# ON THE INFLUENCE OF DENSITY ON THE WAVE PROPAGATION IN FLUIDS

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*Abstract*. In this paper we investigate the influence of the density on the propagation of waves in fluids with exponential and cosine stratification, respectively. We show that the waves are pairs of solitons, each pair consisting of two opposite directions propagating waves of the same mode. The modes of the two pairs are different from each other, and are different from the *m*-th and *n*-th modes. The original waves propagate after interaction without changing their identities, but only the *m*-wave suffers a shift of phase.

*Key words:* wave propagation, fluids, exponential/cosine stratification, solitons.

## 1. INTRODUCTION

Influence of the density on the propagation of waves in fluids represents an important area of research with potentially application to the generation and motion of the Tsunami waves (Imteaz *et al*. [1]).

Linear models on two-layered long wave flow [2–6] and nonlinear aspects extracted from experimental observations [7, 8], were successfully validated by numerical and analytical solutions for multi-layered flow based on Boussinesq-type equations [9, 10] and by piecewise integration of Laplace equation for each individual layer [11].

The Kadomtsev-Petviashvili (KP) equation (2D version of the Korteweg-de Vries equation) describes the motion of long waves of small amplitude with slow dependence on the transverse coordinate [12]. The KP equation coupled with the generalized play operator is able to explain the dilatonic behavior of the soliton interaction and the generation of huge waves in shallow waters [13].

The motion of an incompressible stratified fluid, with different densities in layers has been studied in the case of an exponential stratification by Munteanu and Donescu [14] and Yih [15].

An example of stratified fluid is the atmosphere. Interesting meteorological implications are discussed by Robert [16, 17] in the frame of theoretical and

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experimental analysis of the two-dimensional flow of a stratified fluid over a barrier in a gravity field.

In this paper, we study the propagation of waves in fluids with exponential and cosine stratification, respectively. But, we mention that the governing equations are valid for any arbitrary density stratification.

We show that the motion equations possess a special type of elementary solution. These solutions known as *solitons* have the form of localized waves that conserve their properties even after interaction among them, and then act somewhat like particles. These equations have interesting properties: an infinite number of local conserved quantities, an infinite number of exact solutions expressed in terms of the Jacobi elliptic functions (cnoidal solutions) or the hyperbolic functions (solitonic solutions or solitons), and the simple formulae for nonlinear superposition of explicit solutions [14].

## 2. FORMULATION OF THE PROBLEM

The fluid is situated between two horizontal boundaries spaced at distance $h$ apart. The wave motion takes place in the $(x, y)$ plane of a Cartesian system of coordinates with the origin in the lower boundary and $y$ vertically upward.

Euler motion equations are given by [14, 15]

$$(\overline{\rho}+\rho)\frac{\mathrm{D}u}{\mathrm{D}t} = -p_x\,, \tag{1}$$

$$(\overline{\rho}+\rho)\frac{\mathrm{D}v}{\mathrm{D}t} = -p_y - g(\overline{\rho}+\rho)\,, \tag{2}$$

where $u$ and $v$ are the components of velocity in the positive direction of $x$ and $y$, $p$ is the pressure, $\overline{\rho}(y)$ is the density in the undisturbed fluid, $\rho$ the density perturbation and $g$ the gravitational acceleration, and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\,. \tag{3}$$

The incompressibility equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(\overline{\rho}+\rho) = 0\,, \tag{4}$$

leads to continuity equation

$$u_x + v_y = 0\,. \tag{5}$$

If the stream function is denoted by $\psi$, the components of velocity are expressed as follows

$$u = \psi_y\,, \quad v = -\psi_x\,. \tag{6}$$

From (1) and (2) we obtain

$$(\bar{\rho} + \rho)\frac{\mathrm{D}(\psi_{\mathrm{xx}} + \psi_{\mathrm{yy}})}{\mathrm{D}t} + (\bar{\rho} + \rho)_y \frac{\mathrm{D}\psi_{\mathrm{y}}}{\mathrm{D}t} + p_x \frac{\mathrm{D}\psi_{\mathrm{x}}}{\mathrm{D}t} = g p_x\,. \tag{7}$$

The equation (4) becomes

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \bar{\rho}_y \psi_x\,. \tag{8}$$

