VIBRATIONS ANALYSIS OF A MECHANICAL SYSTEM CONSISTING OF TWO IDENTICAL PARTS

SORIN VLASE, MARIUS PĂUN

Abstract. In many technical applications (like those in automotive engineering or in structural mechanics) the mechanical system studied can be considered composed by two or many identical subsystems or parts. These kind of symmetries of the structure can be used in order to simplify the analysis of the vibrations and permit to reduce the dimension of the differential equations that describe the motion. In the paper we proof that in case of such technical systems with elastic elements composed of two identical subsystems, the eigenvalues of the subsystems associated to the identical parts are eigenvalues of the whole structure as well. The demonstrated property allows the simplification of the calculation of the problem of eigenvectors and eigenvalues for this kind of structure.

Key words: vibration, symmetrical systems, identical parts, eigenvalues, eigenvector.

1. INTRODUCTION

In case of the study of deformations to static loads the systems presenting symmetries have properties allowing the simplification of calculus. In case of dynamic loading and especially in case of vibrations of elastic mechanical systems presenting certain symmetries we shall prove that this type of systems lead to properties finally allowing the simplification of calculus. Underneath we shall present several simple cases suggesting the main property of such systems.

i) Let us consider a very easy problem namely that of a mechanical system consisting of three wheels, two of which being identical. These wheels may have torsional vibrations (Fig. 1). The system analyzed in Fig. 1, called (S), consists of two identical subsystems (S_1^s) and (S_1^d) in liaison with the wheel J_2. J_1 and J_2 represent the moment of inertia of the wheels, k_1 and k_2 being the rigidities of the elastic elements.

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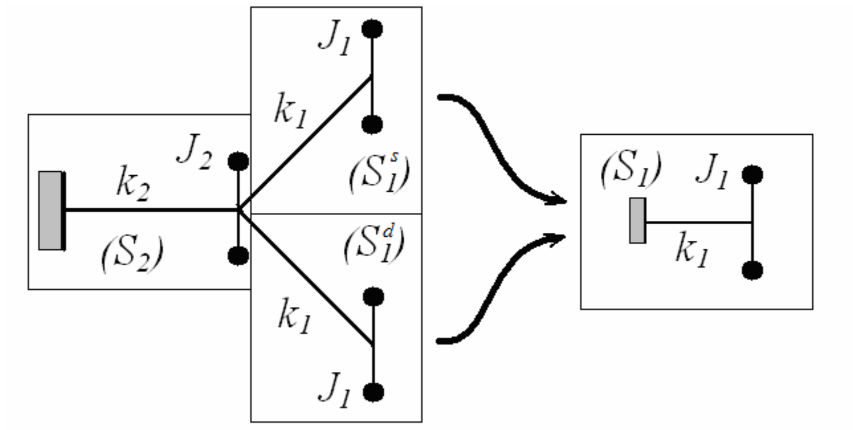


Fig. 1 – A very simple model.

We denote with φ_1^s, φ_1^d and φ_2 the absolute rotation of the wheel with the moment of inertia J_1, J_1 and respectively J_2. The free non-damped vibrations for system (S) may be written as follows [1-4]:

$$J_1\ddot{\varphi}_1^s + k_1(\varphi_1^s - \varphi_2) = 0\,,$$

$$J_1\ddot{\varphi}_1^d + k_1(\ddot{\varphi}_1^d - \varphi_2) = 0\,, \tag{1a}$$

$$J_2\ddot{\varphi}_2 - k_1(\ddot{\varphi}_1^d - \varphi_2) - k_1(\varphi_1^d - \varphi_2) + k_2\varphi_2 = 0$$

or

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\varphi}_1^s \\ \ddot{\varphi}_1^d \\ \ddot{\varphi}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_1 & -k_1 \\ -k_1 & -k_1 & 2k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \varphi_1^s \\ \varphi_1^d \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1b}$$

The characteristic equation allowing the determination of eigenvalues is:

$$\begin{vmatrix} k_1 - \omega^2 J_1 & 0 & -k_1 \\ 0 & k_1 - \omega^2 J_1 & -k_1 \\ -k_1 & -k_1 & 2k_1 + k_2 - \omega^2 J_2 \end{vmatrix} = 0\,. \tag{2}$$

For system (S_1) the eigenvalue of the wheel (J_1) is:

$$\omega_1^2 = \frac{k_1}{J_1}\,. \tag{3}$$

It can be easily proved by direct calculation that ω_1^2 obtained by (3) verifies the equation (2).

ii) A more complicated problem in which a system is considered consisting of two identical subsystems (Fig.2) is presented in [5]. System (S) (Fig.2) consists of two identical subsystems (S_1^s) and (S_1^d) and two non-identical systems (S_2) and (S_3).

