EIGENVALUE ANALYSIS OF CURVED SANDWICH PANELS LOADED IN UNIAXIAL COMPRESSION

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Abstract. An energy formulation of a simply supported cylindrically curved sandwich panel loaded in compression is introduced and the corresponding potential energy function is evaluated. The mechanical model of a cylindrically curved sandwich panel comprises three interacting buckling modes, corresponding to nine degrees of freedom comprising q_s, q_x and q_y components of local snake (*m, n*) and overall (*k, l*) modes, the related local hourglass (*m, n*) mode, Finally, closed-form solutions are presented for the global and local buckling modes.

Key words: curved panels, sandwich panels, composite structures.

1. INTRODUCTION

Sandwich structures exhibit a potentially unstable post-buckling behaviour due to coupling of stable buckling modes [1]. Hunt *et al*. [2] have shown that realistic combinations of face materials with significantly less stiff core materials lead to unstable post-buckling response in the case of sandwich struts. This effect is even more pronounced in the case of core materials with orthotropic properties, such as honeycomb cores [3].

The characterization of the post-buckling behaviour of sandwich panels requires the identification of the relevant buckling modes and the corresponding elastic critical loads.

Sandwich panels can be nowadays produced with arbitrary shapes. The construction sector increasingly uses curved panels in structural applications, clearly extending the classical range of application of cylindrical sandwich panels as shells of revolution in the aeronautical industry.

This paper presents an energy formulation based on the Rayleigh-Ritz method for the calculation of the elastic critical loads of cylindrically curved sandwich panels. Closed-form solutions are obtained for the global and local buckling modes.

2. MODAL DESCRIPTION

Figure 1 illustrates a typical curved sandwich panel loaded in uniaxial compression with simply-supported edges. Figure 1b depicts the particular case of a

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flat panel henceforth denoted a sandwich plate. A sandwich panel loaded in compression exhibits two distinct sets of buckling modes: snake and hourglass modes, assumed to vary sinusoidally along the length of the panel [4], typically illustrated in Fig. 2.

(*i*) *Snake mode*. The snake mode is defined as the combination of three independent degrees of freedom, [4], allowing complete flexibility of mode form between that of pure shear to those of pure bending in each direction with no shear, as represented in expressions (1) to (3).

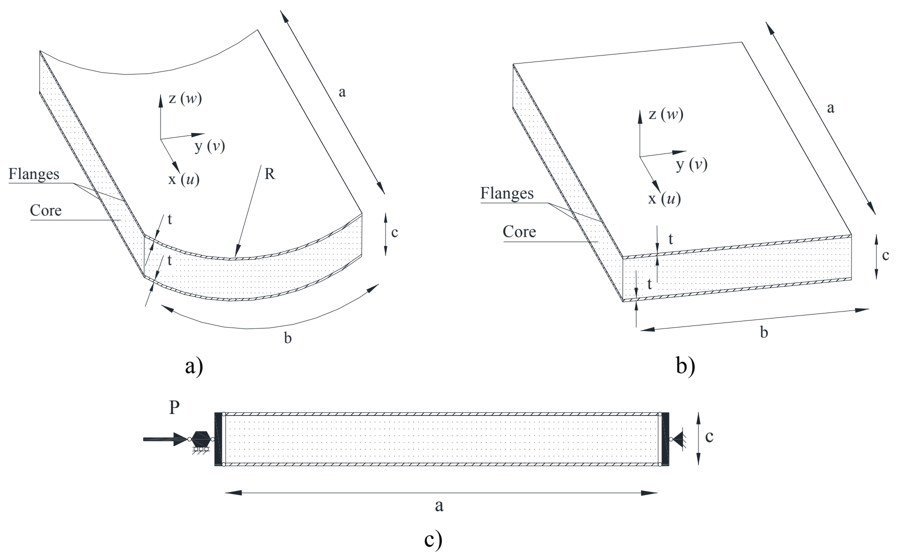


Fig. 1 – Simply supported sandwich panel: flat ($R = \infty$) and curved panel.

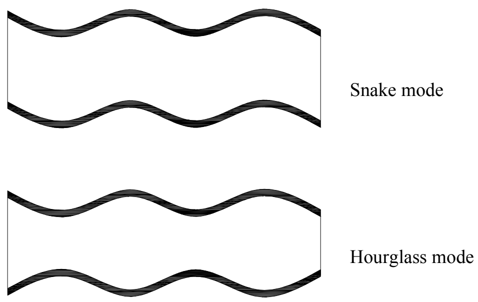


Fig. 2 – Local buckling modes of sandwich panel.

$$w = q_s \frac{a}{i} \sin\left(\frac{i\pi x}{a}\right) \sin\left(\frac{j\pi y}{b}\right) \tag{1}$$

$$u = -zq_x \pi \cos\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right) \tag{2}$$

$$v = -zq_y \frac{a}{b}\frac{j}{i}\pi \sin\left(\frac{i\pi x}{a}\right)\cos\left(\frac{j\pi y}{b}\right). \tag{3}$$

q_s represents the amplitude of the pure shear component of the snake mode. There is bending in each flange over its thickness but no bending action in the section as a whole. The other two displacement fields are introduced to allow for bending strains to develop, consisting of an axial variation in the angle of tilt of each plane section in the *x*- *(q_x)* and *y*-direction (q_y) [1]. Depending on the number of half-waves, the snake mode may appear as a local (large numbers of half wavelength *m, n*) or as a global mode. In the latter case it is denoted overall mode (small numbers of half wavelength *k, l*).

