A NOVEL MECHANICAL MODEL FOR MAGNETIC RESONANCE ELASTOGRAPHY

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In the memory of my professors Dr.C. Beju,
Dr. G. Camenschi and Dr. E. Sóos

Magnetic resonance elastography is a non-invasive, *in vivo* imaging technique used to measure the elasticity of soft tissues subject to mechanical stresses. The resulting strains are measured using MR imaging and the elastic modulus is computed from viscoelastic models of tissue mechanics. In this paper we propose a novel model, called the fractional model of continuum mechanics, which is a generalization of the classical continuum mechanics theory in which the classic integer order spatial derivatives are replaced by fractional order spatial derivatives. The fractional model is a non-local, multi-scale model that can be used to solve both direct and inverse problems of importance in practical applications. In particular, our results to the inverse problem of elastography will show that the proposed model is capable to find the correct elastic moduli without having to perform any of the image processing steps required by the use of the classical continuum mechanics theory.

1. INTRODUCTION

For centuries, palpation has been an important medical diagnostic tool which uses the fact that many diseases change the mechanical properties of tissues. These changes are caused either by exudation of fluids from the vascular into the extra- and intracellular space or by loss of lymphatic systems, as in the case of cancer [5]. The result is an increase in stiffness or elastic modulus of the tissue. Even today it is common for surgeons to find tumors during surgery that have been missed by CT, magnetic resonance (MR), or ultrasound. None of these imaging modalities provide the information about the elastic properties of tissue elicited by palpation. The elastic moduli of various human soft twhaissues are known to vary over a wide range, more than four orders of magnitude [1]. In contrast, most of the physical properties depicted by conventional medical imaging modalities are distributed over a much smaller numerical range. These observations have provided the

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Rev. Roum. Sci. Techn. – Méc. Appl., Tome 55, Nº 1, P. 3–18, Bucarest, 2010

motivation for many researchers to seek a medical imaging technology that can estimate or assess the elastic moduli of tissues. The approaches to date have been to use conventional imaging methods to measure the mechanical response of tissue to static, quasi-static or cyclic mechanical stress. The resulting strains have been measured using ultrasound, CT or MRI and the related elastic modulus has been computed from biomechanical models of tissues. Recent reviews of such work are given in [4] and [5].

Most of the pioneering work in elasticity imaging has been done using ultrasound and either a quasi-static stress model or a dynamic stress model. The methods for delineating tissue elasticity using MRI are based on motion-encoded phase-contrast imaging techniques. The spatial strain distribution is the result of a quasi-static longitudinal stress application or of dynamical, harmonic mechanical excitation. Unlike the ultrasound-based techniques, the magnetic resonance elastography (MRE) method using harmonic shear waves offers direct visualization and quantitative measurement of tissue displacements, high sensitivity to very small motions, a field of view unencumbered by acoustic window requirements, and the ability to obtain full 3D displacement information throughout a three dimensional volume [6].

In order to recover the mechanical properties of biological tissues we need to invert the displacement data measured by the MRE method. This inversion process requires the use of an accurate biomechanical model for tissues. It was noticed experimentally that most biological tissues have incompressible viscoelastic features [2]. The incompressibility assumption for soft tissues is based on the fact that most tissues are made primarily of water. In addition, since the displacements in MRE are very small (on the order of microns), a linear constitutive law is usually assumed [8]. However, despite the richness of the data set, the variety of processing techniques and the simplifications made in the biomechanical model, it remains a challenge to extract accurate results at high resolution in complex, heterogeneous tissues from the intrinsically noisy data. Therefore, any improvement in the MRE data processing with the help of biomechanics and mathematical methods will be of significant importance to the modern medicine. MRE can help in tumor detection, in determination of characteristics of disease, and in assessment of rehabilitation. Also, the elastic parameters found by MRE can be used in the validation of biomechanical models capable of correctly predicting the dynamics of tissues, and the growth of tumors.

In this paper we will focus on the MRE method using harmonic shear waves and, like in [8], assume that biological tissues are almost incompressible, isotropic, linear viscoelastic solids. However, the classic Hooks-like linear relationship between strain and stress used in [8] can describe only the macroscopic behavior of tissues, and thus, any information from microscopic levels, which we believe is present in the measured data due to the advanced imaging techniques used during experiments, is considered as data artifacts or noise which have to be eliminated in order for the model to give reliable results. In the present paper, we propose a new multi-scale, non-local linear model to analyze the MRE data. We call this new model *the fractional model of continuum mechanics* since it is obtained from the classical equations of continuum mechanics by replacing the integer order spatial derivatives used to represent relative displacement and force between any two neighboring particles by fractional order spatial derivatives. This will relax the constraint of differentiability imposed on the displacement fields, and extend the class of allowable displacements to non-continuous fields. We conjecture that the fractional order parameter of the fractional order derivative is a measure of the dynamic deformation of the space scale and thus, by representing the law of motion as a fractional order integral, we represent, in fact, a multi-scale type of motion in which the microscopic and macroscopic levels are smoothly connected by the presence of the integral. In addition, since the fractional derivatives are non-local, the new theory can model many other problems of fundamental importance in mechanics such as those which involve the formation of cracks, phase transitions, or the presence of inclusions or mixtures. Our fractional model of continuum mechanics is *a new, general mathematical framework for mechanics* which should not be confused either with constitutive laws of temporal fractional order used to describe the linear visco-elastic behavior of certain materials [12-16, 17 and references within] or with non-local constitutive laws for one-dimensional elastic materials with microstructure [10, 11].