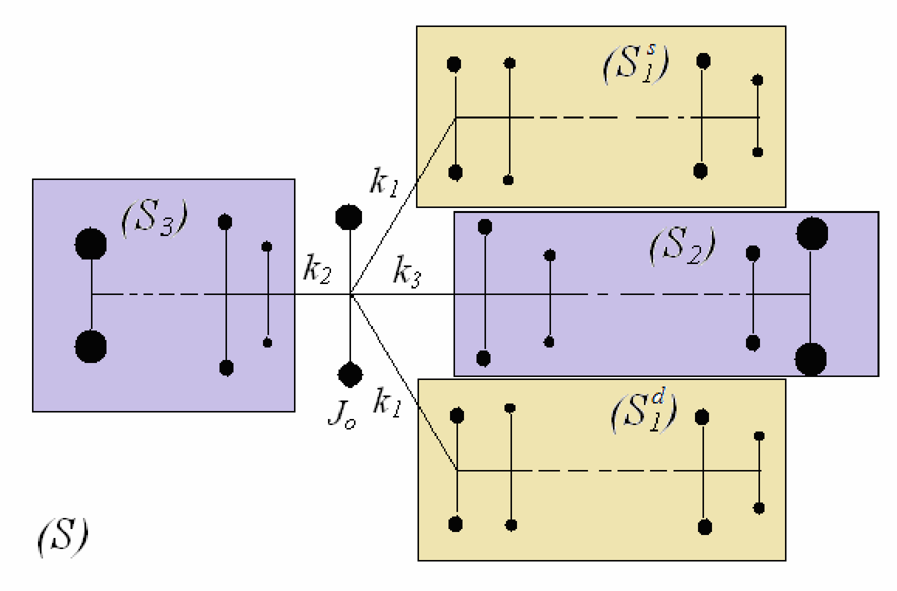


Fig. 2 – The model with wheels of the symmetrical system.

The motion equations for a wheel i with the moment of inertia J_i linked to the wheels $i-1$ and $i+1$ by the elastic elements $k_{i-1,i}$ and $k_{i,i+1}$ respectively have the following (well known) form [1–5]:

$$J_i\ddot{\varphi}_i + k_{i-1,i}(\varphi_i - \varphi_{i-1}) + k_{i,i+1}(\varphi_i - \varphi_{i+1}) = 0 . \tag{4}$$

If the free non-damped vibrations are being considered for the wheel J_o with the absolute rotation angle φ_o, which ensures the connection between the four subsystems, the motion equations are:

$$J_o\ddot{\varphi}_o + k_1(\varphi_o - \varphi_1^s) + k_1(\varphi_o - \varphi_1^d) + k_2(\varphi_o - \varphi_2^a) + k_3(\varphi_o - \varphi_3^a) = 0 , \tag{5}$$

where φ_1^s, φ_1^d, φ_2^a and φ_3^a denote the absolute rotation angle of the adjacent flywheel to the wheel J_o, belonging to the subsystems (S_1^s), (S_1^d), (S_2) and, respectively, (S_3).

Figure 3 shows the identical subsystems (S_1^s) and (S_1^d), both denoted with (S_1); by combination, they form the symmetrical system. The motion equations, describing the free non-damped vibrations for the identical subsystems (S_1^s) and (S_1^d) are, respectively:

$$[J]_1\{\ddot{\varphi}^s\}_1 + [K]_1\{\varphi^s\}_1 = 0\,, \tag{6a}$$

$$[J]_1\{\ddot{\varphi}^d\}_1 + [K]_1\{\varphi^d\}_1 = 0\,. \tag{6b}$$

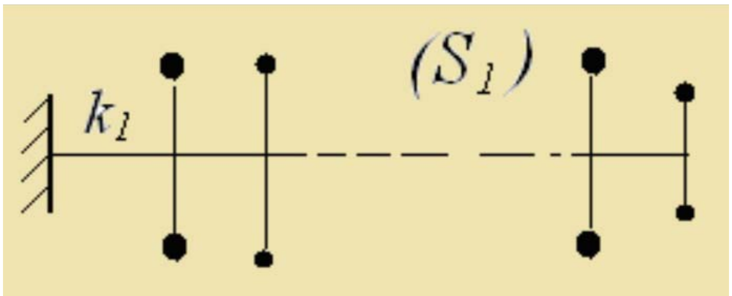


Fig. 3 – The identical subsystems which, by combination, form the symmetrical system.