(ii) *Hourglass mode*. The symmetrical nature of the hourglass mode requires that there must be zero transverse displacements at the middle plane of the sandwich panel, as well as no longitudinal displacements. Thus, the shear strain must be zero at the centre line and maximum at the interface between the core and the flanges. If a linear variation in shear angle is assumed, such that it matches the angle of inclination of the flange at each interface, the hourglass mode is described by [1]:

$$w = q_h \frac{2z}{c}\frac{a}{i}\sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right). \tag{4}$$

The model includes three interacting buckling modes, corresponding to nine degrees of freedom comprising q_s, q_x and q_y components of local snake (*m, n*) and overall (*k, l*) modes, the related local hourglass (*m, n*) mode, and constant and variable end shortening. Furthermore, the longitudinal wave number m is assumed to be even [1]. Specifically the degrees of freedom are as follows:

a_1 – "shear" component of snake mode *(i, j) = (m, n)*
a_2 – corresponding hourglass mode *(i, j) = (m, n)*
a_3 – "shear" component of overall mode *(i, j) = (k, l)*
A_4 – constant end-shortening
a_5 – "tilt" component of snake mode in *x*-direction *(i, j) = (m, n)*
a_6 – "tilt" component of overall mode in *x*-direction *(i, j) = (k, l)*
a_7 – "tilt" component of snake mode in *y*-direction *(i, j) = (m, n)*
a_8 – "tilt" component of overall mode in *y*-direction *(i, j) = (k, l)*
A_9 – variable end-shortening.

3. STRAIN-DISPLACEMENT RELATIONS

As the faces and core materials are different it is necessary to consider separately the stress states in both faces and core. Because of the assumption that the faces behave as thin plates or shells, the direct stress and shear strains related to z-direction are taken as zero (5) and only the strains in the xy-plane are relevant ((6) to (8), where z^* denotes a local coordinate system with its origin at the centre line of each flange). The strain displacement relations for the core correspond to a three dimensional stress state ((9) to (14))

$$\sigma_z = \gamma_{xz} = \gamma_{yz} = 0. \tag{5}$$

It is further assumed that the panels are in the range of shallow shells and DMV (Donnell-Mushtari-Vlasov) nonlinear shell theory is applicable. This theory assumes that the shell shows infinitesimal deformations and moderate rotations. Also the intrinsic geometry of a shallow shell is identical to the geometry of a plane of its projection. This actually represents the first basic assumption of the theory of shallow shells [5]. The second assumption of shallow shells theory is that the effect of transverse shear forces on the in plane equilibrium equations is negligible and the influence of the deflections, w, predominates over the influence of the in plane displacements u and v in the bending response of the shell.

Furthermore, in order to simplify the strain displacements relations a mean radius of curvature is considered (instead of considering one for the core and two other for the flanges) which can lead to errors when computing critical stresses. Nevertheless, this simplification is in line with the geometrical definition of shallow shells.

The next set of equations reflects these assumptions for a sandwich panel.

(*i*) *Flanges*

$$\varepsilon_x = \frac{\partial u}{\partial x} - \frac{1}{a}\int_0^a \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x - z^* \frac{\partial^2 w}{\partial x^2}, \tag{6}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} - z^* \frac{\partial^2 w}{\partial y^2}, \tag{7}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z^* \frac{\partial^2 w}{\partial x \partial y}. \tag{8}$$

(*ii*) *Core*

$$\varepsilon_x = \frac{\partial u}{\partial x} - \frac{1}{a}\int_0^a \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x, \tag{9}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R}, \tag{10}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}, \tag{11}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{12}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \tag{13}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}. \tag{14}$$

4. STRAIN ENERGY EXPRESSIONS

The general strain energy expression is given by (15)

$$U = \frac{1}{2}\int_V \left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_x \gamma_x + \tau_y \gamma_y + \tau_z \gamma_z\right) \mathrm{d}V\,. \tag{15}$$

For the faces, introducing Hooke's law in expression (15) leads to, after some reworking,

$$\begin{aligned} V_f = {} & \frac{\overline{E}_f}{2}\int_0^a \int_0^b \int_{-t/2}^{t/2} \left(\varepsilon_x^2 + \varepsilon_y^2 + 2\varepsilon_x \varepsilon_y\right)\ \mathrm{d}x\mathrm{d}y\mathrm{d}z^* + \\ & + \frac{G_f}{2}\int_0^a \int_0^b \int_{-t/2}^{t/2} \gamma_{xy}^2\ \mathrm{d}x\mathrm{d}y\mathrm{d}z^* \end{aligned} \tag{16}$$

where

$$\overline{E}_f = \frac{E_f}{1-\nu_f^2}\,. \tag{17}$$

The strain energy of the faces may be split into two contributions: pure bending in the faces (V_{fb}) and membrane action in the faces (V_{fm}).

Similarly, for the core the strain energy is given by:

$$\begin{aligned} V_c = {} & \frac{\overline{E}_c}{2}\int_0^a \int_0^b \int_{-c/2}^{c/2} \left[\nu_c^{(1)}\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2\right) + \nu_c^{(2)}\left(\varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z\right)\right]\ \cdot \\ & \cdot \mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{G_c}{2}\int_0^a \int_0^b \int_{-c/2}^{c/2} \left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\ \mathrm{d}x\mathrm{d}y\mathrm{d}z, \end{aligned} \tag{18}$$

where, for simplicity:

$$\overline{E}_c = \frac{E_c}{1-\nu_c^2}, \tag{19}$$

$$\nu_c^{(1)} = \frac{(1-\nu_c)^2}{1-2\nu_c}, \tag{20}$$

$$\nu_c^{(2)} = \nu_c \frac{1-\nu_c}{1-2\nu_c}. \tag{21}$$

5. TOTAL POTENTIAL ENERGY FUNCTION

The total potential function (22) is obtained by adding to the two components of the strain energy (U), the strain energy due to the end-beam of stiffness K_b/a (23) and the work done by the load P (24)

$$V = V_f + V_c + V_b - WDL, \tag{22}$$

$$V_b = \frac{K_b \cdot}{a}\int_0^b A_9 a \sin^2\left(\frac{\pi y}{b}\right)\mathrm{d}y = \frac{3}{16}K_b A_9^2 ab, \tag{23}$$

$$WDL = P\int_0^b u_{x=a}\mathrm{d}x = P\left(A_4 + \frac{1}{2}A_9\right)ab. \tag{24}$$