In the present paper we will use our fractional continuum mechanics model to solve the inverse problem of MRE and show that the proposed model is capable to find the correct elastic moduli without having to perform any of the image processing steps required by the use of the classical continuum mechanics theory. The paper is organized as follows. In the next section we give a brief review of fractional calculus, followed by a section in which we introduce the fractional continuum mechanics model. In section 4 we present the forms of the equations of motion in the classic and the fractional models of continuum mechanics which are used in the inverse problem of MRE. In section 5 we show how our model performs on a two-dimensional MRE data set. The paper ends with a section of conclusions and further work. The present paper is an expansion of the work presented by the author in [9].

2. BRIEF REVIEW OF FRACTIONAL CALCULUS

In this section we present a few fractional calculus results taken from [16, 18] that we will use in the development of the fractional continuum mechanics model.

Definition 1. If $f \in L^1([a,b])$ and $\alpha \in (-\infty,1)$ then:

1. $I_{a+}^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \dfrac{f(s)\mathrm{d}s}{(t-s)^{1-\alpha}}$ is called *the left-sided Riemann-Liouville fractional integral of order* α and,

$I_{b-}^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^b \dfrac{f(s)\mathrm{d}s}{(s-t)^{1-\alpha}}$ is the *right-sided Riemann-Liouville fractional integral of order* α.

2.

$$D_{a+}^{\alpha} f(t) = \begin{cases} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} I_{a+}^{m-\alpha} f(t) = \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} \displaystyle\int_a^t \dfrac{f(s)\mathrm{d}s}{(t-s)^{\alpha+1-m}}, & \text{if } \alpha \in (m-1, m), m = 1,2,3,\ldots \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \text{if } \alpha = m,\ m = 1,2,3\ldots \\ I_{a+}^{-\alpha} f(t) = \dfrac{1}{\Gamma(-\alpha)} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} \displaystyle\int_a^t \dfrac{f(s)\mathrm{d}s}{(t-s)^{\alpha+1}}, & \text{if } \alpha < 0 \end{cases}$$

is called the *left-sided Riemann-Liouville fractional derivative of order* α. Similarly, we can introduce the right-sided Riemann-Liouville fractional derivative of order α. In addition, $D_{a+}^{\alpha+\beta} f(t) = D_{a+}^{\max(\alpha,\beta)} \left(D_{a+}^{\min(\alpha,\beta)} f(t) \right)$.

3. ${}_{a+}R_{b-}^{\alpha} f(t) = I_{a+}^{\alpha} f(t) + I_{b-}^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^b \dfrac{f(s)\mathrm{d}s}{|t-s|^{1-\alpha}}$ is called the *Riesz integral*, where $\Gamma(s) = \displaystyle\int_0^{\infty} \mathrm{e}^{-t} t^{s-1} \mathrm{d}t$ is the gamma function.

Some important properties of fractional order integrals and derivatives are listed in the following proposition:

Proposition 1.

1. *If* $f \in L^p([a,b]),\ 1 \le p < \infty$ *and* $0 < \alpha < 1$ *then* $D_{a+}^{\alpha} I_{a+}^{\alpha} f(t) = f(t),\ \forall t \in [a,b]$.

2. *If* $f \in L^1([a,b])$ *and there exists* $g \in L^p([a,b]),\ 1 \le p < \infty$ *such that* $f = I_{a+}^{\alpha} g$ *and* $0 < \alpha < 1$, *then* $I_{a+}^{\alpha} D_{a+}^{\alpha} f(t) = f(t),\ \forall t \in [a,b]$.

3. *If* $f \in L^2(\ )$ *then* $\Im\left(D^{\alpha}_{0+} f(t)\right) = (\mathrm{i}\omega)^{\alpha}\,\Im(f(t))$, *where* $\Im(f(t)) = \int f(t) e^{-\mathrm{i}\omega t}\mathrm{d}t$ *is the Fourier transform.*

4. *If* $I^{\alpha}_{-} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \frac{f(s)\mathrm{d}s}{(t-s)^{1-\alpha}}$, $I^{\alpha}_{+} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} \frac{f(s)\mathrm{d}s}{(s-t)^{1-\alpha}}$ *converge, then* $I^{\alpha}_{-} f(t) = I^{\alpha}_{+} g(t')$ *where* $g(t') = f(-t'),\ t' = -t$, and $D^m I^{m-\alpha}_{-} f(t) = I^{m-\alpha}_{-} D^m f(t)$, $D^m I^{m-\alpha}_{+} f(t) = I^{m-\alpha}_{+} D^m f(t)$ *for* $m-1 < \alpha \le m,\ m = 1, 2, 3...$

Definition 1 can be generalized to ^{n} as follows:

Definition 2. If $\vec{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) > 0$ and $f \in L^p\left(\ ^n\right),\ 1 \le p < \min\limits_{\underline{k=1,n}} \left(\frac{1}{\alpha_k}\right)$ then:

