left\| X[\overline{u}(\theta)] \right\|}. \tag{47}$$

By derivation of the expression (46) and taking into account:

$$\begin{aligned} &\frac{\mathrm{d}\overline{u}(\theta)}{\mathrm{d}\theta} = X[\overline{u}(\theta)] \\ &\frac{\mathrm{d}X[\overline{\mathrm{u}}(\theta)]}{\mathrm{d}\theta} = \left.\frac{\partial X[\overline{u}(\theta)]}{\partial x}\right|_{x=u(\theta)} X[\overline{u}(\theta)] = A(\theta)X[\overline{u}(\theta)], \end{aligned} \tag{48}$$

we finally obtain:

$$\frac{\mathrm{d}p}{\mathrm{d}\theta}X[\overline{u}(\theta)] + \sum_{v=2}^{n}\frac{\mathrm{d}p^{v}}{\mathrm{d}\theta}\xi_{v} + p^{v}\frac{\mathrm{d}\xi_{v}}{\mathrm{d}\theta} = \sum_{v=2}^{n} p^{v}A(\theta)\xi_{v}. \tag{49}$$

Multiplying the equation (49) with ξ^{*} we have:

$$\frac{\mathrm{d}p^{\mu}}{\mathrm{d}\theta} = \xi_{\mu}^{*}\sum_{v=2}^{n}\left[A(\theta)\xi_{v}(\theta) - \frac{\mathrm{d}\xi_{v}(\theta)}{\mathrm{d}\theta}\right]p^{v} = \sum_{v=2}^{n} b_{\mu v}(\theta)p^{v}. \tag{50}$$

Proposition 1. *The normal components of the variations v, are verifying the normal variations system*.

Proposition 2. $\frac{\mathrm{d}\overline{u}}{\mathrm{d}\theta} = X\left[\overline{u}\left(\theta\right)\right]$ *is a solution of the variation system (44).*

Proposition 3. *The matrix K has also the monodromy matrix U and, thus, its eigenvalues are multipliers of the variations system.*

Proposition 4. *The functions* $p_{\nu}^{\mu}\left(\theta\right) \quad \nu = 2,...,n$ *represents the terms of the fundamentals matrix for the normal variations system (43).*

Proposition 5. *The normal variations system has the eigenvakues of* K_2 *as multipliers. There values are the eigenvalues of K without the multiplier l corresponding to the periodical solution* $X\left[\overline{u}\left(\theta\right)\right]$.

From the Proposition 3 and Proposition 5 it might be concluded:

Proposition 6. *The normal variation system multipliers are obtained from the system multipliers eli8minating the multiplier l.*

The notion of orbital stability is connected with the periodic stability.

Let Γ be the integral curve corresponding to system (1) in the phase space. The distance between the point *x* and Γ is given by:

$$p\left(x,\Gamma\right) = \inf_{x\in\Gamma}\left\|x - z\right\|.$$

Definition 1. *We shall state the periodical solution* $x = u(t)$ *of the system (15) has the orbital stability if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *that if* $p(x_0,\Gamma) < \delta$ *then* $p(x(t,t_0),\Gamma) < \varepsilon$ *for each t, the results obtained in the previous developments are concluded by the following:*

Theorem. *If* $(n-1)$ *multipliers of the variation system, corresponding to the periodical solution* $x = u(t)$*, are inside the unit circle it follows that the periodical solution* $x = \overline{u}(t)$ *is orbitaly stable.*

In the sense of the orbital stability, the statement of theorem is equivalent with:

Admitting that $x_1\left(t\right)$ is an other solution of system (1) for which $x_1\left(t_0\right)$ is sufficient close to curve $x = \overline{u}(t)$ then, there exists *c* so that:

$$\lim_{t\to\infty}\left[x_1\left(t\right) + \overline{u}\left(t + c\right)\right] = 0. \tag{52}$$

The demonstration of this theorem is not the subject of this study.

5. CONCLUSIONS

The research performed so far [1], has approached the critical cases problem using the Liapunov theory. Thus the construction of a Liapunov function is providing an answer regarding the asymptotic stability or the instability of the trivial solution of the considered case, but it can not be stated whether the system is stable if asymptotic stability does not exist.

In [4] the stability of the three dimensional halo periodic orbits is analysed. A group of theorems regarding differential equations with periodic coefficients is used to determine the stability motion for the motion.

In this way we are leaded to the stability analysis of the periodical solution of the transformed system, which means the periodic solutions stability and, thus, the stability of the quasi-linear system trivial solution.

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