Introducing the last term of the strain displacement relations ((6) to (8)) into (16) gives, after summing the contributions from both flanges, the strain energy of flange bending of the panel:

$$\begin{aligned} V_{bf,panel} = V_{bf,plate} = \frac{t^3\pi^4}{12}\Bigg[\frac{\overline{E}_f}{4}\Bigg(a_3^2\left(\frac{k^2}{a^2}+\frac{l^4a^2}{b^4k^2}+2\nu_f\frac{l^2}{b^2}\right)+ \\ \left(a_1^2+a_1^2\right)\left(\frac{m^2}{a^2}+\frac{n^4a^2}{b^4m^2}+2\nu_f\frac{n^2}{b^2}\right)\Bigg)+\frac{G_f}{b^2}\left(a_3^2l^2+\left(a_1^2+a_2^2\right)n^2\right)\Bigg]ab\cdot \end{aligned} \tag{25}$$

Similarly, introducing the first terms of the strain displacement relations ((6) to (8)) into (16) yields the flange membrane energy of the panel

$$V_{mf,panel} = V_{mf,plate} + \frac{1}{2}\overline{E}_f t\left[\frac{a^2}{2R^2}\left(\frac{1}{m^2}\left(a_1^2+a_2^2\right)+\frac{1}{k^2}a_3^2\right)+ \right.$$
$$+\nu_f\frac{1}{R}\left(\frac{c\pi^2}{2}a_2a_5-\frac{a}{k^2l}\left(1-(-1)^k\right)\left(1-(-1)^l\right)\times\right.$$
$$\left.\left.\times\left(\frac{4+2(-1)^l}{3}a_3^3-\frac{2n^2}{l^2-4n^2}\left(a_1^2+a_2^2\right)a_3+\frac{4a_3\left(2A_9-A_4\left(l^2-4\right)\right)}{\pi^2\left(l^2-4\right)}\right)\right)\right]\cdot ab \tag{26}$$

where

$$V_{mf,plate} = \frac{1}{2}\overline{E}_f t\left[2A_4^2+2A_4A_9+\frac{3}{4}A_9^2-\frac{1}{2}\pi^2A_4\left(a_1^2+a_2^2+a_3^2\right)-\right.$$
$$-\pi\ A_9\left(\left(a_1^2+a_2^2\right)K_n+a_3^2K_l\right)+\frac{3}{64}\pi^4\left(\begin{array}{l}a_1^4+a_2^4+a_3^4+6a_1^2a_2^2+\frac{4+K_{nl}}{3}\cdot\\ \cdot\left(a_1^2a_3^2+a_2^2a_3^2\right)\end{array}\right)-$$
$$-2\pi^2c\frac{n^2}{4n^2-l^2}\left(1-(-1)^k\right)\ \left(1-(-1)^l\right)\ \left(\frac{1}{a\ l}a_1a_2a_6+\nu_f\frac{a\ l}{b^2k^2}a_1a_2a_8\right)+ \tag{27}$$
$$+\left(\frac{c\pi^2}{2b}\right)^2\left.\left(\left(\frac{b^2}{2a^2}\left(m^2a_5^2+k^2a_6^2\right)+\frac{a^2}{2b^2}\left(\frac{n^4}{m^2}a_7^2+\frac{l^4}{k^2}a_8^2\right)+\nu_f\left(n^2a_5a_7+l^2a_6a_8\right)\right)\right]\cdot$$
$$\cdot ab+\frac{1}{2}\pi^2G_f t\left[\frac{a^2}{3b^2}A_9^2+\left(\frac{c\pi}{2b}\right)^2\left(l^2\left(a_6+a_8\right)^2+n^2\left(a_5+a_7\right)^2\right)\right]ab.$$

Equation (26) highlights the specific contributions from curvature, compared to a flat panel. The strain energy of the core results from (18), introducing the strain displacement relations (9) to (14)

$$V_{c,panel} = V_{c,plate} + \overline{E}_c c\ \left[v_c^{(1)}\frac{a^2}{8R^2}\left(\frac{1}{m^2}\left(a_1^2+\frac{1}{3}a_2^2\right)+\frac{1}{k^2}a_3^2\right)+\right.$$
$$+v_c^{(2)}\frac{1}{R}\left(\frac{a^2}{2m^2c}a_1a_2+\frac{c\pi^2}{24}a_2a_5-\frac{a}{k^2l}\left(1-(-1)^k\right)\ \left(1-(-1)^l\right)\times\right.$$
$$\left.\left.\times\left(\frac{2+(-1)^l}{6}a_3^3-\frac{n^2}{6\left(l^2-4n^2\right)}\left(3a_1^2+a_2^2\right)\ a_3+\frac{a_3\left(2A_9-A_4\left(l^2-4\right)\right)}{\pi^2\left(l^2-4\right)}\right)\right)\right].\cdot ab \tag{28}$$

