

# NEW PERIODIC LEM SOLUTIONS FOR THE NONLINEAR PENDULUM\*

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The linear equivalence method (LEM), introduced by the author, allows the study of nonlinear dynamical systems in a linear frame. By using the method of averaging along with the normal LEM representations, there are obtained new LEM periodic solutions for the free and forced nonlinear rigid pendulum.

## 1. INTRODUCTION

The linear equivalence method (LEM) was introduced by the author [19] to the aim of studying both qualitatively and quantitatively the nonlinear dynamical systems and their solutions. LEM does not require nonlinearities of particular type; moreover, the specific LEM representations of the solutions [20, 21] emphasize the influence of higher order effects within the solution by a finite matrix calculus that does not modify the previous steps.

Starting directly from the classic model of the nonlinear pendulum, there were previously obtained the normal LEM solutions for the free, damped and forced pendulum [14, 15, 18]. The scope of this paper is to present new representations based on LEM for the periodic solutions of the free and forced nonlinear pendulum using the averaged model; this emphasizes better the periodicity.

A brief presentation of an interesting recent application of LEM concludes this paper. It concerns the wobble soliton behaviour of a heavy elastica (Chiroiu and Munteanu [0]). The authors put into evidence a mathematical model presenting similitudes with that of the pendulum; they show via LEM that the ODE governing the large deformations of a heavy elastica of uniform density and cross-section anchored on one end allows wobble-soliton solutions; the well known results of Greenhill and Wang are thus special cases of this study.

## 2. LEM REVISITED

The linear equivalence method – LEM – initially introduced by the author for first order polynomial differential systems [19], could be extended to canonic first

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order ODEs, with right side analytic with respect to the unknown function (see *e.g.* [20, 21, 22]). As throughout the paper we deal only with ODS with constant coefficients, of null free term, we shall consider only ODSs of the form

$$\mathcal{F}(\mathbf{y}) \equiv \dot{\mathbf{y}} - \mathbf{f}(\mathbf{y}) = \mathbf{0}, \quad \mathbf{f}(\mathbf{y}) = [f_j(\mathbf{y})]_{j=\overline{1,n}}, \quad f_j(\mathbf{y}) = \sum_{|\mu|=1}^{\infty} f_{j\mu} \mathbf{y}^{\mu}, \quad (1)$$

where  $f_{j\mu}$ ,  $j = \overline{1,n}$ ,  $|\mu| \in \mathcal{U}$ , are all of them real constants.

As it was mentioned for the first time in [19], LEM considers an exponential mapping depending on  $n$  parameters –  $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathfrak{R}^n$  – namely

$$v(t, \xi) \equiv e^{\langle \xi, \mathbf{y} \rangle}, \quad (2)$$

that associates to the nonlinear ODS two linear equivalents [19, 21, 22]:

a) a linear PDE, always of first order with respect to  $t$

$$\mathcal{L}v(t, \xi) \equiv \frac{\partial v}{\partial t} - \langle \xi, \mathbf{f}(\mathbf{D}) \rangle v = 0, \quad (3)$$

and

b) a linear, while infinite, first order ODS

$$\frac{dv_{\gamma}}{dt} = \sum_{j=1}^n \gamma_j \sum_{|\mu|=1}^{m_j} f_{j\mu} v_{\gamma+\mu-\mathbf{e}_j}, \quad \mathbf{e}_j = (\delta_i^j)_{i=\overline{1,n}}. \quad (4)$$

The LEM equivalent (3) was obtained by differentiating (2) with respect to  $t$  and replacing the derivatives  $\dot{y}_j$  from the nonlinear ODS.