For the subsystem (S_2) and (S_3) the motion (Fig. 4) equations have the form:

$$[J]_2\{\ddot{\varphi}\}_2 + [K]_2\{\varphi\}_2 = 0 \tag{7}$$

and

$$[J]_3\{\ddot{\varphi}\}_3 + [K]_3\{\varphi\}_3 = 0 \tag{8}$$

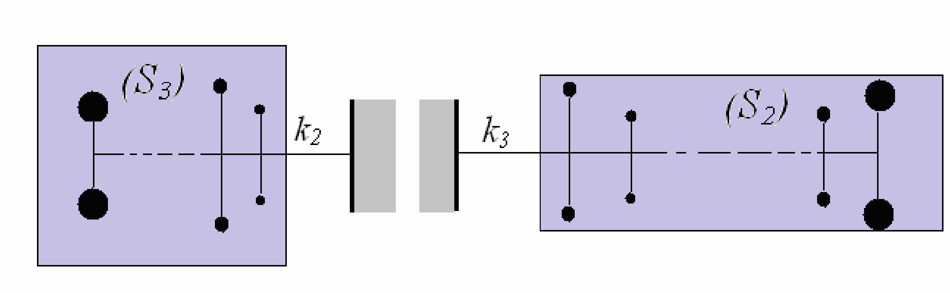


Fig. 4 – The other two parts (S_2) and (S_3) of the system.

For the entire system, the free non-damped vibrations shall be described by the following system of equations:

$$\begin{bmatrix} [J]_1 & 0 & 0 & 0 & 0 \\ 0 & [J]_1 & 0 & 0 & 0 \\ 0 & 0 & [J]_2 & 0 & 0 \\ 0 & 0 & 0 & [J]_3 & 0 \\ 0 & 0 & 0 & 0 & J_o \end{bmatrix} \begin{Bmatrix} \{\ddot{\varphi}^s\}_1 \\ \{\ddot{\varphi}^d\}_1 \\ \{\ddot{\varphi}\}_2 \\ \{\ddot{\varphi}\}_3 \\ \ddot{\varphi}_o \end{Bmatrix} +$$

$$+ \begin{bmatrix} [K]_1 & & & & \\ 0 & [K]_1 & & SIM & \\ 0 & 0 & [K]_2 & & \\ 0 & 0 & 0 & [K]_3 & \\ 0\ldots-k_1 & 0\ldots-k_1 & 0\ldots-k_2 & 0\ldots-k_3 & 2k_1+k_2+k_3 \end{bmatrix} \begin{Bmatrix} \{\varphi^s\}_1 \\ \{\varphi^d\}_1 \\ \{\varphi\}_2 \\ \{\varphi\}_3 \\ \varphi_o \end{Bmatrix} = 0 \qquad (9)$$

or, briefly:

$$[J]\{\ddot{\varphi}\} + [K]\{\varphi\} = 0\,, \qquad (10)$$

where $\{\varphi\}$ represents the vector of wheels absolute angles of rotations, $[J]$ the inertia matrix and $[K]$ the rigidity matrix.

The characteristic equation for the presented system is:

$$\Delta = \det([K] - \omega^2 [M])\,. \qquad (11)$$

In the papers [5, 6] the following property has been demonstrated:

The eigenvalues for the (S_1) systems are eigenvalues for system (S) as well.

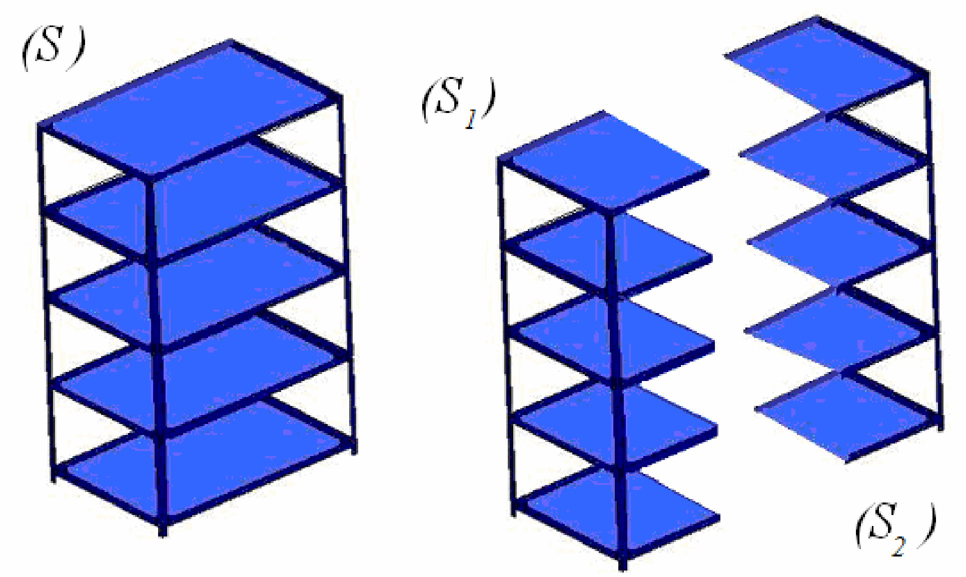
The result has been proved by a laborious method and uses the fact that the connection between the (S_1) systems and the rest of the system is being achieved by one element k_1. If the connection is carried out by several elements, the calculation becomes too complicated for obtaining a result by direct calculation.