where

$$V_{c,plate} = \frac{1}{2}\overline{E}_c c\left[v_c^{(1)}\left(A_4^2 + A_4 A_9 + \frac{3}{8}A_9^2 + \frac{a^2 a_2^2}{c^2 m^2} - \frac{1}{4}\pi^2 A_4\left(a_1^2 + \frac{1}{3}a_2^2 + a_3^2\right) - \right.\right.$$
$$-\frac{1}{2}\pi\ A_9\left(\left(a_1^2 + \frac{1}{3}a_2^2\right)K_n + a_3^2 K_l\right) + \frac{3}{128}\pi^4\left(\begin{array}{l} a_1^4 + \frac{1}{5}a_2^4 + a_3^4 + 2a_1^2 a_2^2 + \\ + \frac{4+K_{nl}}{3}\left(a_1^2 a_3^2 + \frac{1}{3}a_2^2 a_3^2\right)\end{array}\right) -$$
$$-\frac{1}{3}\pi^2 c\frac{n^2}{4n^2 - l^2}\left(1-(-1)^k\right)\ \left(1-(-1)^l\right)\left(\begin{array}{l} v_c^{(1)}\frac{1}{a\ l}a_1 a_2 a_6 + \\ + v_c^{(2)}\frac{a\ l}{b^2 k^2}a_1 a_2 a_8\end{array}\right) +$$
$$+\left(\frac{c\pi^2}{4\sqrt{3}b}\right)^2\left(\frac{b^2}{a^2}\left(m^2 a_5^2 + k^2 a_6^2\right) + \frac{a^2}{b^2}\left(\frac{n^4}{m^2}a_7^2 + \frac{l^4}{k^2}a_8^2\right)\ \right)\ \Bigg) + \tag{29}$$
$$\left. + v_c^{(2)}\left(\frac{c\pi^2}{4\sqrt{6}b}\right)^2\left(n^2 a_5 a_7 + l^2 a_6 a_8\right)\ \right]ab +$$
$$+\frac{1}{2}\pi^2 G_c c\left[\frac{a^2}{6b^2}A_9^2 + \left(\frac{c\pi}{4\sqrt{3}b}\right)^2\left(l^2\left(a_6 + a_8\right)^2 + n^2\left(a_5 + a_7\right)^2\right) + \right.$$
$$\left. + \frac{1}{4}\left(\left(a_1 - a_5\right)^2 + \frac{1}{3}a_2^2 + \left(a_3 - a_6\right)^2 + \frac{a^2}{b^2}\left(\begin{array}{l}\frac{n^2}{m^2}\left(\left(a_1 - a_7\right)^2 + \frac{1}{3}a_2^2\right) + \\ + \frac{l^2}{k^2}\left(a_3 - a_8\right)^2\end{array}\right)\ \right)\ \right]\cdot ab,$$

and (see Annex A)

$$K_n = \begin{cases} 3\pi/8 & \text{if } n = 1 \\ \pi/4 & \text{if } n > 1 \end{cases},\quad K_l = \begin{cases} 3\pi/8 & \text{if } l = 1 \\ \pi/4 & \text{if } l > 1 \end{cases} \text{ and } K_{nl} = \begin{cases} 2 & \text{if } n = l \\ 0 & \text{if } n \neq l \end{cases} \tag{30}$$
$$\wedge\ \ n \geq l.$$

The degrees of freedom associated with the total end shortening can be eliminated from further consideration by using the corresponding equilibrium equations and solving them simultaneously with respect to A_4 and A_9.

$$A_4 = \frac{P}{2\overline{E}_f t + \overline{E}_c c v_c^{(1)}} + \frac{\pi}{8}\left[\left(\pi - 3\left(4K_n - \pi\right)\ \lambda_{plate}\right)a_1^2 + \right.$$

$$+\left(\pi-3\left(4K_n-\pi\right)\ \lambda_{plate}\right)\ \left(1-\frac{2}{3}\Psi_{plate}\right)a_2^2+\left(\pi-3\left(4K_l-\pi\right)\ \lambda_{plate}\right)a_3^2\Bigg]- \\ -\frac{a}{\pi^2R}\left(1-(-1)^k\right)\ \left(1-(-1)^l\right)\left(\frac{\Psi_{panel}}{k^2l}+3\lambda_{panel}\right)a_3, \tag{31}$$

$$A_9=\frac{3\pi}{4}\lambda_{plate}\begin{bmatrix}\left(4K_n-\pi\right)a_1^2+\left(4K_n-\pi\right)\ \left(1-\frac{2}{3}\Psi_{plate}\right)a_2^2+\\+\left(4K_l-\pi\right)a_3^2\end{bmatrix}+ \\ +\frac{6a}{\pi^2R}\left(1-(-1)^k\right)\ \left(1-(-1)^l\right)\lambda_{panel}a_3, \tag{32}$$

where

$$\Psi_{panel}=\frac{8\overline{E}_ftv_f+\overline{E}_ccv_c^{(2)}}{2\overline{E}_ft+\overline{E}_ccv_c^{(1)}}, \tag{33}$$

$$\lambda_{panel}=\frac{b^2\left(8\overline{E}_ftv_f\left(l^2-3\right)+\overline{E}_ccv_c^{(2)}l^2\right)}{k^2l\left(l^2-4\right)\left[2a^2\pi^2\left(2G_ft+G_cc\right)+\frac{3}{2}b^2\left(3K_b+2\overline{E}_ft+\overline{E}_ccv_c^{(1)}\right)\right]}, \tag{34}$$

$$\Psi_{plate}=\frac{\overline{E}_ccv_c^{(1)}}{2\overline{E}_ft+\overline{E}_ccv_c^{(1)}}, \tag{35}$$

$$\lambda_{plate}=\frac{b^2\left(2\overline{E}_ft+\overline{E}_ccv_c^{(1)}\right)}{2a^2\pi^2\left(2G_ft+G_cc\right)+\frac{3}{2}b^2\left(3K_b+2\overline{E}_ft+\overline{E}_ccv_c^{(1)}\right)}. \tag{36}$$

Introducing equations (31) and (32) into (22) (the latter divided by *a.b*) gives the reduced form of the total potential energy function of the panel in seven degrees of freedom expanded about a point on the fundamental path, denoted by *F*, written in general form.