1. The *left-sided mixed Riemann-Liouville fractional integral of order* $\vec{\alpha}$ is given by:

$$I^{\vec{\alpha}}_{\vec{a}+} f\left(\vec{x}\right) = I^{\alpha_1}_{a_1+} I^{\alpha_2}_{a_2+} ... I^{\alpha_n}_{a_n+} f(x_1, x_2, ..., x_n)$$
$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)...\Gamma(\alpha_n)} \int_{a_1}^{x_1}\int_{a_2}^{x_2} ... \int_{a_n}^{x_n} \frac{f(t_1, t_2, ..., t_n)\mathrm{d}t_1\mathrm{d}t_2...\mathrm{d}t_n}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}...(x_n-t_n)^{1-\alpha_n}}.$$

2. An *operator of the Riesz polypotential type of order* α is given by:

$$K^{\vec{\alpha}} f\left(\vec{x}\right) = R^{\alpha_1} R^{\alpha_2} ... R^{\alpha_n} f(x_1, x_2, ..., x_n)$$
$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)...\Gamma(\alpha_n)} \int_{^n} \frac{f(t_1, t_2, ..., t_n)\mathrm{d}t_1\mathrm{d}t_2...\mathrm{d}t_n}{|x_1-t_1|^{1-\alpha_1}|x_2-t_2|^{1-\alpha_2}...|x_n-t_n|^{1-\alpha_n}}.$$

3. The *Riesz potential of order* $\alpha > 0$ is given by:

$$R^{\alpha} f\left(\vec{x}\right) = \int_{^n} k_{\alpha}\left(\vec{x}-\vec{y}\right) f\left(\vec{y}\right)\mathrm{d}y_1\mathrm{d}y_2...\mathrm{d}y_n\ ,$$

where the Riesz kernel has the form:

$$k_{\alpha}\left(\vec{x}\right) = \frac{1}{\gamma_n(\alpha)} \begin{cases} |\vec{x}|^{\alpha-n}, & \alpha - n \ne 0, 4, 6, ... \\ |\vec{x}|^{\alpha-n} \ln\frac{1}{|\vec{x}|}, & \alpha - n = 0, 4, 6, ... \end{cases}$$, with $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + ... x_n^2}$ and

$$\gamma_n(\alpha) = \begin{cases} \dfrac{2^{\alpha}\pi^{n/2}\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, & \alpha \neq n+2k, \alpha \neq -2k \\ 1, & \alpha = -2k,\ k=1,2,3... \\ (-1)^{(n-\alpha)/2}\pi^{n/2}2^{\alpha-1}\left(\dfrac{\alpha-n}{2}\right)!\ \Gamma(\alpha/2), & \alpha = n+2k. \end{cases}$$

In the following section we introduce our fractional continuum mechanics model.

3. THE FRACTIONAL CONTINUUM MECHANICS MODEL

We assume that a continuous medium occupies at time t = 0 a region Ω which is a subset of $\mathbb{R}^3$ and, at time t > 0, occupies a region $\Omega_t \subset \mathbb{R}^3$, where Ω and Ω_t are assumed to be bounded, open, and connected. Let P be an arbitrary particle of an object, $\vec{X} = \left(X_K\right)_{K=\overline{1,3}}$ be the position of the particle at t = 0 and $\vec{x} = \left(x_k\right)_{k=\overline{1,3}}$ be the position of the same particle P at t > 0.

Definition 3. Let α be a 3×3 matrix whose elements $\alpha_{Ii},\ I,i=1,2,3$ are continuous functions of $t>0$ such that $-\infty < \alpha_{Ii}(t) \leq 1,\ \forall t>0,\ \forall I,i=1,\ 2,3$. Let $\vec{\chi}(\cdot;t,\alpha(t)):\Omega \to \vec{\chi}(\Omega,t,\alpha(t)) \equiv \Omega_{t,\alpha(t)}$ be a family of functions in $L^1(\Omega)$. The *motion (deformation) of order* $\alpha(t)$ of a body is determined by the position $\vec{x}$ of the material points in space as a function defined for every $t>0$ and every $\alpha_{Ii}(t) \in (-\infty,1],\ I,i=1,2,3$ by:

$$\vec{x} = \begin{cases} \mathbf{K}_{\vec{X}}^{\mathbf{1}-\boldsymbol{\alpha}(t)}\vec{\chi}(\vec{X},t), & \text{if } -\infty < \alpha_{Ii}(t) < 1, \quad I,i=1,2,3 \\ \vec{\chi}(\vec{X},t), & \text{if } \alpha_{Ii}(t)=1, \quad I,i=1,2,3 \end{cases} \tag{1}$$