The usual notation  $f_j(\mathbf{D}_{\xi})$  stands for the formal operator

$$f_j(\mathbf{D}_{\xi}) = \sum_{|\mu|=1}^{\infty} f_{j\mu} \frac{\partial^{|\mu|}}{\partial \xi^{\mu}}. \quad (5)$$

The formal scalar product in (3) is expressed as

$$\sum_{j=1}^n \xi_j f_j(\mathbf{D}_{\xi}) \equiv \langle \xi, \mathbf{f}(\mathbf{D}) \rangle. \quad (6)$$

The second LEM equivalent, the system (4), is obtained from the first one, by searching the unknown function  $v$  in the class of analytic in  $\xi$  functions, uniformly with respect to  $t$ .

$$v(t, \xi) = 1 + \sum_{|\gamma|=1}^{\infty} v_{\gamma}(t) \frac{\xi^{\gamma}}{\gamma!}. \quad (7)$$

The LEM system (4) may be also written in matrix form

$$\mathfrak{D}\mathbf{V} \equiv \frac{d\mathbf{V}}{dt} - \mathbf{A}\mathbf{V} = \mathbf{0}, \quad \mathbf{V} = (\mathbf{V}_j)_{j \in \mathcal{O}}, \quad \mathbf{V}_j = (v_\gamma)_{|\gamma|=j}. \quad (8)$$

The LEM matrix  $\mathbf{A}$  is constant in this case and has a special form, being column-finite; in the case of a polynomial operator (see, e.g., [0]),  $\mathbf{A}$  is also row-finite. The cells  $\mathbf{A}_{ss}$  on the main diagonal are square, of  $s+1$  rows and columns; they are generated by the coefficients of the linear part of the operator – namely, those  $f_{j\mu}$  for which  $|\mu|=1$ . More precisely, the diagonal cells contain the coefficients of the linear part, on the next upper diagonal we find cells containing the coefficients of the second degree in  $\mathbf{y}$ , etc. In the case of polynomial operators of degree  $m$ , the associated LEM matrix is band-diagonal, the band being made up of  $m$  lines. One can express the LEM matrix as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \dots & \mathbf{A}_{1m} & \mathbf{A}_{1,m+1} & \dots \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} & \dots & \mathbf{A}_{2m} & \mathbf{A}_{2,m+1} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \dots & \mathbf{A}_{3m} & \mathbf{A}_{3,m+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \quad (9)$$

In the case of a nonhomogeneous ODS, the LEM matrix also contains a band of underdiagonal cells, generated by the free terms.

It should be mentioned that this particular form of the LEM matrix permits the calculus by block partitioning, which represents a considerable simplification.

Consider now for (4) or, equivalently, (8), the initial conditions

$$\mathbf{y}(t_0) = \mathbf{y}_0, \quad t_0 \in I. \quad (10)$$

By LEM, they are transferred to

$$v(t_0, \xi) = e^{(\xi, \mathbf{y}_0)}, \quad \xi \in \mathfrak{R}^n, \quad (11)$$

a condition that must be associated to the PDE (3), and

$$v_\gamma(t_0) = \mathbf{y}_0^\gamma, \quad |\gamma| \in \mathcal{O}, \quad (12)$$

indicating an initial condition for the system (4) or, equivalently, (8). For the matrix form, the initial conditions (12) become

$$\mathbf{V}(t_0) = (\mathbf{y}_0^\gamma)_{|\gamma| \in \mathcal{O}}. \quad (13)$$

**THEOREM 14.2.2.** [0] *The solution of the nonlinear initial problem (8), (10) i) coincides with the first  $n$  components of the infinite vector*

$$\mathbf{V}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{V}_0, \quad (14)$$

where the exponential matrix

$$e^{\mathbf{A}(t-t_0)} = \mathbf{E} + \frac{(t-t_0)}{1!} \mathbf{A} + \frac{(t-t_0)^2}{2!} \mathbf{A}^2 + \dots + \frac{(t-t_0)^k}{k!} \mathbf{A}^k + \dots, \quad (15)$$

can be computed by block partitioning, each step involving finite sums;  
ii) coincides with the series

$$y_j(t) = y_{j0} + \sum_{l=1}^{\infty} \sum_{|\gamma|=l} u_{j\gamma}(t) y_0^\gamma, \quad j = \overline{1, n}, \quad (16)$$