The coefficients $V_{ij}^{\ F}$ are given in Annex B.

$$V_{panel}=\frac{1}{2}V_{11}^Fa_1^2+\frac{1}{2}V_{22}^Fa_2^2+\frac{1}{2}V_{33}^Fa_3^2+\frac{1}{2}V_{55}^Fa_5^2+\frac{1}{2}V_{66}^Fa_6^2+\frac{1}{2}V_{77}^Fa_7^2+ \\ +\frac{1}{2}V_{88}^Fa_8^2+V_{12}^Fa_1a_2+V_{15}^Fa_1a_5+V_{17}^Fa_1a_7+V_{25}^Fa_2a_5+V_{57}^Fa_5a_7+V_{36}^Fa_3a_6+$$

$$
\begin{aligned}
&+V_{38}^{F}a_3a_8+V_{68}^{F}a_6a_8+V_{333}^{F}a_3^3+V_{126}^{F}a_1a_2a_6+V_{128}^{F}a_1a_2a_8+V_{113}^{F}a_1^2a_3+\\
&+V_{223}^{F}a_2^2a_3+\frac{1}{24}V_{1111}^{F}a_1^4+\frac{1}{24}V_{2222}^{F}a_2^4+\frac{1}{24}V_{3333}^{F}a_3^4+\frac{1}{4}V_{1122}^{F}a_1^2a_2^2+\\
&+\frac{1}{4}V_{1133}^{F}a_1^2a_3^2+\frac{1}{4}V_{2233}^{F}a_2^2a_3^2+\frac{1}{2}\left(P-P^F\right)\left(V_{11}^{'F}a_1^2+V_{22}^{'F}a_2^2+V_{33}^{'F}a_2^2\right).
\end{aligned}
\tag{37}
$$

The equilibrium equations are obtained by stetting to zero first-order derivatives of the total potential function [7]:

$$
\frac{\partial V}{\partial a_i}=0\,. \tag{38}
$$

It can be seen that the fundamental path (pre-buckling solution) is non trivial with respect to the chosen degrees of freedom, yielding out of plane deformations along the fundamental path. However, for a flat panel (R tends to infinity) this behavioural feature disappears and the fundamental path becomes trivial ($a_i = 0$).

6. CRITICAL LOADS

The critical loads are obtained by setting to zero the determinant of the second derivatives with respect to each of the degrees of freedom a_i in turn, evaluated along the fundamental path,

$$
\left|V_{s,ij}^{F}\right|=\begin{vmatrix}
V_{11}^{F} & V_{12}^{F} & 0 & V_{15}^{F} & 0 & V_{17}^{F} & 0\\
V_{12}^{F} & V_{22}^{F} & 0 & V_{25}^{F} & 0 & 0 & 0\\
0 & 0 & V_{33}^{F} & 0 & V_{36}^{F} & 0 & V_{38}^{F}\\
V_{15}^{F} & V_{25}^{F} & 0 & V_{55}^{F} & 0 & V_{57}^{F} & 0\\
0 & 0 & V_{36}^{F} & 0 & V_{66}^{F} & 0 & V_{68}^{F}\\
V_{17}^{F} & 0 & 0 & V_{57}^{F} & 0 & V_{s,77}^{F} & 0\\
0 & 0 & V_{38}^{F} & 0 & V_{68}^{F} & 0 & V_{88}^{F}
\end{vmatrix}=0\,. \tag{39}
$$

Solving equation (39) yields the three critical loads corresponding to the three buckling modes: snake, hourglass and overall

$$
\begin{aligned}
P_{s,panel}^{C}&=\frac{1}{3}\pi^2\overline{E}_f t^3\left(\frac{m^2}{2a^2}+\frac{a^2n^4}{2b^4m^2}+\frac{n^2}{b^2}\right)+\frac{2}{\pi^2R^2}\overline{E}_f t\frac{a^2}{m^2}+\\
&+G_c c\left(1+\frac{a^2n^2}{b^2m^2}\right)-\frac{4}{\pi^2}\times V_s\,,
\end{aligned}
\tag{40}
$$

$$P^{C}_{h,panel}=\left[\frac{1}{3}\pi^{2}\overline{E}_{f}t^{3}\left(\frac{m^{2}}{2a^{2}}+\frac{a^{2}n^{4}}{2b^{4}m^{2}}+\frac{n^{2}}{b^{2}}\right)+\frac{2}{\pi^{2}R^{2}}\overline{E}_{f}t\frac{a^{2}}{m^{2}}+\right.$$
$$\left.+\frac{1}{3}G_{c}c\left(1+\frac{a^{2}n^{2}}{b^{2}m^{2}}\right)+\frac{4}{\pi^{2}}\times\overline{E}_{c}\frac{a^{2}}{cm^{2}}v^{(1)}_{c}+\frac{4}{\pi^{2}}\times V_{h}\times\left(1-\frac{2}{3}\Psi_{plate}\right)^{-1}\right], \tag{41}$$

$$P^{C}_{o,panel}=\frac{1}{3}\pi^{2}\overline{E}_{f}t^{3}\left(\frac{k^{2}}{2a^{2}}+\frac{a^{2}l^{4}}{2b^{4}k^{2}}+\frac{l^{2}}{b^{2}}\right)+G_{c}c\left(1+\frac{a^{2}l^{2}}{b^{2}k^{2}}\right)+$$
$$+\frac{1}{\pi^{2}R^{2}}\left(2\overline{E}_{f}t+\overline{E}_{c}cv^{(1)}_{c}\right)\frac{a^{2}}{k^{2}}-\frac{4a^{2}}{\pi^{6}R^{2}}\left(1-(-1)^{k}\right)^{2}\left(1-(-1)^{l}\right)^{2}\times \tag{42}$$
$$\times\left[\Psi_{panel}\frac{\overline{E}_{c}cv^{(2)}_{c}+4\overline{E}_{f}t\left(v_{f}-2\right)}{k^{4}l^{2}}+3\lambda_{panel}\frac{\overline{E}_{c}cv^{(2)}_{c}l^{2}-4\overline{E}_{f}tv_{f}\left(l^{2}-6\right)}{l\left(l^{2}-4\right)^{2}}\right]+\frac{4}{\pi^{2}}\times V_{o},$$

where

$$V_{s}=\frac{1}{2\left(-V_{57}^{2}+V_{55}V_{77}\right)}\left[V_{17}^{2}V_{55}-2V_{15}V_{17}V_{57}+V_{15}^{2}V_{77}+V_{25}^{2}V_{77}+\right.$$
$$+\left(\left(-V_{17}^{2}V_{55}+2V_{15}V_{17}V_{57}-V_{15}^{2}V_{77}-V_{25}^{2}V_{77}\right)^{2}-4\left(-V_{57}^{2}+V_{55}V_{77}\right)\right. \tag{43}$$
$$\left.\left.\left(V_{17}^{2}V_{25}^{2}-2V_{12}V_{17}V_{25}V_{57}+V_{12}^{2}V_{57}^{2}-2V_{12}V_{15}V_{25}V_{77}-V_{12}^{2}V_{55}V_{77}\right)\right)^{1/2}\right],$$