2. VIBRATIONS OF MECHANICAL SYSTEMS WITH SYMMETRIES

Let us consider now a complex problem, that of determining the eigenvalues for the structure from Fig. 5, a system (S) formed of two identical subsystems (S_1). Using the classical methods of mechanics [7, 8] it is possible to write the equations of the free non-damped vibrations for the entire structure, as follows:

$$\begin{bmatrix} [m_a] & 0 & [m_{ab}] \\ 0 & [m_a] & [m_{ab}] \\ [m_{ab}]^T & [m_{ab}]^T & [m_b] \end{bmatrix} \{\ddot{X}\} + \begin{bmatrix} [k_a] & 0 & [k_{ab}] \\ 0 & [k_a] & [k_{ab}] \\ [k_{ab}]^T & [k_{ab}]^T & [k_b] \end{bmatrix} \{X\} = 0, \quad (12)$$

where $[m_a]$, $[k_a]$, $[m_b]$, $[k_b]$ are symmetrical square matrices, of random size.



a) b)

Fig. 5 – A system composed by two identical parts.

For half a structure (the subsystem (S_1)) the equations becomes:

$$[m_a]\{\ddot{Y}\} + [k_a]\{Y\} = 0 . \quad (13)$$

By numerical calculation, using the method of finite elements [9–11] the following property has been established:

PROPERTY 1. *The eigenvalues for system (13) are eigenvalues for system (12) as well.*

That means the solutions of algebraic equations:

$$\det\left([k_a] - \omega^2 [m_a]\right) = 0 \quad \left(\text{or} \quad \left|[k_a] - \omega^2 [m_a]\right| = 0\right) \quad (14)$$

are also solutions of the algebraic equation:

$$\det\left(\begin{bmatrix} [k_a] & 0 & [k_{ab}] \\ 0 & [k_a] & [k_{ab}] \\ [k_{ab}]^{\mathrm{T}} & [k_{ab}]^{\mathrm{T}} & [k_b] \end{bmatrix} - \omega^2 \begin{bmatrix} [m_a] & 0 & [m_{ab}] \\ 0 & [m_a] & [m_{ab}] \\ [m_{ab}]^{\mathrm{T}} & [m_{ab}]^{\mathrm{T}} & [m_b] \end{bmatrix}\right) = 0 \quad (15)$$

or

$$\begin{vmatrix} [k_a]-\omega^2[m_a] & 0 & [k_{ab}]-\omega^2[m_{ab}] \\ 0 & [k_a]-\omega^2[m_a] & [k_{ab}]-\omega^2[m_{ab}] \\ [k_{ab}]^{\mathrm{T}}-\omega^2[m_{ab}]^{\mathrm{T}} & [k_{ab}]^{\mathrm{T}}-\omega^2[m_{ab}]^{\mathrm{T}} & [k_b]-\omega^2[m_b] \end{vmatrix} = 0. \quad (16)$$

We could write:

$$\left|[k_a]-\omega^2[m_a]\right| = 0 \quad \Rightarrow$$

$$\begin{vmatrix} [k_a]-\omega^2[m_a] & 0 & [k_{ab}]-\omega^2[m_{ab}] \\ 0 & [k_a]-\omega^2[m_a] & [k_{ab}]-\omega^2[m_{ab}] \\ [k_{ab}]^{T}-\omega^2[m_{ab}]^{\mathrm{T}} & [k_{ab}]^{T}-\omega^2[m_{ab}]^{\mathrm{T}} & [k_b]-\omega^2[m_b] \end{vmatrix} = 0.$$

That implies: the polynomial in ω^2 from equation (16) is divided by the polynomial in ω^2 from equation (14).

The demonstration of this property represents the main contribution of the paper.

3. THE EIGENVALUES OF THE SUBSYSTEM (S_1') ARE EIGENVALUES FOR THE SYSTEM (S) AS WELL

The Property 1 is valid in a more general context. This fact will be demonstrated as follows.

PRELIMINARY CONSIDERATIONS. Let us consider $M = (m_{ij})_{i,j\in\{1,2,\dots,n\}}$, $U = \left(m_{ij}\right)_{i\in\{i1,i2,..,ik\}\subset\{1,2,..,n\},\ j\in\{j1,j2,..,jk\}\subset\{1,2,..,n\}}$ a sub-matrix of M and $\alpha = \det(U)$. For the chosen matrix U we consider the complementary matrix $\overline{U} = (m_{ij})_{i\in\{1,2,\dots,n\}\setminus\{i1,\dots,ik\},\ j\in\{1,2,\dots,n\}\setminus\{j1,\dots,jk\}}$. We call an algebraic complement of the minor α (the cofactor of α) the determinant with sign $\overline{\alpha} = (-1)^{i1+\dots+ik+j1+\dots+jk}\det(\overline{U})$.

Considering the lines $i1,..,ik$ as being fixed, we have:

$$\det(M)=\sum_{1\le j1<...<jk\le n}\alpha\bar{\alpha}\,. \tag{17}$$

This formula generalizes the Laplace expansion formula according to a line of the matrix M determinant.