In the above definition we denoted by **1** the 3×3 matrix made of ones and by

$$\mathbf{K}_{\vec{X}}^{\mathbf{1}-\boldsymbol{\alpha}(t)}\cdot = \begin{pmatrix} K_{\vec{X}}^{\vec{1}-\vec{\alpha_1}(t)}\cdot & 0 & 0 \\ 0 & K_{\vec{X}}^{\vec{1}-\vec{\alpha_2}(t)}\cdot & 0 \\ 0 & 0 & K_{\vec{X}}^{\vec{1}-\vec{\alpha_3}(t)}\cdot \end{pmatrix}.$$

the integral operator whose main diagonal is made of operators of Riesz polypotential type of order $\vec{1}-\vec{\alpha_i}(t) = (1-\alpha_{1i}, 1-\alpha_{2i}, 1-\alpha_{3i}),\ i=1,2,3$ along each spatial direction, where $\overrightarrow{\alpha_i} = (\alpha_{1i}, \alpha_{2i}, \alpha_{3i})$, $i = 1, 2, 3$ are the rows of matrix $\boldsymbol{\alpha}$. Thus, by definition 2.2 of the Riesz polypotential, the components of the motion defined by equation (1) are:

$$x_i = K_{\vec{X}}^{\vec{1}-\overrightarrow{\alpha_i}(t)} \chi_i(\vec{X},t) = R_{X_1}^{1-\alpha_{1i}(t)} R_{X_2}^{1-\alpha_{2i}(t)} R_{X_3}^{1-\alpha_{3i}(t)} \chi_i(\vec{X},t) =$$

$$= \frac{1}{\Gamma(1-\alpha_{1i}(t))\Gamma(1-\alpha_{2i}(t))\Gamma(1-\alpha_{3i}(t))} \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \frac{\chi_i(\vec{Y},t)\mathrm{d}Y_1\mathrm{d}Y_2\mathrm{d}Y_3}{|X_1-Y_1|^{\alpha_{1i}(t)}|X_2-Y_2|^{\alpha_{2i}(t)}|X_3-Y_3|^{\alpha_{3i}(t)}}. \tag{2}$$

The interval of integration $H = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ defines the *horizon* of the reference position $\vec{X}$ made of all the material points that contribute to the deformation of $\vec{X}$ into $\vec{x}$. In general, H may depend on the order α, but for simplicity we assume that H and α are independent. We assume further that the integration in (2) is done on the entire ^{3} by prolonging, if necessary, the function $\vec{\chi}$ by zero outside the finite horizon H.

We conjecture that the parameter matrix $\boldsymbol{\alpha}(t)$ measures the dynamic deformation of the space scale as a function of time and therefore it may be related to an intrinsic deformation field present in a body at any given time. This deformation field may be due to electro-magnetic, thermal, and/or chemical processes that take place in objects *independently* of any existing mechanical stresses. By representing the law of motion as fractional order integrals we represent, in fact, a multi-scale type of motion in which the microscopic and macroscopic levels are smoothly connected by the presence of the integrals. Note that the case $\boldsymbol{\alpha}(t) = \mathbf{1},\ \forall t > 0$ could be seen as describing macroscopic motions caused merely by mechanical loading in objects without microstructure and corresponds to the classic continuum mechanics case. For simplicity, we assume further that α is a constant matrix with $-\infty < \alpha_{Ii} \le 1,\ I, i = 1,2,3$.

Definition 4. A deformation of order α is *homogeneous* if $\alpha_{Ii} = \alpha = \text{constant} > 0,\ \forall I, i = 1, 2, 3$ and the deformation is defined using the Riesz potential:

$$\vec{x} = R^{1-\alpha}\vec{\chi}(\vec{X},t) = \iiint_{\ ^3} k_\alpha(\vec{X}-\vec{Y})\vec{\chi}(\vec{Y},t)\mathrm{d}Y_1\mathrm{d}Y_2\mathrm{d}Y_3\ , \tag{3}$$

where the Riesz kernel is given in definition 2.3.

Definition 5. We define the *deformation of order* α, $-\infty < \alpha_{Ii} \le 1,\ I, i = 1, 2, 3$ as:

$$\mathbf{F}_\alpha = \left(\frac{\partial x_k}{\partial X_K}\right)_{k,K=\overline{1,3}}, \tag{4}$$

where the law of motion is given either by equation (1) or by equation (3).

From (1) (or (3)), (4) and definition 1.2, $\mathbf{F}_\alpha$ is a mixture of fractional order derivatives and integrals. For example, we can have the following representations of elements of $\mathbf{F}_\alpha$:

$$\frac{\partial x_k}{\partial X_K} = \begin{cases} D_{X_K}^{\alpha_{Kk}} R_{X_I}^{1-\alpha_{Ik}} R_{X_J}^{1-\alpha_{Jk}} \chi_k(\vec{X},t), & 0<\alpha_{Kk},\alpha_{Ik},\alpha_{Jk}<1,\ I\neq J\neq K \\ R_{X_K}^{-\alpha_{Kk}} R_{X_I}^{1-\alpha_{Ik}} R_{X_J}^{1-\alpha_{Jk}} \chi_k(\vec{X},t), & \alpha_{Kk}<0,\ 0<\alpha_{Ik},\ \alpha_{Jk}<1,\ I\neq J\neq K \\ \dfrac{\partial \chi_k}{\partial X_K}, & \alpha_{1k}=\alpha_{2k}=\alpha_{3k}=1. \end{cases}$$

Note that if $\vec{U}\left(\vec{X},t\right)$ is the displacement vector in the reference configuration and $\vec{u}\left(\vec{x},t\right)=\mathbf{K}_{\vec{X}}^{\mathbf{1-\alpha}(t)}\left[\vec{\chi}\left(\vec{X}+\vec{U},t\right)-\vec{\chi}\left(\vec{X},t\right)\right]$ is the displacement vector in the actual configuration then the following is true: $\vec{u}=\mathbf{F}_\alpha\vec{U}$.