$$V_{h}=\frac{1}{2\left(-V_{57}^{2}+V_{55}V_{77}\right)}\left[V_{17}^{2}V_{55}-2V_{15}V_{17}V_{57}+V_{15}^{2}V_{77}+V_{25}^{2}V_{77}-\right.$$
$$-\left(\left(-V_{17}^{2}V_{55}+2V_{15}V_{17}V_{57}-V_{15}^{2}V_{77}-V_{25}^{2}V_{77}\right)^{2}-4\left(-V_{57}^{2}+V_{55}V_{77}\right)\right. \tag{44}$$
$$\left.\left.\left(V_{17}^{2}V_{25}^{2}-2V_{12}V_{17}V_{25}V_{57}+V_{12}^{2}V_{57}^{2}-2V_{12}V_{15}V_{25}V_{77}-V_{12}^{2}V_{55}V_{77}\right)\right)^{1/2}\right],$$

$$V_{o}=\frac{{V^{F}_{38}}^{2}V^{F}_{66}-2V^{F}_{36}V^{F}_{38}V^{F}_{68}+{V^{F}_{36}}^{2}V^{F}_{88}}{{V^{F}_{68}}^{2}-V^{F}_{66}V^{F}_{88}}. \tag{45}$$

7. RESULTS AND DISCUSSION

Although the expressions for the buckling loads of a sandwich panel were obtained analytically with closed form solution, they are complex, making it difficult to establish how these critical loads vary with the different parameters.

Nevertheless, because *m* and *k* figure always in the denominator as well as the numerator, the sandwich panel will yield minima for certain values of *m* and *k*. If only the first terms of the expressions are considered it can be seen that the transverse wavenumbers n and l always figure in the numerator only, suggesting that the minimum value is obtained for n and l = 1, as can be easily checked numerically (Fig. 3b) for example).

To exemplify the formulation developed in this study and to illustrate the behaviour of curved sandwich panels loaded in compression some typical examples were selected from the literature ([1] and [5]). This component is longitudinally compressed, simply supported plate with aspect ratio 2, with length a = 508 mm, width b = 254 mm, flange thickness t = 0.508 mm, core thickness c = 5.08 mm, flange Young's modulus E_f = 68 947.57 N/mm2, flange Poisson's ratio v_f = 0.3, core Young's modulus E_c = 198.57 N/mm2, core shear modulus G_c = 82.74 N/mm^2 and core Poisson's ratio v_c = 0.2. Here, to enhance the interactive effect, by bringing closer together the critical loads for the overall and hourglass modes, it is considered a core thickness of c = 15.24 mm [1]. In this example a similar panel is calculated having a radius of curvature equal to R = 1000 mm.

The variation of the critical loads are shown plotted against longitudinal (*m*) and transverse (*n*) wavenumbers in Fig. 3, where P_s denotes the critical load for the snake mode and P_h denotes the critical load for the hourglass mode. As discussed before, both modes exhibit a monotonically increasing variation with the transverse wavenumber *n*.

Longitudinally, and assuming that *n* and *l* are equal to 1, the software Mathematica [8] was used to compute the minimum critical loads (Fig. 3a) and the corresponding wavenumbers (Table 1).

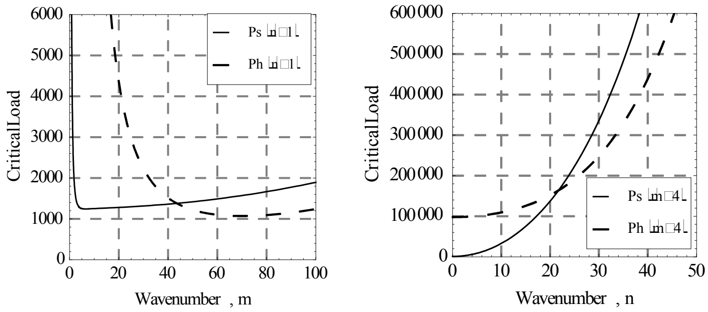
It is important to highlight that in contrast to what happens to the sandwich plate, the minimum values of the critical loads depend on the radius of curvature. It is possible to prove that whenever *R* tends to infinite, *m* tends to 4.

The variation of the critical loads with width *b*, plotted in Fig. 4a, b at constant values of *m*, *n* and *R*, which correspond to the minimum critical load for the snake mode (m = 4, n = 1 and R = infinite; m = 70, n = 1 and R = 1000) and for the hourglass mode (m = 6, n = 1 and R = infinite; m = 70, n = 1 and R = 1000). It noticeable that while the variation of snake mode is quite abrupt for low values of *b*, the hourglass mode is hardly sensitive to variations of width, reflecting the local nature of its corresponding wavelength, already small compared to the length of the panel [1].