where  $u_{j\gamma}(t)$  are solutions of the finite linear ODSs

$$\frac{d\mathbf{U}_k}{dt} = \mathbf{A}_{1k}^T \mathbf{U}_1 + \mathbf{A}_{2k}^T \mathbf{U}_2 + \dots + \mathbf{A}_{kk}^T \mathbf{U}_k, \quad k = \overline{1, l}, \quad \mathbf{U}_s(t) = [u_\gamma(t)]_{|\gamma|=s}, \quad (17)$$

that satisfy the Cauchy conditions

$$\mathbf{U}_1(t_0) = \mathbf{e}_j, \quad \mathbf{U}_s(t_0) = \mathbf{0}, \quad s = \overline{2, l}, \quad (18)$$

T standing for transposed.

The representation (14) generalizes a similar one, previously obtained for polynomial ODSs. The corresponding result is very much alike a solution of a linear ODS with constant coefficients. There is more: the computation is even easier due to the fact that the eigenvalues of the diagonal cells are always known [22]. The generalized representation (16) is also called the *normal LEM representation* [20, 21, 22] and it was used in many applications requiring the qualitative behavior of the solution (see e.g. [4, 14, 22]).

### 3. THE NONLINEAR PENDULUM

Consider a rigid pendulum of length  $l$ , fixed up at a point, carrying a bob of mass  $m$  and free to oscillate in a vertical plane. In a simplified model, the equation of motion of the pendulum subjected to a periodic perturbation  $a \cos \theta t$  is

$$ml\ddot{y} + mg \sin y = a \cos \theta t. \quad (19)$$

The unknown function  $y$  is the angle of deviation from the vertical (equilibrium) position. Equation (19) may be also written as

$$\ddot{y} + \omega^2 \sin y = A \cos \theta t, \quad \omega^2 = \frac{g}{l}, \quad A = \frac{a}{ml}. \quad (20)$$

It is known that the equation (20), of Duffing's type, allows a unique periodic solution, of period  $2\pi/\theta$ , for  $\omega < \theta$  and any  $A$ .

To get the normal LEM solution of the above model, we add the Cauchy conditions

$$y(0) = \alpha, \quad \dot{y}(0) = 0. \quad (21)$$

#### 4. THE AVERAGED SYSTEM

To apply LEM to equation (20) we shall consider the equivalent first order system

$$\begin{aligned} \dot{y} &= z, \\ \dot{z} &= -\omega^2 \sin y + A \cos \theta t, \end{aligned} \quad (22)$$

where  $\theta$  is the frequency of the perturbation (forcing). A periodic solution, of period  $T$  close to that of the perturbation, is to be expected, therefore one can try to find solutions of the form

$$\begin{aligned} y &= u \cos \theta t - v \sin \theta t, \\ z &= -u \sin \theta t - v \cos \theta t. \end{aligned} \quad (23)$$

If we introduce these expressions in (22), we get a new nonlinear ODS of first order, having  $u$  and  $v$  as unknown functions, to which we apply the method of averaging, to avoid secular terms and non-constant coefficients. We finally get the averaged system

$$\begin{aligned} \dot{u} &= \left( \theta - \frac{1}{2} \right) v - \omega^2 \frac{v}{\sqrt{u^2 + v^2}} J_1 \left( \sqrt{u^2 + v^2} \right), \\ \dot{v} &= - \left( \theta - \frac{1}{2} \right) u + \omega^2 \frac{u}{\sqrt{u^2 + v^2}} J_1 \left( \sqrt{u^2 + v^2} \right) - \frac{A}{2}, \end{aligned} \quad (24)$$

where  $J_1(x)$  is the Bessel function of first order.

The corresponding Cauchy data are, obviously,

$$u(0) = \alpha, \quad v(0) = 0. \quad (25)$$

To find normal LEM representations for the averaged system (24), we shall perform a translation using criticality points, satisfying

$$- \left( \theta - \frac{1}{2} \right) u_0 + \omega^2 J_1(u_0) + \frac{A}{2} = 0. \quad (26)$$

The solutions of this functional equation depend on the forcing. The zeros  $u_0$  of (26) form at most a finite set.