Yih [15] has been studied the equations (7) and (8) in the case of exponential stratification

$$\bar{\rho} = \rho_0 \exp(-\alpha y)\,. \tag{9}$$

In this paper we intend to study not only the Yih case, but also the case of cosine stratification

$$\bar{\rho} = \rho_0 \cos(\beta y)\,. \tag{10}$$

To solve the governing equations (7) and (8), we apply the cnoidal method [14]

$$\psi(x, y, t) = \frac{2}{\lambda}\frac{\mathrm{d}^2}{\mathrm{dx}^2}\log\Theta(\eta_1, \eta_2)\,, \tag{11}$$

where $\Theta$ is the *theta function* defined as

$$\Theta(\eta_1, \eta_2) = \sum_{M \in (-\infty, \infty)} \exp\left(\mathrm{i}\left( M_1\eta_1 + M_2\eta_2 + \frac{1}{2}\sum_{i,j=1}^{2} M_i B_{ij} M_j \right)\right), \tag{12}$$

$$\exp B_{ij} = \left(\frac{\omega_i - \omega_j}{\omega_i + \omega_j}\right)^2, \quad \exp B_{ii} = \omega_i^2\,. \tag{13}$$

$$\eta_j = k_j x - \omega_j t + \phi_j\,, \quad j = 1, 2\,. \tag{14}$$

In (12), $k_j$ are the wave numbers, $\omega_j$ are the frequencies and $\phi_j$ are phases.

For the case (9), the solutions for the $m^{\mathrm{th}}$ and $n^{\mathrm{th}}$ modes of the equations (10) are obtained from applying (12)–(14), to the lowest order [15]

$$\psi_m = (-1)^m a_m \mathrm{sech}^2\left[\gamma_m\left(x - \frac{1}{m\pi}t\right)\right]\left(\sin m\pi y + \frac{\alpha}{2} y \sin m\pi y\right),\ \theta_m = m\pi\psi_m, \quad (15)$$

$$\psi_n = (-1)^n a_n \mathrm{sech}^2\left[\gamma_n\left(x - \frac{1}{n\pi}t\right)\right]\left(\sin n\pi y + \frac{\alpha}{2} y \sin n\pi y\right),\quad \theta_n = n\pi\psi_n, \quad (16)$$

$$\gamma_m^2 = \frac{m\pi\alpha a_m}{9},\ \text{for odd}\ m,\quad \gamma_m^2 = \frac{m\pi\alpha^2 a_m}{36},\ \text{for even}\ m, \quad (17)$$

and similarly for $\gamma_n^2$.

For the case (10), we have the solutions for the $m^{\text{th}}$ and $n^{\text{th}}$ modes of the equations (7) and (8) to the lowest order

$$\psi_m = (-1)^m b_m\left(\mathrm{cn}^2\left[\gamma_m\left(x - \frac{1}{m\pi}t\right)\right] + \mathrm{sech}^2\left[\gamma_m\left(x - \frac{1}{m\pi}t\right)\right]\right)\left(\mathrm{cn} m\pi y + \beta y \mathrm{cn} m\pi y\right),$$

$$\theta_m = \frac{1}{6} m\pi\psi_m, \quad (18)$$

$$\psi_m = (-1)^n b_n\left(\mathrm{cn}^2\left[\gamma_n\left(x - \frac{1}{n\pi}t\right)\right] + \mathrm{sech}^2\left[\gamma_m\left(x - \frac{1}{n\pi}t\right)\right]\right)\left(\mathrm{cn} n\pi y + \beta y \mathrm{cn} n\pi y\right),$$

$$\theta_n = \frac{1}{6} n\pi\psi_n, \quad (19)$$

$$\gamma_m^2 = \frac{m\pi\beta b_m}{12},\ \text{for odd}\ m,\quad \gamma_m^2 = \frac{m\pi\beta^2 b_m}{48},\ \text{for even}\ m, \quad (20)$$

and similarly for $\gamma_n^2$.

### 3. RESULTS

It is convenient to discuss the propagation of waves for the particular case of $n = 2m$. In both cases (9) and (10), these waves may propagate in the same direction or in opposite directions. The waves are pairs of solitons, each pair consisting of two opposite directions propagating solitary waves of the same mode. The modes of the two pairs are different from each other, and are different from the $m^{\text{th}}$ and $n^{\text{th}}$ modes of the waves. The original waves propagate after interaction without changing their identities, but only the $m$-wave suffers a shift of phase. We have taken $n = 6$ and $m = 3$. The depth of the fluid is 75 m, the wave length is

$L = 390\,\mathrm{m}$, and the dimensionless amplitude is the amplitude reported to the wave amplitude of 4 m.