Table 1

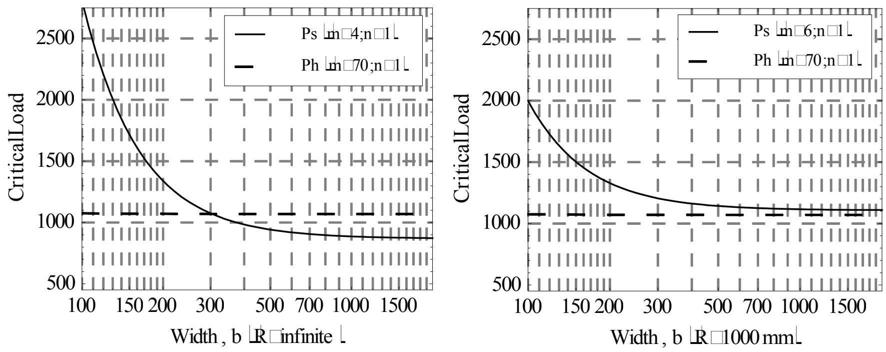
Minimum values of critical loads (P_s and P_h)

Radius of curvature	Snake mode		Hourglass mode	
	Critical Load	Longitudinal wavenumber, *m*	Critical Load	Longitudinal wavenumber, *m*
infinite	1157.53 N	4	1070.26 N	70
1000	1244.26 N	6	1070.77 N	70



a) b)

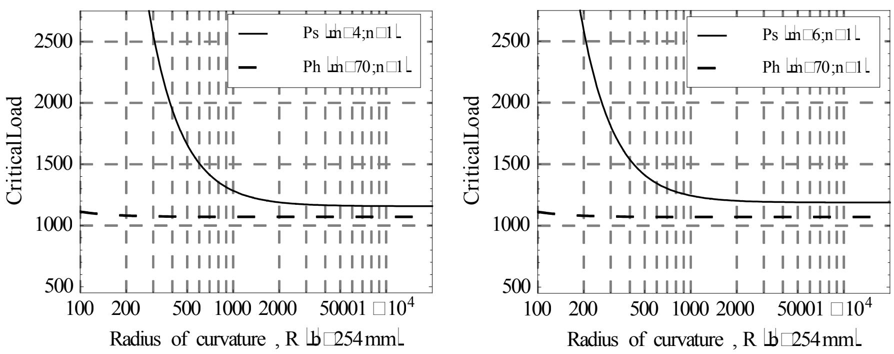
Fig. 3 – Critical load variations with longitudinal and transverse wavenumbers (R = 1000 mm).



a) b)

Fig. 4 – Critical load variations with width, *b*.

The conclusion on the effect of curvature in the critical behaviour of the sandwich panel is immediately drawn from Fig. 5: the bigger the curvatures (low values of R) the higher the critical loads are.



a) b)

Fig. 5 – Critical load variations with width radius of curvature, R.

8. CONCLUSIONS

The energy formulation presented in this paper provides closed-form solutions for the critical loads of curved sandwich panels. Despite the complexity of the obtained expressions, they are easily programmed in a spreadsheet. Furthermore, the formulation already includes the nonlinear terms required for a post-buckling evaluation, an issue that is currently actively being pursued by the authors.

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ANNEX A

In the membrane strain energy of the flanges, K_n, K_l and K_{nl} (30) correspond to simplifications of the following general expressions

$$K_n = \frac{2n\ \left(-1+n^2\right)\pi + \sin\left(2n\pi\right)}{8n\ \left(-1+n^2\right)};\quad K_l = \frac{2l\ \left(-1+l^2\right)\pi + \sin\left(2l\pi\right)}{8l\ \left(-1+l^2\right)} \tag{46}$$

$$K_{nl} = \frac{\sin\left(2\pi\ (l-n)\right)}{\pi\left(l-n\right)}. \tag{47}$$

It is noted that equation (42) represents an indeterminate expression for n or l =1 that is solved using a limit approximation (48). In all other possible cases (l >1) the value of K_l and K_n is easily computed and is equal to 1/4.

$$\begin{aligned} &\lim_{l\to 1}\frac{2l\ \left(-1+l^2\right)\pi + \sin\left(2l\pi\right)}{8l\ \left(-1+l^2\right)} = \frac{3\pi}{8} \\ &\lim_{l\to Int[>1]}\frac{2l\ \left(-1+l^2\right)\pi + \sin\left(2l\pi\right)}{8l\ \left(-1+l^2\right)} = \frac{\pi}{4}. \end{aligned} \tag{48}$$

Equation (47) is also indeterminate for $l = n$. Again a solution is found using a limit approximation, and in all cases it should be noted that n is always greater than l. It is concluded (from (49)) that whenever n = 1 K_{nl} is equal to 2; whenever n > l K_{nl} is equal to 0

$$\begin{aligned}&\lim_{l\to n}\frac{\sin(2\pi(l-n))}{\pi(l-n)}=2\\ n>1:\;&\lim_{l\to 1}\frac{\sin(2\pi(l-n))}{\pi(l-n)}=0\end{aligned}\quad . \tag{49}$$

ANNEX B

(i) Sandwich plate $V_{p,ij}{}^{F}$ terms

$$\begin{aligned}V_{p,11}^{F}&=\frac{1}{4}\pi^2G_c c\left(1+\frac{a^2n^2}{b^2m^2}\right)+\frac{1}{6}\pi^4G_f t^3\frac{n^2}{b^2}-\\ &-\frac{1}{12}\pi^4\overline{E}_f t^3\left(\frac{m^2}{2a^2}+\frac{a^2n^4}{2b^4m^2}+\frac{n^2}{b^2}v_f\right)-\frac{1}{4}\pi^2P^F,\end{aligned} \tag{50}$$

$$\begin{aligned}V_{p,22}^{F}&=\frac{1}{12}\pi^2G_c c\left(1+\frac{a^2n^2}{b^2m^2}\right)+\frac{1}{6}\pi^4G_f t^3\frac{n^2}{b^2}-\\ &-\frac{1}{12}\pi^4\overline{E}_f t^3\left(\frac{m^2}{2a^2}+\frac{a^2n^4}{2b^4m^2}+\frac{n^2}{b^2}v_f\right)+\overline{E}_c\frac{a^2}{c\ m^2}v_c^{(1)}-\\ &-\frac{1}{4}\pi^2P^F\left(1-\frac{2}{3}\Psi_{plate}\right),\end{aligned} \tag{51}$$

$$\begin{aligned}V_{p,33}^{F}&=\frac{1}{4}\pi^2G_c c\left(1+\frac{a^2l^2}{b^2k^2}\right)+\frac{1}{6}\pi^4G_f t^3\frac{l^2}{b^2}-\\ &-\frac{1}{12}\pi^4\overline{E}_f t^3\left(\frac{k^2}{2a^2}+\frac{a^2l^4}{2b^4k^2}+\frac{l^2}{b^2}v_f\right)-\frac{1}{4}\pi^2P^F,\end{aligned} \tag{52}$$