### 5. NEW PERIODIC LEM SOLUTIONS FOR THE FORCED PENDULUM

Let us apply LEM to get third order effects. We perform the change of functions  $U = u - u_0, V = v$  and then truncate the system up to order 3.

The corresponding homogeneous ODS in  $U, V$  will be

$$\begin{aligned}\dot{U} &= \varphi_1 V - 2b_1 UV - A_1 U^2 V - B_1 V^3, \\ \dot{V} &= -\varphi_3 U + a_1 U^2 + b_1 V^2 + A_1 U^3 + B_1 UV^2,\end{aligned}\quad (27)$$

where we used the notations

$$a = \frac{u_0 J_1'(u_0) - J_1(u_0)}{u_0^3}, \quad \varphi_1 = \theta - \frac{1}{2} - \omega^2 \frac{J_1(u_0)}{u_0}, \quad \varphi_3 = \varphi_1 - \omega^2 a u_0^2 \quad (28)$$

and

$$A_1 = \frac{\omega^2}{2} \left( a - \frac{J_1(u_0)}{u_0} \right), \quad B_1 = \frac{\omega^2 a}{2}, \quad a_1 = u_0 A_1, \quad b_1 = u_0 B_1. \quad (29)$$

Let us also put

$$\Omega^2 = \varphi_1 \varphi_3. \quad (30)$$

The truncated LEM matrix is in this case

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} \end{bmatrix}, \quad (31)$$

where

$$\mathbf{A}_{11} = \begin{bmatrix} 0 & \varphi_1 \\ -\varphi_3 & 0 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} 0 & -2b_1 & 0 \\ a_1 & 0 & b_1 \end{bmatrix}, \quad \mathbf{A}_{22} \equiv \begin{bmatrix} 0 & 2\varphi_1 & 0 \\ -\varphi_1 & 0 & \varphi_1 \\ 0 & -2\varphi_3 & 0 \end{bmatrix}, \quad (32)$$

and

$$\mathbf{A}_{23} = \begin{bmatrix} 0 & -4b_1 & 0 & 0 \\ a_1 & 0 & -b_1 & 0 \\ 0 & 2a_1 & 0 & 2b_1 \end{bmatrix}, \quad \mathbf{A}_{33} = \begin{bmatrix} 0 & 3\varphi_1 & 0 & 0 \\ -\varphi_3 & 0 & 2\varphi_1 & 0 \\ 0 & -2\varphi_3 & 0 & \varphi_1 \\ 0 & 0 & -3\varphi_3 & 0 \end{bmatrix}, \quad (33)$$

$$\mathbf{A}_{13} = \begin{bmatrix} 0 & -A_1 & 0 & -B_1 \\ A_1 & 0 & B_1 & 0 \end{bmatrix}.$$

The normal LEM solution for the Cauchy data  $U(0)=\beta, V(0)=0$ , where  $\beta = \alpha - u_0$ , is written in the form

$$U(t) \cong \beta u_{10}^{(1)} + \beta^2 u_{20}^{(1)} + \beta^3 u_{30}^{(1)}, \quad V(t) \cong \beta u_{10}^{(2)} + \beta^2 u_{20}^{(2)} + \beta^3 u_{30}^{(2)}, \quad (34)$$

where  $u_{k0}^{(j)}, j=1,2$  are first components of the vectors  $\mathbf{U}_k(t)$  that satisfy the ODS

$$\dot{\mathbf{U}}_1 = \mathbf{A}_{11}^T \mathbf{U}_1, \quad \dot{\mathbf{U}}_2 = \mathbf{A}_{22}^T \mathbf{U}_2 + \mathbf{A}_{12}^T \mathbf{U}_1, \quad \dot{\mathbf{U}}_3 = \mathbf{A}_{33}^T \mathbf{U}_3 + \mathbf{A}_{23}^T \mathbf{U}_2 + \mathbf{A}_{13}^T \mathbf{U}_1, \quad (35)$$

and the null Cauchy conditions  $\mathbf{U}_2(0) = \mathbf{0}, \mathbf{U}_3(0) = \mathbf{0}$  for both  $j=1, 2$  and

$$\mathbf{U}_1(0) = [1, 0]^T \text{ for } j=1, \quad \mathbf{U}_1(0) = [0, 1]^T \text{ for } j=2. \quad (36)$$

The eigenvalues of  $\mathbf{A}_{11}^T$  are  $\pm i\Omega$ .