If $\vec{\chi}\in L^p(H),\ 1\le p<\infty$, where H is the horizon, has a unique inverse then, from proposition 1.1, we obtain for $0<\alpha_{Kk}\le 1,\ \forall K,k=1,2,3$:

$$\vec{X}=\vec{\chi}^{-1}(\mathbf{D}_{\vec{X}}^{\mathbf{1-\alpha}}\vec{x},t),$$

with

$$\mathbf{D}_{\vec{X}}^{\mathbf{1-\alpha}}\cdot=\begin{pmatrix} D_{X_1}^{1-\alpha_{11}}D_{X_2}^{1-\alpha_{21}}D_{X_3}^{1-\alpha_{31}}\cdot & 0 & 0 \\ 0 & D_{X_1}^{1-\alpha_{12}}D_{X_2}^{1-\alpha_{22}}D_{X_3}^{1-\alpha_{32}}\cdot & 0 \\ 0 & 0 & D_{X_1}^{1-\alpha_{13}}D_{X_2}^{1-\alpha_{23}}D_{X_3}^{1-\alpha_{33}}\cdot \end{pmatrix}.$$

Since the properties in proposition 1.1 and 1.2 are not valid for $-\infty<\alpha_{Kk}<0,\ \forall K,k=1,2,3$, an inversion of the motion (1) is not possible in this case, which means that the spatial memory of each material point during such a motion is lost, this deformation process is irreversible.

While the definitions of most of the strain tensors remain the same as in the classical theory of continuum mechanics, the definition of contact forces (and hence of the stresses) will change to accommodate our new definition of deformation.

Definition 6. If $\mathrm{d}\vec{P}$ is a contact force acting on the deformed area element $\mathrm{d}\vec{a}=\vec{n}\mathrm{d}a$, where $\vec{n}$ is the unit outer normal to the element of area $\mathrm{d}a$, then the α*-contact force* is by definition:

$$\mathrm{d}\overrightarrow{P_\alpha} = \mathbf{K}_{\vec{x}}^{\mathbf{1-\alpha}} \mathrm{d}\vec{P}(\vec{x},t)\,, \tag{5}$$

and *the stress vector of order* α is given by:

$$\vec{t}_\alpha(\vec{x},t) = \lim_{\mathrm{d}a\to 0} \frac{\mathrm{d}\overrightarrow{P_\alpha}(\vec{x},t)}{\mathrm{d}a} = \mathbf{K}_{\vec{x}}^{\mathbf{1-\alpha}}\, \vec{t}(\vec{x},t)\,, \tag{6}$$

where $\vec{t}(\vec{x},t) = \lim_{\mathrm{d}a\to 0} \frac{\mathrm{d}\vec{P}(\vec{x},t)}{\mathrm{d}a}$ is the classic true stress vector and $K_{\vec{x}}^{\mathbf{1-\alpha}}$ is the Riesz polypotential from definition 2.2.

The α-contact forces can be seen as interaction forces at long distances: the multi-scaling parameter α of the motion of an object determines what parts of the object are linked and interact during the deformation of order α. We can introduce now the following definition:

Definition 7. The *Cauchy stress tensor of order* α is by definition:

$$\mathbf{T}_\alpha = \mathbf{K}_{\vec{x}}^{\mathbf{1-\alpha}}\, \mathbf{T}\,, \tag{7}$$

where **T** is the classical Cauchy stress tensor.

From (7) it follows that α can be seen, for example, as a measure of anisotropy of electro-magnetic, thermal, and/or chemical nature, while the purely mechanical anisotropy is present only in the expression of **T**. Thus our model allows materials to be mechanically isotropic and electrically, thermally, and/or chemically anisotropic. We will say that an object is *totally isotropic* if $\alpha_{Kk} = \alpha = \text{const},\ \forall K,\ k = 1, 2, 3$ and the classic Cauchy tensor **T** depends only on the invariants of the deformation tensors. If the material is homogeneous, then we replace the Riesz polypotential used in (7) by the Riesz potential as in (3).

We give now the new forms of the equations of motion. The Eulerian forms of the global equations of motion of a deformed object occupying domain Ω_t and of mass density $\rho(\vec{x},t)$ which is subjected to an α-contact force given by a stress vector of order α, $\overrightarrow{t_\alpha}(\vec{x},t)$, acting over the surface area of $\partial\Omega_t$ and a body force $\vec{b}(\vec{x},t)$ acting over the volume Ω_t are given by:

Principle of *linear momentum*:

$$\int_{\partial\Omega_t} \overrightarrow{t_\alpha}\mathrm{d}a + \int_{\Omega_t} \vec{b}\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\mathrm{d}v\,. \tag{8}$$

Principle of *angular momentum*:

$$\int_{\partial\Omega_t} \vec{x} \times \overrightarrow{t_{\boldsymbol{\alpha}}} \mathrm{d}a + \int_{\Omega_t} \vec{x} \times \vec{b}\, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho \left(\vec{x} \times \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \right) \mathrm{d}v . \tag{9}$$