$$V_{p,55}^{F}=\frac{1}{4}\pi^2G_c c+\left(\frac{c\pi^2}{2\sqrt{2}}\right)^2\left[\begin{aligned}&\frac{m^2}{a^2}\left(\overline{E}_f t+\frac{1}{6}\overline{E}_c c v_c^{(1)}\right)+\\ &+\frac{n^2}{b^2}\left(G_f t+\frac{1}{6}G_c c\right)\end{aligned}\right], \tag{53}$$

$$V_{p,66}^{F}=\frac{1}{4}\pi^{2}G_{c}c+\left(\frac{c\pi^{2}}{2\sqrt{2}}\right)^{2}\left[\begin{array}{l}\frac{k^{2}}{a^{2}}\left(\overline{E}_{f}t+\frac{1}{6}\overline{E}_{c}cv_{c}^{(1)}\right)+\\+\frac{l^{2}}{b^{2}}\left(G_{f}t+\frac{1}{6}G_{c}c\right)\end{array}\right], \tag{54}$$

$$V_{p,77}^{F}=\frac{1}{4}\pi^{2}G_{c}c\frac{a^{2}n^{2}}{b^{2}m^{2}}+\left(\frac{c\pi^{2}}{2\sqrt{2}\ b}n\right)^{2}\left[\frac{a^{2}n^{2}}{b^{2}m^{2}}\left(\overline{E}_{f}t+\frac{1}{6}\overline{E}_{c}cv_{c}^{(1)}\right)+\right.$$
$$\left.+\left(G_{f}t+\frac{1}{6}G_{c}c\right)\right], \tag{55}$$

$$V_{p,88}^{F}=\frac{1}{4}\pi^{2}G_{c}c\frac{a^{2}l^{2}}{b^{2}k^{2}}+\left(\frac{c\pi^{2}}{2\sqrt{2}\ b}l\right)^{2}\left[\frac{a^{2}l^{2}}{b^{2}k^{2}}\left(\overline{E}_{f}t+\frac{1}{6}\overline{E}_{c}cv_{c}^{(1)}\right)+\right.$$
$$\left.+\left(G_{f}t+\frac{1}{6}G_{c}c\right)\right], \tag{56}$$

$$V_{p,15}^{F}=V_{p,36}^{F}=-\frac{1}{4}\pi^{2}G_{c}c, \tag{57}$$

$$V_{p,17}^{F}=\frac{1}{4}\pi^{2}G_{c}c\frac{a^{2}n^{2}}{b^{2}m^{2}}, \tag{58}$$

$$V_{p,38}^{F}=-\frac{1}{4}\pi^{2}G_{c}c\frac{a^{2}l^{2}}{b^{2}k^{2}}, \tag{59}$$

$$V_{p,57}^{F}=\left(\frac{c\pi^{2}}{2\sqrt{2}\ b}n\right)^{2}\left(\overline{E}_{f}tv_{f}+\frac{1}{6}\overline{E}_{c}cv_{c}^{(2)}+G_{f}t+\frac{1}{6}G_{c}c\right), \tag{60}$$

$$V_{p,68}^{F}=\left(\frac{c\pi^{2}}{2\sqrt{2}\ b}l\right)^{2}\left(\overline{E}_{f}tv_{f}+\frac{1}{6}\overline{E}_{c}cv_{c}^{(2)}+G_{f}t+\frac{1}{6}G_{c}c\right). \tag{61}$$

(ii) Sandwich panel V_{ij}^{F} terms

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$$V_{11}^{F}=V_{p,11}^{F}+\frac{1}{2R^{2}}\overline{E}_{f}t\frac{a^{2}}{m^{2}}, \tag{62}$$

$$V_{12}^{F} = \frac{1}{2R}\overline{E}_c v_c^{(2)}\frac{a^2}{m^2}, \tag{63}$$

$$V_{22}^{F} = V_{p,22}^{F} + \frac{1}{2R^2}\overline{E}_f t\frac{a^2}{m^2}, \tag{64}$$

$$V_{33}^{F} = V_{p,33}^{F} + \frac{1}{4R^2}\left(2\overline{E}_f t + \overline{E}_c c v_c^{(1)}\right)\frac{a^2}{k^2} - \frac{a^2}{\pi^4 R^2}\left(1-(-1)^k\right)^2\left(1-(-1)^l\right)^2 \times$$
$$\times\left[\Psi_{panel}\frac{\overline{E}_c c v_c^{(2)} + 4\overline{E}_f t\left(v_f - 2\right)}{k^4 l^2} + 3\lambda_{panel}\frac{\overline{E}_c c v_c^{(2)} l^2 - 4\overline{E}_f t v_f\left(l^2 - 6\right)}{l\left(l^2-4\right)^2}\right], \tag{65}$$

$$V_{55}^{F} = V_{p,55}^{F} \tag{66}$$

$$V_{66}^{F} = V_{p,66}^{F} \tag{67}$$

$$V_{66}^{F} = V_{p,77}^{F} \tag{68}$$

$$V_{88}^{F} = V_{p,88}^{F} \tag{69}$$

$$V_{15}^{F} = V_{36}^{F} = V_{p,15}^{F} \tag{70}$$

$$V_{17}^{F} = V_{p,17}^{F} \tag{71}$$

$$V_{25}^{F} = \frac{1}{24R}c\pi^2\left(6\overline{E}_f t v_f + \overline{E}_c c v_c^{(2)}\right), \tag{72}$$

$$V_{38}^{F} = V_{p,38}^{F} \tag{73}$$

$$V_{57}^{F} = V_{p,57}^{F} \tag{74}$$

$$V_{68}^{F} = V_{p,68}^{F}. \tag{75}$$

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