From the general LEM, it is known that the eigenvalues of  $\mathbf{A}_{22}^T$  are  $0, \pm 2i\Omega$  and those of  $\mathbf{A}_{33}^T$  are  $\pm i\Omega, \pm 3i\Omega$ . Using the Laplace transform, we can solve by blocks the above ODS, firstly getting

$$\begin{aligned} u_{10}^{(1)}(t) &= \cos \Omega t, \\ 6\Omega^2 u_{20}^{(1)}(t) &= 3(b_1 \varphi_3 + a_1 \varphi_1) + 2a_1 \varphi_1 \cos \Omega t + (3b_1 \varphi_3 + a_1 \varphi_1) \cos 2\Omega t, \\ 288\Omega^4 u_{30}^{(1)}(t) &= -96(b_1 \Omega^2 + a_1 \varphi_1^2) + [8(69b_1^2 \varphi_3^2 + 24a_1 b_1 \Omega^2 + 47a_1^2 \varphi_1^2) + \\ &\quad + 9(-3B_1 \varphi_3^3 - B_1 \varphi_3 \Omega^2 + A_1 \varphi_1 \Omega^2 + 3A_1 \varphi_3 \Omega^2)] \cos \Omega t + \\ &\quad + 32(3a_1 b_1 \Omega^2 + a_1^2 \varphi_1^2) \cos 2\Omega t + [96(9b_1^2 \varphi_3^2 + 10a_1 b_1 \Omega^2 + a_1^2 \varphi_1^2) + \\ &\quad + 144(3B_1 \varphi_3^3 + B_1 \varphi_3 \Omega^2 - A_1 \varphi_1 \Omega^2 - 3A_1 \varphi_3 \Omega^2)] \cos 3\Omega t + \\ &\quad + [48(9b_1^2 \varphi_3^2 + 6a_1 b_1 \Omega^2 + 5a_1^2 \varphi_1^2) + \\ &\quad + 18(3B_1 \varphi_3^3 + B_1 \varphi_3 \Omega^2 + 3A_1 \varphi_1 \Omega^2 + A_1 \varphi_3 \Omega^2)] \Omega t \sin \Omega t, \end{aligned} \quad (37)$$

and then

$$\begin{aligned} \Omega u_{10}^{(2)}(t) &= -\varphi_3 \sin \Omega t, \quad 3\Omega u_{20}^{(2)}(t) = a_1 (\sin \Omega t + \sin 2\Omega t), \\ 288\Omega^5 u_{30}^{(2)}(t) &= [2\varphi_3(-81b_1^2 \varphi_3^2 - 18a_1 b_1 \Omega^2 + 31a_1^2 \varphi_1^2) + \\ &\quad + 144(3B_1 \varphi_3^4 + 3B_1 \varphi_3^2 \Omega^2 + 8A_1 \Omega^4)] \sin \Omega t - 64a_1^2 \varphi_1 \Omega^2 \sin 2\Omega t + \\ &\quad + [-18(b_1^2 \varphi_3^3 + a_1^2 \varphi_1 \Omega^2) + 9(A_1 \Omega^2 - B_1 \varphi_3^2)(\varphi_3^2 + 3\Omega^2)] \sin 3\Omega t + \\ &\quad + [24\varphi_3(9b_1^2 \varphi_3^2 + 6a_1 b_1 \Omega^2 + 5a_1^2 \varphi_1^2) + \\ &\quad + 144(3B_1 \varphi_3^4 + B_1 \varphi_3^2 \Omega^2 + 3A_1 \Omega^4 + A_1 \varphi_3^2 \Omega^2)] \Omega t \sin \Omega t. \end{aligned} \quad (38)$$

We replace now these expressions in (34) to get  $U$  and  $V$ , then we compute  $u$  and  $v$  accordingly. The final step is to introduce  $u$  and  $v$  in (23), thus getting the LEM solution of order 3 of the forced pendulum problem.