The quasi-local forms of the above equations are:

$$\nabla_{\vec{x}} \cdot \mathbf{T}_{\boldsymbol{\alpha}} + \vec{b} - \rho \frac{\mathrm{d}^2 \vec{x}}{\mathrm{d}t^2} = \vec{0} \tag{10}$$

following from equation (8), where $\nabla_{\vec{x}} \cdot = \left(\frac{\partial \cdot}{\partial x_1}, \frac{\partial \cdot}{\partial x_2}, \frac{\partial \cdot}{\partial x_3} \right)$, and respectively:

$$\mathbf{T}_{\boldsymbol{\alpha}} = \mathbf{T}_{\boldsymbol{\alpha}}^{\mathrm{T}}, \text{ and } \alpha_{K1} = \alpha_{K2} = \alpha_{K3} \equiv \alpha_K, \; K = 1, 2, 3, \tag{11}$$

which follows from (9), definition (7), and the symmetry of the classic Cauchy stress tensor. The presence of the fractional order integrals in equations (10) and (11) makes these forms quasi-local in nature. We note that in order to conserve the angular momentum the matrix $\boldsymbol{\alpha}$ should be a vector $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. However, for ferofluids, molecular fluids or solids with microstructure in electromagnetic field, the individual molecules or microstructural particles may be spinning and thus for such materials the conservation of angular momentum is not required [19-21]. In this case $\mathbf{T}_{\boldsymbol{\alpha}}$ does not need to be a symmetric tensor and thus $\boldsymbol{\alpha}$ can be an arbitrary 3×3 matrix. We will show that for the MRE problem the best results are obtained with a non-symmetric Cauchy stress tensor of order $\boldsymbol{\alpha}$.

If (11) is satisfied, we introduce the following fractional order gradient which is a generalized form of the fractional gradient from [22]:

$$\nabla_{\vec{x}}^{\vec{\alpha}} \cdot = \left(D_{x_1}^{\alpha_1} R_{x_2}^{1-\alpha_2} R_{x_3}^{1-\alpha_3} \cdot \;,\; D_{x_2}^{\alpha_2} R_{x_1}^{1-\alpha_1} R_{x_3}^{1-\alpha_3} \cdot \;,\; D_{x_3}^{\alpha_3} R_{x_1}^{1-\alpha_1} R_{x_2}^{1-\alpha_2} \cdot \right), \tag{12}$$

and thus it can be shown as in [22] that equation (10) becomes:

$$\nabla_{\vec{x}}^{\vec{\alpha}} \cdot \mathbf{T} + \vec{b} - \rho \frac{\mathrm{d}^2 \vec{x}}{\mathrm{d}t^2} = \vec{0} . \tag{13}$$

When $\vec{\alpha} = (1, 1, 1)$, equation (13) becomes the classic equation of motion of the continuum mechanics theory. When $\alpha_K < 0, \; \forall K = 1, 2, 3$ then, from definition 1.2, it follows that the equation of motion (13) contains only triple integrals of the classic Cauchy stress tensor and *no spatial derivatives*. In addition, the strain tensors contain only fractional order integrals and no spatial derivatives so in this case equation (13) is a non-local integral equation of motion.

A particularly interesting case is when $\mathbf{T}_{\boldsymbol{\alpha}}$ is not symmetric and $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & 1 & 1 \\ 1 & \alpha_{22} & 1 \\ 1 & 1 & \alpha_{33} \end{pmatrix}$. In this case we denote by $\vec{\alpha} = (\alpha_{11}, \alpha_{22}, \alpha_{33})$ and introduce the fractional gradient given in [22]:

$$\nabla_{\vec{x}}^{\vec{\alpha}} \cdot = \left(D_{x_1}^{\alpha_{11}} \cdot,\ D_{x_2}^{\alpha_{22}} \cdot,\ D_{x_3}^{\alpha_{33}} \cdot \right). \tag{14}$$

We obtain again the equation of motion (13). We have presented this particular case in [9] and will consider it in this paper, as well. If $\alpha_{jj} < 0,\ \forall j = 1, 2, 3$ then, according to definition 1.2 of the fractional order derivative, equation (13) will be a non-local integral equation containing only single integrals of the classic stress tensor:

$$\sum_{j=1}^{3} \frac{1}{\Gamma(-\alpha_{jj})} \int_H \frac{T_{ij}(\vec{y},t)}{\left| x_j - y_j \right|^{1+\alpha_{jj}}} \mathrm{d}y_j + b_i - \rho \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = 0, \quad i = 1, 2, 3. \tag{15}$$

4. THE INVERSE PROBLEM OF MRE

In this section we formulate the inverse problem of MRE using the new fractional model of continuum mechanics. We assume that the soft tissues are almost incompressible, linear viscoelastic solids, $\vec{b} = \vec{0}$, and $\rho = 1\ \mathrm{g/cm}^3$ [8]. Under oscillatory loading, equation (13) takes the following form in the frequency domain:

$$\Lambda \nabla_{\vec{x}}^{\vec{\alpha}} \left(\nabla_{\vec{x}} \cdot \vec{U} \right) + M \left[\nabla_{\vec{x}} \left(\nabla_{\vec{x}}^{\vec{\alpha}} \cdot \vec{U} \right) + \left(\nabla_{\vec{x}} \cdot \nabla_{\vec{x}}^{\vec{\alpha}} \right) \vec{U} \right] = -\omega^2 \vec{U}, \tag{16}$$

where $\vec{U}$, Λ, M are the temporal Fourier transforms of the displacement field, and of the Lamé coefficients λ and μ, respectively. We denoted by $\omega = 2\pi f$, where f is the frequency of oscillations. When $\alpha_1 = \alpha_2 = \alpha_3 = 1$, equation (16) becomes the well-known Navier equation in the frequency domain used in [8]:

$$(\Lambda + M) \nabla_{\vec{x}} \left(\nabla_{\vec{x}} \cdot \vec{U} \right) + M \left(\nabla_{\vec{x}} \cdot \nabla_{\vec{x}} \right) \vec{U} = -\omega^2 \vec{U}. \tag{17}$$