$\bar{U}$ is a matrix $(n-k)\times(n-k)$ and considering V its square sub-matrix of indices $\{m1,...,mp\}$, $\{l1,...,lp\}$ of determinant β and correspondingly matrix $\bar{V}$ we may write:

$$\det(\bar{U})=\sum_{1\le l1;..;lp\le n-k}\beta\cdot(-1)^{m1+...+mp+l1+...lp}\det(\bar{V})\,.$$

There from we have:

$$\det(M)=\sum_{1\le j1<..<jk\le n}\ \sum_{1\le l1<..<lp\le n-k}\alpha\beta\left(-1\right)^{i1+...+ik+j1+...+jk+m1+...+mp+l1+...+lp}\det(\bar{V})=$$

$$=\sum_{j}\sum_{l}\alpha\beta\gamma\,. \tag{18}$$

PROPOSITION 1. *Consider the square polynomial matrices with complex coefficients, of size n, denoted A, B, C, L, Z=O_n and matrix* $M=\begin{pmatrix}A & Z & B\\ Z & A & B\\ L & L & C\end{pmatrix}$. *Then* $\det(M)$ *is dividable by* $\det(A)$.

Proof. Considering an expansion of type (18) with minors of the n^{th} order having elements on the first n lines we have[12]:

$$\det(M)=\sum_{1\le j1;..;jn\le 3n}\ \sum_{1\le l1;..;\ln\le 2n}\alpha\beta\gamma\,.$$

We shall prove the sentence demonstrating that for this special type of matrix if a term of the previous sum is not dividable by $\det(A)$, then there is a term $\alpha'\beta'\gamma'$ in expansion so that $\alpha\beta\gamma+\alpha'\beta'\gamma'=0$.

We shall analyze one by one the possible cases. We shall highlight for the start the columns of the blocks which intervene in matrix M namely $A=(A_1 \dots A_n)$, $B=(B_1 \dots B_n)$, $C=(C_1 \dots C_n)$, $L=(L_1 \dots L_n)$, $Z=(Z_1 \dots Z_n)=(0 \dots 0)$.

– for $j_1=1$, $j_n=n$ we have $\alpha=\det(A)$

– for $j_1=2n+1$, $j_n=3n$ we have $\alpha=\det(B)$ and

$$\bar{\alpha}=\det\begin{pmatrix}Z & A\\ L & L\end{pmatrix}=-\det(L)\cdot\det(A)$$

– for the rest we notice that

1) if there is an index $jk \in \{n+1,...,2n\}$ then column k from α is null thus $\alpha\beta\gamma = 0$.

2) α is non-null if $\{j1,..., jk\} \subset \{1,2,...,n\}$ and $\{jk+1,..., jn\} \subset \{2n+1,...,3n\}$ and in this case $\alpha = \det(Aj1...Ajk\, Bik+1...Bin)$ where $ik+l = jk+l-2n$. For such a fixed α we have three possibilities for β namely:

   I) β has a column 0 thus $\beta = 0$

   II) $\beta = \det(A)$

   III) $\beta = \det(As1...Asl\, Brl+1...Brn)$. In this case we can determine in a unique way the matrix $\overline{V}$ for each of the two possible versions:

      a) if there is $t \in \{1,..,n\} \setminus \{j1,..jk, s1,..,sl\}$ then $\overline{V}$ contains twice the column Lt thus $\alpha\beta\gamma = 0$

      b) if $\{1,...,n\} = \{j1,...jk, s1,..., sl\}$ then we consider $\alpha' = (\det(As1...Asl\, Brl+1...Brn)$, $\beta' = \det(Aj1...Ajk\, Bik+1...Bin)$ and γ' will be a determinant having the same C type columns located in the same position as in γ and the L type columns will be the same but permutated as far as the position is concerned. A direct calculation of signs will lead to $\alpha\beta\gamma + \alpha'\beta'\gamma' = 0$.

Thus the sentence has been proved.

Taking $L=B^{\mathrm{T}}$ and with $Z=Z^{\mathrm{T}}$ we can apply this proposition to obtain the property for:

$$M = \begin{pmatrix} A & Z & B \\ Z^{\mathrm{T}} & A & B \\ B^{\mathrm{T}} & B^{\mathrm{T}} & C \end{pmatrix}.$$

4. CASE STUDY

In the following we will perform the calculus of the eigenvalues and eigenvectors for a transmission of the military track with two identical engines. One of the engine is used for the transportation of the vehicle. Both engines are used to act a device where is necessary more power. The inertia and rigidities are computed using the relation given by [13]. In Fig. 6 is presented the kinematic scheme of the transmission and in Fig. 7 the mechanical model. Table 1 shows the inertia and rigidities of the elements of the model.

The eigenvalues problem is solved using the Matlab programming medium. In Table 2 are presented the eigenvalues and eigenmodes for the whole structure, including both engines. In Table 3 are presented the eigenvalues and eigenmodes for the mechanical model of the symmetric part (only one engine). It is easy to observe that the eigenvalues for the this second model can be found among the eigenvalues of the whole system. This example illustrate the property proved in the section 3.