Fig. 1 shows the variation of the amplitude with respect to $x/L$ at the top surface after 50 seconds, for both models of stratification, and similar input data.

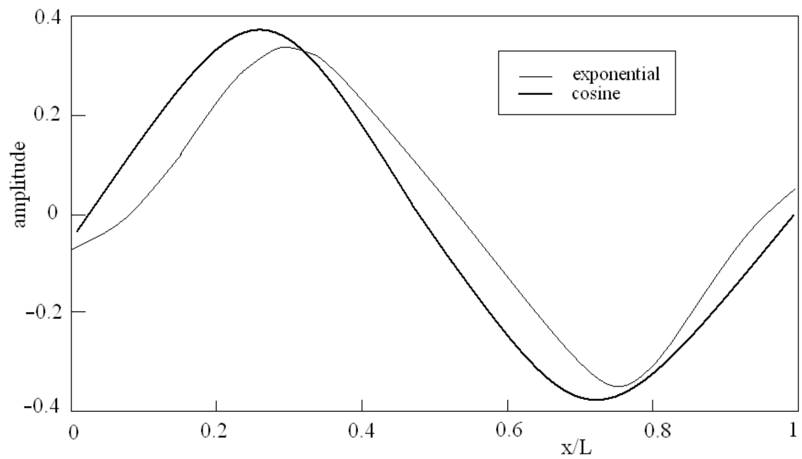


Fig. 1 – Comparison of stratification models for the top surface of the fluid at $t = 50$ seconds.

Figs. 2 and 3 illustrate the interaction between two opposite directions propagating solitary waves for the top surface, of the same mode, for both models of stratification in the interval of 6 and 9 seconds, respectively. We see that the original waves propagate after interaction without changing their identities.

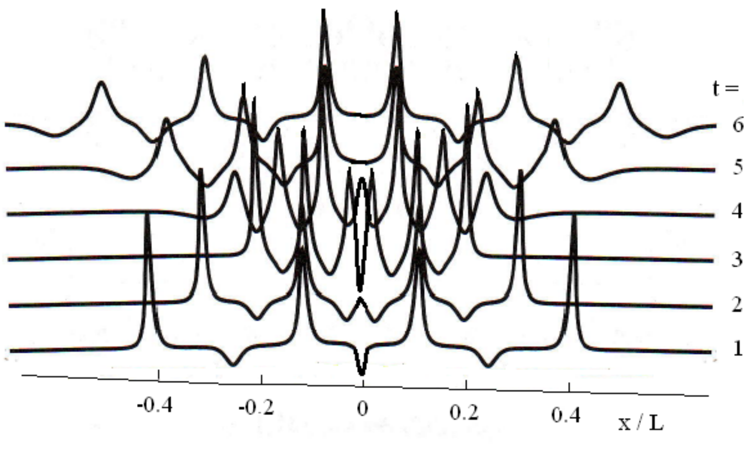


Fig. 2 – Interaction between two solitary waves for the top surface in the case of exponential stratification.

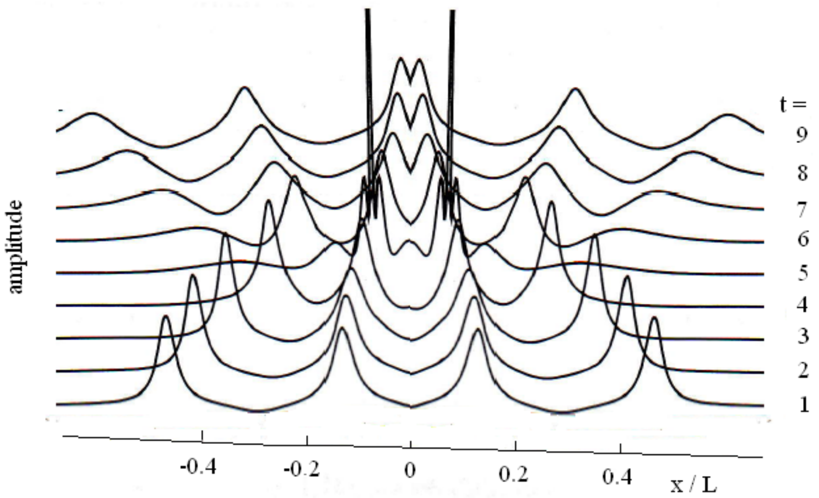


Fig. 3 – Interaction between two solitary waves for the top surface in the case of cosine stratification.

