VALIDATION OF NEW RIGID BODY DYNAMICS FORMULATION USING ROTATION MATRICES ELEMENTS AS DEPENDENT PARAMETERS – DOUBLE PENDULUM CASE STUDY

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Abstract. The method of using the rotation matrix elements as dependent parameters for the 3D rotation of a rigid body is currently very rarely used in multibody dynamics. Nevertheless, this redundant parameterization of 3D rotations presents advantages, as well as disadvantages, with respect to classical independent parameters or less dependent parameters (e.g., quaternions). Thus, when using rotation matrix elements as dependent parameters, the dynamics of a rigid multibody system consists of solving an algebro-differential equations system comprising 12 scalar differential equations plus 6 algebraic scalar orthogonality equations per solid, plus the algebraic equations characterizing the articulations between the linked solids of the multibody system. The disadvantage of such an increased number of parameters/equations for our method is fully compensated by the fact that the dynamics equations can be written in a systematic way, being structurally similar for each solid. This paper validates our new rigid body dynamics formulation on the double pendulum case study, proposing a simplified version of the Lagrange multipliers elimination method. More precisely, a two-step elimination method is proposed to solve the algebraic part of the algebro-differential equations system.

Key words: rigid body dynamics, rotation matrix elements, orthogonality condition, Lagrange multipliers, algebro-differential system, double pendulum.

1. INTRODUCTION: "ART OF PARAMETERIZATION"

The choice of the parameters used to locate the position of the different rigid bodies is the first but crucial step in multibody dynamics. This "art of parameterization" has a major influence on solving the multibody dynamics equations. Obviously, there is no perfect choice of parameters, one has always to evaluate the pros and cons of the choice made, by taking into account several criteria: the number of parameters; the algebraic simplicity of the rotation matrix expression as

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a function of the chosen parameters; the degree of nonlinearity of the dynamic equations obtained by using the considered parameterization; the computational time for solving the dynamic equations; the absence of singularities in the rotation matrix representation; the geometrical interpretation of the chosen parameters.

The formulation using rotation matrices elements (full 3×3 matrices) as dependent parameters of a 3D rotations is used here. Considering 9 parameters for 3 rotational degrees of freedom means redundancy, as a consequence 9−3=6 constraint equations per solid are necessary. Lagrange multipliers are introduced in order to take into account these rigidity/orthogonality constraints. Our formulation introduces also other Lagrange multipliers corresponding to the constraints characterizing the joints between the different solids composing the multibody system.

It is usually less known, but Stuelpnagel [1] proved since 1964 that five is the minimum number of parameters necessary so that to a 3D rotation **R** corresponds a unique set of parameters, *i.e.*, the rotation group SO(3) is parameterized in such a way that a global description without singularities is obtained. Thus, the parameterizations using 3 independent parameters (Euler angles, Bryant angles, rotation vector, Rodrigues parameters etc.) or 4 dependent parameters (quaternions, linear parameters) present singularities in the representation of 3D rotations. Instead, our formulation using rotation matrices elements as dependent parameters of a 3D rotations presents no singularities. Other parameterizations without singularities are: natural coordinates [2–4], a set of two orthonormal base vectors (the third base vector being the cross product of the first two orthonormal base vectors), etc.

On the other hand, even if the number of equations is reduced when using 3 independent parameters, the symbolic complexity of the model equations can be extremely high [5], which leads to numerical schemes that are too slow for a real-time implementation. Using dependent coordinates, the rotational dynamics of multibody systems turns out to be represented by model equations having a low symbolic complexity, which simplifies and fastens the numerical resolution.