Since for biological tissues $\Lambda >> M$, we will use the same approximation as in [8] and neglect the Λ-term in equation (16). Assuming that the displacement field $\vec{U}$ is known from the MRE measurements, we want to find M by inverting the following equation:

$$M\left[\nabla_{\vec{x}}\left(\nabla_{\vec{x}}^{\vec{\alpha}}\cdot\vec{U}\right)+\left(\nabla_{\vec{x}}\cdot\nabla_{\vec{x}}^{\vec{\alpha}}\right)\vec{U}\right]=-\omega^2\vec{U}. \quad (18)$$

We introduce the shear wave speed as [8]:

$$c_s(f)=\sqrt{\frac{2(\mathrm{Re}(M)^2+\mathrm{Im}(M)^2)}{\mathrm{Re}(M)+\sqrt{\mathrm{Re}(M)^2+\mathrm{Im}(M)^2}}}. \quad (19)$$

In the next section we compare the shear wave speed values obtained from inverting equation (18) with $\nabla_{\vec{x}}^{\vec{\alpha}}$ given by (12) and, respectively, (14) and from inverting equation (17) where the $(\Lambda+M)$-term has been neglected as in [8].

5. RESULTS

In order to test our new model, we used the displacement field obtained from a two dimensional MRE-type shear test experiment run on a gel phantom containing four cylindrical inclusions of stiffer gel [7]. The outside gel was 1.5% Agar and the cylinders were composed of 10% B-gel (bovine). The cylinders were approximately 5, 10, 16, and 25 mm in diameter. The field of view was 20 cm, and the slice thickness was 5 mm. Eight offsets through time were acquired at a frequency of f = 100 Hz. The expected values of the shear wave speed are of 3–4 kPa in the background and 10–12 kPa in the cylindrical inclusions. The magnitude image and the image of the phase difference are shown in Fig. 1.

If we assume that the Cauchy stress tensor of order α is symmetric and use the generalized fractional gradient given by (12) in the inversion of equation (18), then the values of the shear wave speed (19) are very large, increasing with decreasing values of α_1,α_2. In Fig. 2 we show the shear wave speed values when $\alpha_1=0.85,\ \alpha_2=0.75$ (left image) and when $\alpha_1=0.85,\ \alpha_2=-0.75$ (right image) and their corresponding value bars. From the value bars, we note that the shear wave speed values are at least 10 times bigger than the expected values when $\alpha_1=0.85,\ \alpha_2=0.75$ and more than 100 times bigger when $\alpha_1=0.85,\ \alpha_2=-0.75$.

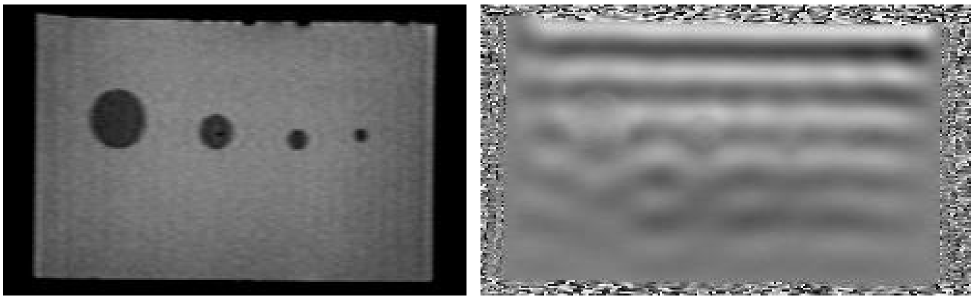

Fig. 1 – The magnitude (left) and the phase-difference (right) images.

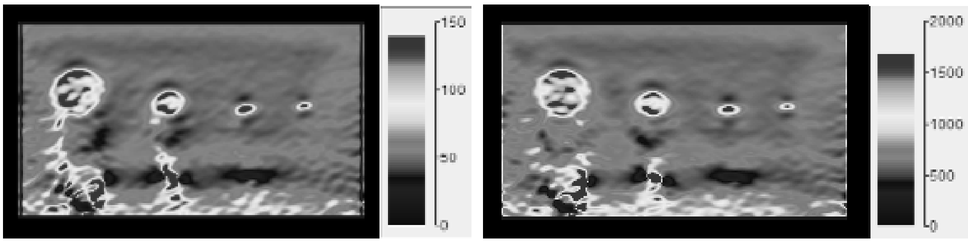

Fig. 2 – The elastograms and their corresponding value bars for $\alpha_1 = 0.85,\ \alpha_2 = 0.75$ (left) and $\alpha_1 = 0.85,\ \alpha_2 = -0.75$ (right).