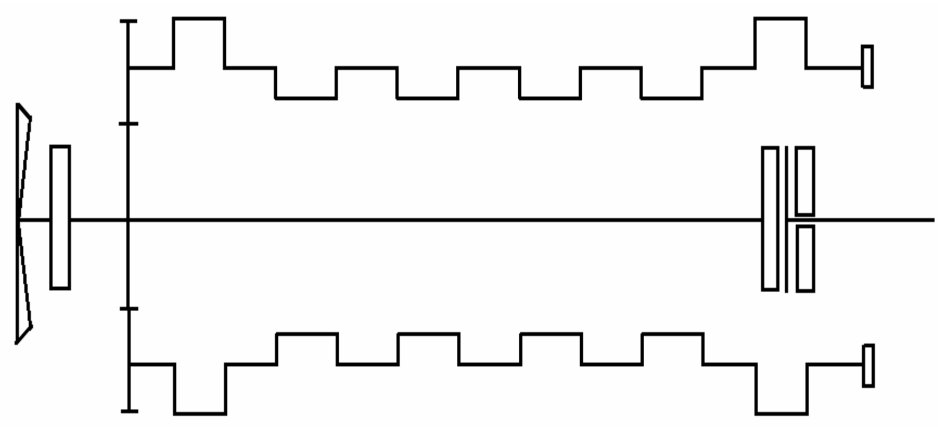


Fig. 6 – Kinematic scheme for the system enginees-transmission for a vehicle with two identical engines.

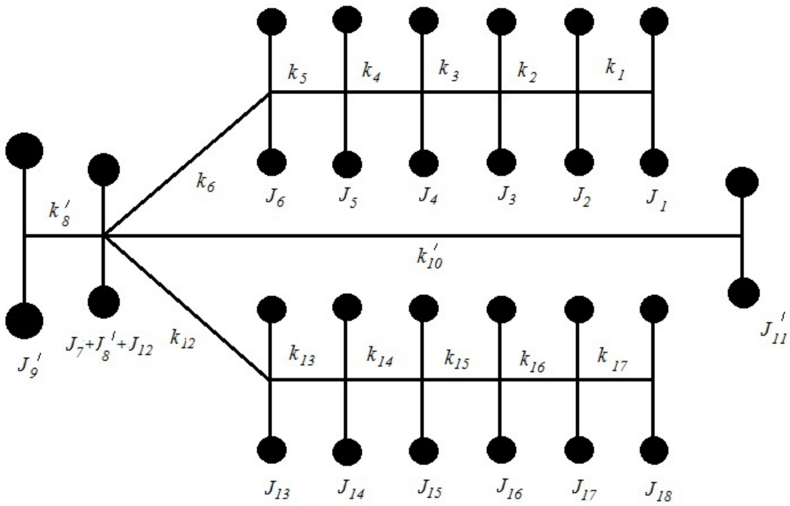


Fig. 7 – Mathematical model.

Table 1

Inertia and rigidities

	Inertia Moment	kg·m^2	Between	Rigidity	N·m
1	J_1	0.1048	1–2	k_1	$2.56\cdot10^6$
2	J_2	0.0638	2–3	k_2	$2.56\cdot10^6$
3	J_3	0.1048	3–4	k_3	$2.528\cdot10^6$
4	J_4	0.1048	4–5	k_4	$2.56\cdot10^6$
5	J_5	0.0638	5–6	k_5	$2.56\cdot10^6$
6	J_6	0.1048	6–7	k_6	$2.08685\cdot10^7$
7	$J_7 + J_8' + J_{12}$	1.81182	7–8	k_8'	$12.666\cdot10^6$
8	J_9'	3.41895	7–9	k_{10}'	$4.5961\cdot10^4$
9	J_{11}'	3.70752	7–10	k_{12}	$2.08665\cdot10^7$
10	J_{13}	0.1048	10–11	k_{13}	$2.56\cdot10^6$
11	J_{14}	0.0638	11–12	k_{14}	$2.56\cdot10^6$
12	J_{15}	0.1048	12–13	k_{15}	$2.528\cdot10^6$
13	J_{16}	0.1048	13–14	k_{16}	$2.56\cdot10^6$
14	J_{17}	0.0638	14–15	k_{17}	$2.56\cdot10^6$
15	J_{18}	0.1048			