## 6. NEW PERIODIC LEM SOLUTIONS FOR THE FREE PENDULUM

In the case of the free pendulum, therefore for  $A = 0$ , we have

$$\ddot{y} + \omega^2 \sin y = 0. \quad (39)$$

This ODE is equivalent to the following first order ODS

$$\dot{y} = z, \quad \dot{z} = -\omega^2 \sin y. \quad (40)$$

Suppose we wish to get the periodic solution of initial data

$$y(0) = \alpha, \quad z(0) = 0. \quad (41)$$

The period  $T$  corresponding to this solution allows a series expansion of the form

$$T(\alpha) = \frac{2\pi}{\omega} \left( 1 + \frac{\alpha^2}{16} + \frac{11}{3 \cdot 2^{10}} \alpha^4 + \dots \right). \quad (42)$$

This dependence is obtained in the literature by using developments for elliptic integrals. It was also obtained in a more general frame, by using LEM [0]. We perform firstly a rotation

$$\begin{aligned} y &= u \cos \varphi t - v \sin \varphi t, \\ z &= -u \sin \varphi t - v \cos \varphi t, \end{aligned} \quad (43)$$

where  $\varphi = 2\pi/T$ . Introducing this in (40), we again get a nonlinear ODS of first order, having  $u$  and  $v$  as unknown functions. Applying the method of averaging, we obtain

$$\begin{aligned} \dot{u} &= \left( \varphi - \frac{1}{2} \right) v - \omega^2 \frac{v}{\sqrt{u^2 + v^2}} J_1 \left( \sqrt{u^2 + v^2} \right), \\ \dot{v} &= - \left( \varphi - \frac{1}{2} \right) u + \omega^2 \frac{u}{\sqrt{u^2 + v^2}} J_1 \left( \sqrt{u^2 + v^2} \right). \end{aligned} \quad (44)$$

This system is homogeneous, analytic and odd with respect to  $u$  and  $v$ ; we do not need a translation in this case. The associated LEM matrix is

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{13} & \mathbf{A}_{15} & \dots \\ \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{35} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{55} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \quad (45)$$

It is seen that the LEM matrix does not contain cells with even indices. This is due to the fact that the nonlinear operator is odd. For such operators, the associated LEM matrix is always of the form (45).



The consequence of this special form of the LEM matrix is that the normal LEM representation will contain only odd powers of the Cauchy data.

Now, denote by

$$\varphi' = \varphi - \frac{\omega^2 + 1}{2}. \quad (46)$$

The square cells on the main diagonal of  $\mathbf{A}$  are

$$\mathbf{A}_{2j-1,2j-1} = \begin{bmatrix} 0 & (2j-1)\varphi' & 0 & \dots & 0 & 0 \\ -\varphi' & 0 & (2j-2)\varphi' & \dots & 0 & 0 \\ 0 & -2\varphi' & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \varphi' \\ 0 & 0 & 0 & \dots & -(2j-1)\varphi' & 0 \end{bmatrix}. \quad (47)$$

The first diagonal cell

$$\mathbf{A}_{11} = \begin{bmatrix} 0 & \varphi' \\ -\varphi' & 0 \end{bmatrix} \quad (48)$$

has the eigenvalues

$$\lambda_{1,2} = \pm i\varphi'. \quad (49)$$

Thus, the normal LEM representation will contain trigonometric functions, possibly multiplied by polynomials. According to the general LEM, the eigenvalues of  $\mathbf{A}_{2j-1,2j-1}$  are, respectively,

$$\lambda_{1,2} = \pm i\varphi', \lambda_{3,4} = \pm 3i\varphi', \dots, \lambda_{2j-1,2j} = \pm(2j-1)i\varphi'. \quad (50)$$