We assume now that the Cauchy stress tensor of order α is not symmetric with α having the particular form discussed at the end of section 3 such that we can use the fractional gradient given by (14) in the inversion of equation (18). In Fig. 3 we show the values of the shear wave speed calculated using equation (19) with M given by equation (18) when $\alpha_{11} = 0.85,\ \alpha_{22} = -0.75$ (left image) and with M given by equation (17) of the classical continuum mechanics theory where the first term was neglected (right image). Both methods found the expected values of 3–4 kPa in the background and of 10–12 kPa in the inclusions. However, while the classical model was able to find just a few of the correct values in the inclusions, most of the values being around 8 kPa and even lower in the smallest cylinder, our fractional model was able to find all the values in the inclusions in the range of 10 kPa to 12 kPa. Also, as it can be seen in Fig. 4 and Table 1, the classical model is unstable (with values as high as 600 kPa in one of the inclusions), while our model is very stable. There is no need to post-process the elastogram obtained with our fractional model. In addition, our numerical simulations show that the best results are obtained with a non-symmetric Cauchy stress tensor of order α.

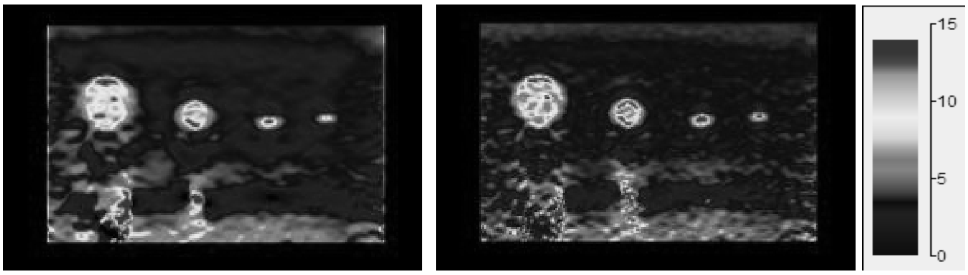

Fig. 3 – The elastograms obtained using our model (left) and the classical model (right). The value bar is the same in both cases.

6. CONCLUSIONS

In the present paper we introduced a new mechanical model that we called the fractional continuum mechanics model to solve the inverse problem of MR elastography. Our preliminary results show that under the assumption that the newly defined Cauchy stress tensor of order α is not symmetric, the model is capable to correctly recover the shear wave speed values, is robust in the presence of noise and thus does not require using any of the image smoothing (denoising) methods that are needed by the classical model. In addition we notice that the proposed model is not limited only to the inverse problem of MRE, it has the potential to play an important role in solving other direct and inverse problems relevant to practical applications. In our future work we plan to investigate the performance of the model in finding the shear wave speed from *in vivo* two and three dimensional MRE data obtained from shear experiments on biological tissues with lesions. We also intend to study the possible links between the fractional order α and the biochemical processes that take place in a living biological tissue.

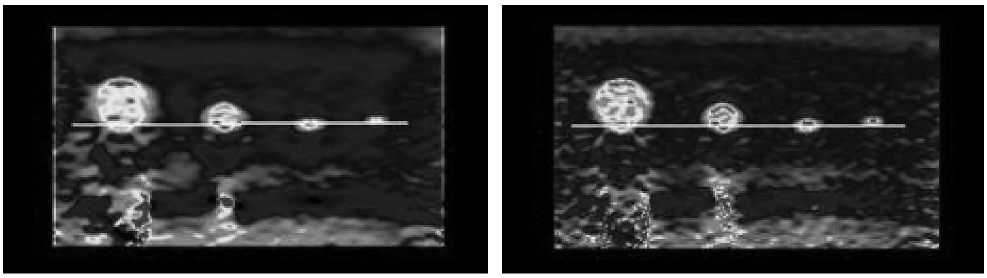

Fig. 4

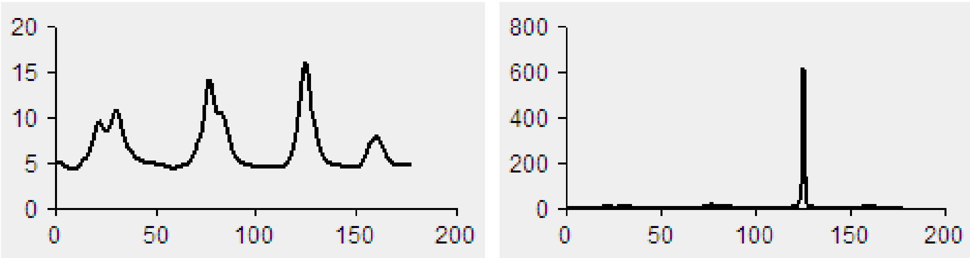

Fig. 4 – One dimensional profiles: our fractional model (left column) and the classical model (right column).

Table 1

The mean and standard deviation of the shear wave speed inside the cylinders using the classical and fractional models

Cylinder Diameter (mm)	Classical Model of Continuum Mechanics		Fractional Model of Continuum Mechanics	
	Mean (kPa)	Standard Deviation (kPa)	Mean (kPa)	Standard Deviation (kPa)
5	6.7	4.0	7.2	2.1
10	19.3	89.4	7.9	2.9
16	6.9	3.0	8.6	2.1
25	6.4	1.9	9.5	1.8

Acknowledgement. I would like to thank dr. R. Ehman and dr. A. Manduca for the financial support and many helpful discussions while visiting the MRE lab at Mayo Clinic.

Received on October 19, 2009

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