Table 2

Eigenvalues and eigenvectors for the whole system

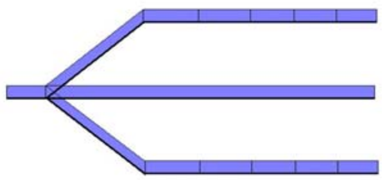
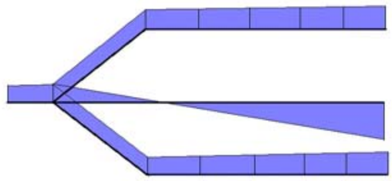
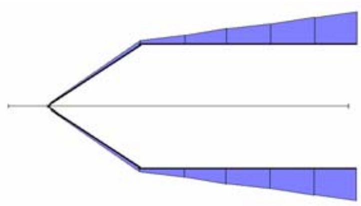
No.	Eigenvalues (Eigenrotation) [rot/min]	Corresponding eigenvector
1	0	
2	1338	
3	14062	

Table continued

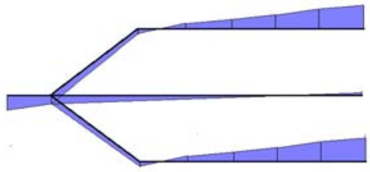
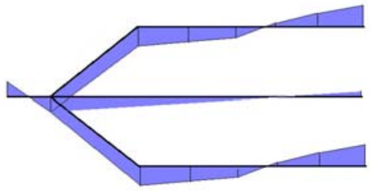
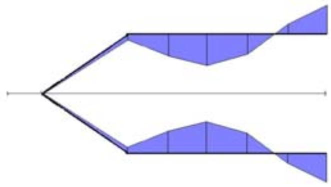
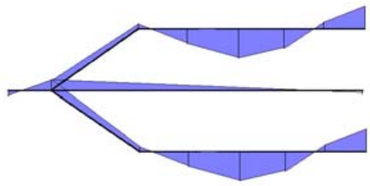
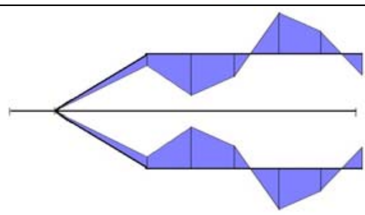
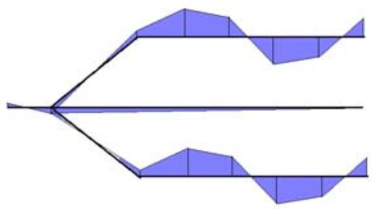
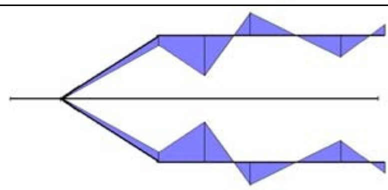
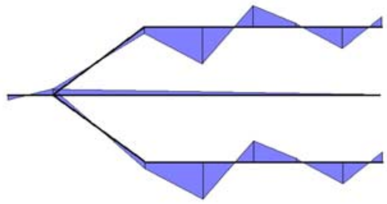
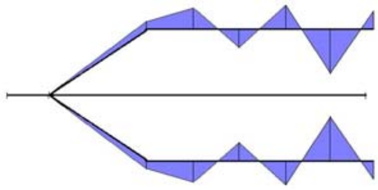
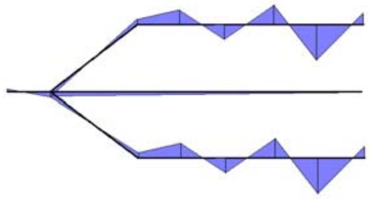
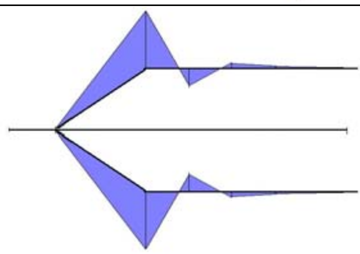
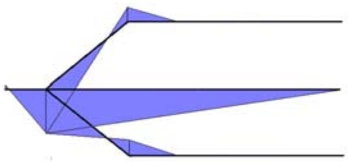
4	14564	
5	30417	
6	40278	
7	41483	
8	67067	
9	67561	
10	92674	