Let us stop to 5<sup>th</sup> order effects. To get the normal LEM representation, we must solve the system

$$\frac{d\mathbf{U}_1}{dt} = \mathbf{A}_{11}^T \mathbf{U}_1, \frac{d\mathbf{U}_3}{dt} = \mathbf{A}_{13}^T \mathbf{U}_1 + \mathbf{A}_{33}^T \mathbf{U}_3, \frac{d\mathbf{U}_5}{dt} = \mathbf{A}_{15}^T \mathbf{U}_1 + \mathbf{A}_{35}^T \mathbf{U}_3 + \mathbf{A}_{55}^T \mathbf{U}_5, \quad (51)$$

where  $\mathbf{U}_3(0) = \mathbf{0}$ ,  $\mathbf{U}_5(0) = \mathbf{0}$  and

$$\mathbf{U}_1(0) = [1, 0]^T, \quad (52)$$

for the LEM solution of  $u$  and

$$\mathbf{U}_1(0) = [0, 1]^T, \quad (53)$$

for the LEM solution of  $v$ .

In (51), the diagonal cells are given by (47) and the rectangular matrices read

$$\mathbf{A}_{13}^T = \frac{\omega^2}{16} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}_{15}^T = \frac{\omega^2}{384} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 2 \\ -2 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{A}_{35}^T = \frac{\omega^2}{16} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 1 & 0 & -3 \\ 3 & 0 & -1 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (54)$$

Knowing the eigenvalues of the diagonal cells, we can immediately invert the Laplace images or, alternatively, use LEM basis [25]. By either method, we get

$$\begin{aligned} u(t) &\cong \alpha \cos \varphi' t - \alpha^3 \frac{\omega^2}{16} t \sin \varphi' t + \alpha^5 \left[ - \left( \frac{\omega^2}{16} \right)^2 t^2 \cos \varphi' t + \frac{\omega^2}{384} t \sin \varphi' t \right], \\ v(t) &\cong -\alpha \sin \varphi' t - \alpha^3 \frac{\omega^2}{16} t \cos \varphi' t + \alpha^5 \left[ \left( \frac{\omega^2}{16} \right)^2 t^2 \sin \varphi' t + \frac{\omega^2}{384} t \cos \varphi' t \right]. \end{aligned} \quad (55)$$

Introducing this in (43), we get a new LEM representation for the free pendulum, that emphasizes better the periodicity of the solution.

## 7. WOBBLE SOLITONS FOR HEAVY ELASTICA

In [9], Chiroiu and Munteanu show that a heavy elastica cantilever, of uniform density and cross-section, anchored on one end, allows wobble soliton-like solutions. To prove this, they set up the following model

$$\frac{d^2 \theta}{ds^2} = K^3 (1-s) \sin \theta, \quad K = \left( \frac{\rho L^3}{EI} \right)^{1/3}, \quad (56)$$

where  $s$  is the arc length from the origin and  $\theta(s)$  is the local angle of inclination;  $EI$  is the flexural rigidity,  $L$  – the length of elastica and  $\rho$  its density. This model has some formal similitudes with that of the pendulum.

The boundary conditions are

$$\theta(0) = \alpha, \quad \frac{d\theta}{ds}(L) = 0, \quad (57)$$

$\alpha$  being the end angle.

This problem was treated in [9] by using LEM. The corresponding LEM solutions for this problem for 6<sup>th</sup> order effects contains a function of the form

$F = 4 \operatorname{Im}[\ln(F + iG)]$ , similar to the wobble-function called by Kälbermann the wobble soliton.

The LEM solutions are well fit for a qualitative study; by using them, the authors emphasized the flipping process of the elastica deformations. The cantilever is flipped to the other side as the torque changes sign. In the flipping process, the energy is dissipated through oscillations, since the distorted kinks decay to wobble solitons. The results are relevant to a study of stability of the solutions and are in good agreement with the classic results, previously obtained in a linear frame or via perturbation methods.

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