Figs. 4 and 5 represent the variation of the wave amplitude with respect to $x/L$ and time, at the top surface in the interval of 1–6 seconds, for both models of stratification.

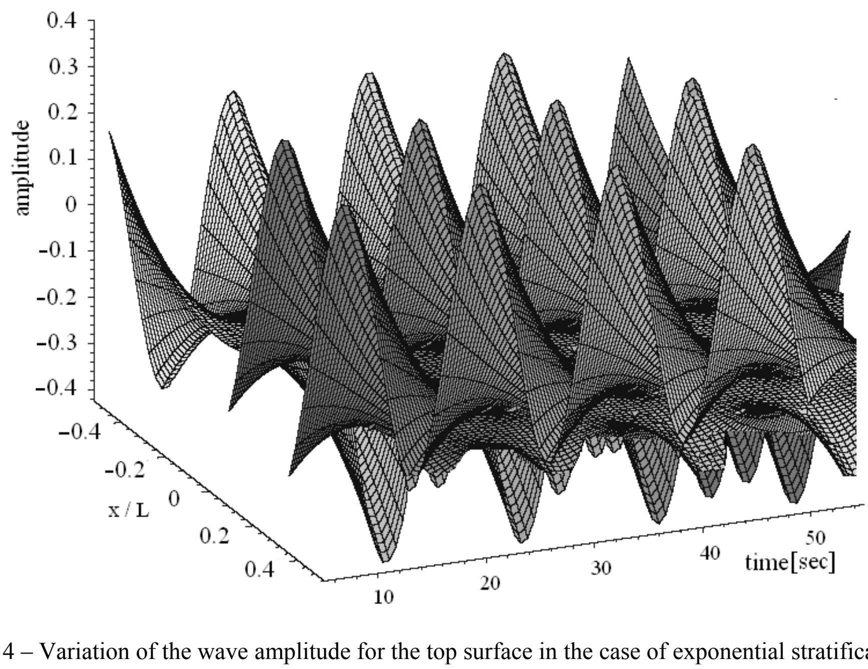


Fig. 4 – Variation of the wave amplitude for the top surface in the case of exponential stratification.

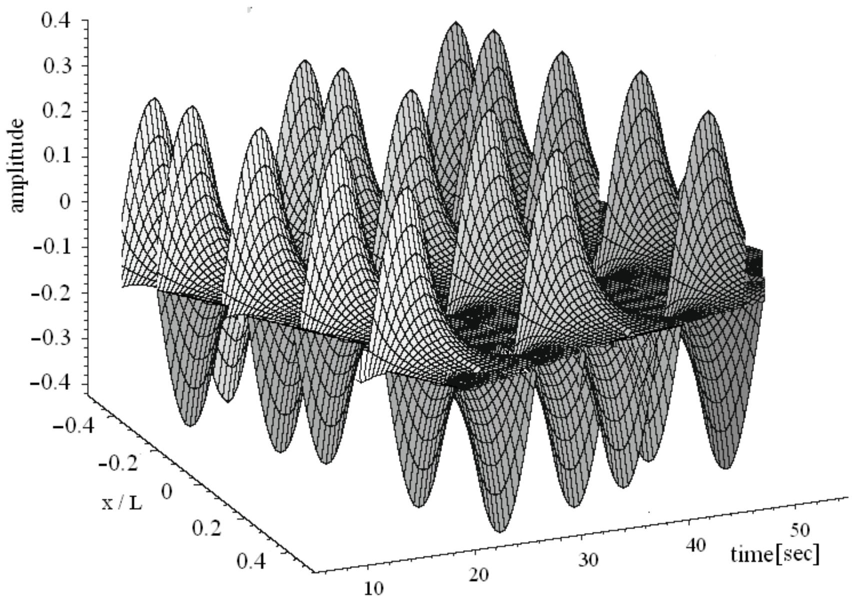


Fig. 5 – Variation of the wave amplitude for the top surface in the case of cosine stratification.

We see from Figs. 4 and 5 that the wave propagation at the top surface has a complex structure. Agreements of our results are very good with results reported by Imteaz *et al.* [1]. It can be concluded that our model is capable to produce realistic results. For in depth surfaces, the picture of the wave propagation is simpler, because the intermediate surfaces gets dampen down with the course of time. These are because of the complicated interactions from the adjacent layers accordingly with [1]. For the top surface, the wave phase changes each 8 seconds and the wave moves a distance of half of the wave length within 8 seconds. Also, within 10 seconds to 50 seconds, many wave crests are formed.

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