Table continued

11	93394	
12	100959	
13	101039	
14	144921	
15	151079	

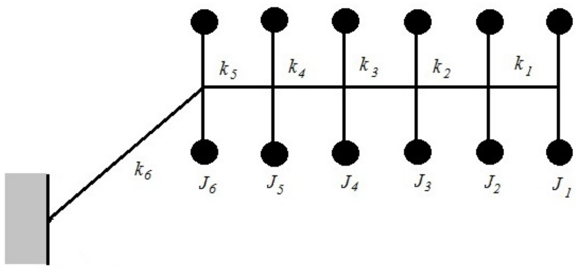
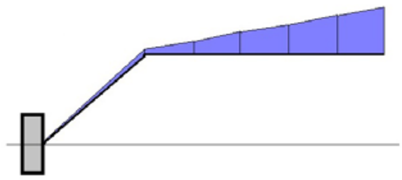
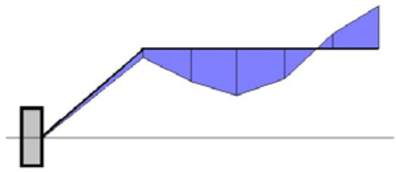
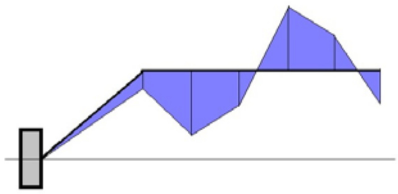
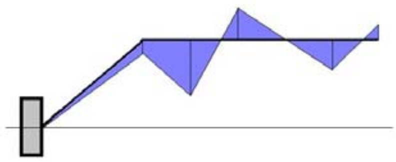
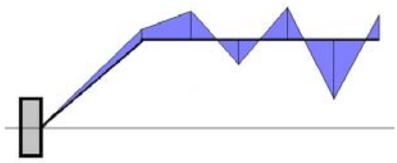
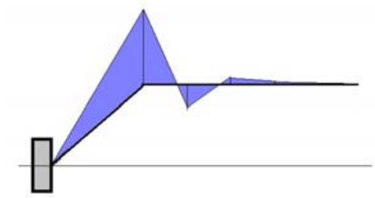


Fig. 8 – The symmetric part of the system.

Table 3

Eigenvalues and eigenvectors for the symmetric part

No.	Eigenrotation [rot/min]	Corresponding eigenvector
1	14062	
2	40278	
3	67067	
4	92674	
5	100959	
6	144921	

5. CONCLUSIONS

We have proved in the paper that in case of mechanical systems with elastic elements composed of two identical subsystems, the eigenvalues of the subsystems associated to the symmetrical part are eigenvalues of the whole structure as well. This fact also leads to other results namely that eigenvectors present certain symmetries. The demonstrated property allows the simplification of the calculation of the problem of eigenvectors and eigenvalues for this kind of structure. Thus we can first determine the eigenvalues of a sub-structure then these eigenvalues can be eliminated from the motion equations of the entire structure. Thus the size of the systems that should be solved decreases and allows an easier solving of this kind of system. The eigenvectors will also present symmetries allowing an intuitive understanding of vibration phenomena in the system.

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REFERENCES

1. VLASE, S., *A method of eliminating Lagrangian multipliers from the equation of motion of interconnected mechanical systems*, ASME Transaction – Journal of Applied Mechanics, **54**, *1*, pp. 235–237, 1987.
2. MEIROVITCH, L., *Elements of vibration analysis* (2nd ed.), McGraww-Hill, New York, 1970.
3. VLASE, S., *Elimination of Lagrangian multipliers*, Mechanics Research Communications, **14**, *1*, pp. 17–22, 1987.
4. NICOARĂ, D., SECARĂ, E., MIHĂLCICĂ, M., *Study of rotor-bearing systems using Campbell diagram*, Proceedings of the 1st International Conference on Manufacturing Engineering, Quality and Production Systems (MEQAPS'09), Volume II, Braşov, Romania, 2009, pp. 393–396.
5. MANGERON, D., GOIA, I., VLASE, S., *Symmetrical branched systems vibration*, Memoirs of the Scientific Sections of the Romanian Academy, Series IV, **XII**, *1*, 1991.
6. VLASE, S., TEODORESCU-DRĂGHICESCU, H., SCUTARU, L.M., *On the eigenvalues of the elastic systems with some symmetries*, 6th International DAAAM Baltic Conference – Industrial Engineering, Tallinn, Estonia, 2008.
7. STAICU, Şt., *Dynamics analysis of a two-module hybrid parallel manipulator*, Ro. J. Techn. Sci. – Appl. Mechanics, **59**, *3*, pp. 293–303, 2014.
8. STAICU, Şt., OCNARESCU, C., UNGUREANU,L., *Dynamics modelling of a parallel flight simulator*, Ro. J. Techn. Sci. – Appl. Mechanics, **58**, *3*, pp. 273–285, 2013.
9. VLASE, S., TEODORESCU, P.P., ITU, C., SCUTARU, M.L., *Elasto-dynamics of a solid with a general "rigid" motion using FEM model. Part II. Analysis of a double cardan joint*, Romanian Journal of Physics, **58**, *7–8*, pp. 882–892, 2013.
10. VLASE, S., DĂNĂŞEL, C., SCUTARU, M.L., MIHĂLCICĂ, M., *Finite Element Analysis of a two-dimensional linear elastic systems with a plane "rigid motion"*, Romanian Journal of Physics, **59**, *5–6*, pp. 476–487, 2014.
11. SCUTARU, M.L., VLASE, S., *Some properties of motion equations describing the nonlinear dynamical response of a multibody system with flexible elements*, Journal of Applied Mathematics, doi:10.1155/2012/628503, 2012.
12. HORN, R.A., JOHNSON, Ch.R., *Matrix Analysis*, Cambridge University Press, 1985.
13. HARIS, C.M., CREDE, C.E., *Shock and vibration handbook* (2nd ed.), McGraw-Hill, 1976.