2. "ROTATIONLESS FORMULATION": REPRESENTATION OF 3D ROTATIONS BY THE 9 ROTATION MATRICES ELEMENTS. STATE OF THE ART

The choice of rotation matrices elements as 9 dependent parameters for a 3D rotation is not widely spread in mechanics, nevertheless several authors have preferred to use this highly redundant parameterization. For example in the early 1960^{s}, the practical problem of satellite attitude estimation conducted logically to the "TRIAD algorithm" [6], where the attitude is determined from two vector measurements (the third base vector could be the cross product of the first two base vectors). In fact, it had come naturally to use direction cosines of objects as observed in a satellite fixed frame of reference and direction cosines of the same objects in a known frame of reference. This parametization is called to be "without any knowledge of the attitude dynamics model", or in other words it is a "rotationless formulation".

This problem of attitude determination from a set of two or more vector measurements was immediately set up as an optimization problem called Wahba's problem or "problem 65–1" [7]. Farrell and Stuelpnagel *et al*. proposed a first solution to Wahba's problem, by computing "a least squares estimate of the rotation matrix which carries the known frame of reference into the satellite fixed frame of reference" [8]. It was already proved by the same Stuelpnagel [1] that at least two vector measurements (6 parameters, greater than the minimum number of parameters) are necessary to avoid any singularity in the rotation representation. More recently, Markley [9] proposed a different solution to Wahba's problem based on the singular value decomposition of a 3×3 matrix. Nowadays, Izadi and Sanyal et al. use the discrete Lagrange-d'Alembert principle, showing that "Wahba's cost function for attitude determination from vector measurements can be generalized and cast as a Morse function on the Lie group of rigid body rotations. A kinetic energy-like term, quadratic in the angular velocity estimation errors, can be used along with this artificial potential to construct a discrete Lagrangian dependent on state estimation errors" [10].

Other applications of the rotation matrix estimation problem are in stereophotogrammetry and in robotics [11]. In this context of using orthonormal matrices to represent rotations, Horn *et al*. present "a closed-form solution to the least-square problem of absolute orientation, one that does not require iteration" [11]. Their method "uses manipulation of 3×3 matrices and their eiganvalue-eigenvector decomposition, showing that the best scale is the ratio of the root-mean square deviations of the measurements from their respective centroids". Their closed-form solution may seem relatively complex, but can be easily computed with nowadays CPU performance and availability of algebraic packages.

More recently, a conservative time integration formulation for rigid bodies is developed by Krenk and Nielsen [12], based on a convected set of orthonormal base vectors. The base vectors are represented in terms of their absolute coordinates, idea which is similar to the natural coordinates concept [2–4]. "The equations of motion are obtained via Hamilton's equations including the zero-strain conditions as well as external constraints via Lagrange multipliers" [12].

Simo and Wong [13] are well-known authors using in mechanics of deformable media the same idea of 3D rotations representation by preserving full 3×3 matrices as parameters. They develop an unconditionally stable implicit algorithm "which exactly preserves energy and the total spatial angular momentum in incremental force-free motions". Their second order accurate algorithm is "directly applicable to transient dynamic calculations of geometrically exact rods and shells" [13]. Of course, for deformable bodies the orthogonality constraints imposed to rotation matrices will no more be entirely fulfilled.

In the same context of "rotationless formulation" of SO(3), i.e., the representation of 3D rotations by preserving all 9 elements of the rotation matrix as parameters, the work of Betsch, Steinmann, Uhlar *et al*. [14–18] is also well-known in multibody dynamics. They propose “energy-momentum consistent time-stepping schemes for finite-dimensional mechanical systems with holonomic constraints” [15, 18], obeying “major conservation laws of the underlying continuous system such as conservation of energy and angular momentum” [14]. Starting from the time discretization yielding to an “index three differential-algebraic equations (DAEs) corresponding to the constrained mechanical system” [15], “size-reductions are performed by eliminating the constraint forces, to lower the computational costs and improve the numerical conditioning” [17]. More precisely, the “discrete Lagrange multipliers are eliminated by using a discrete null space matrix” [15]. Numerous numerical examples validate their method: double spherical pendulum, gyro top, cylindrical and planar pairs and a six-body link, planar revolute pair with torsional spring, screw joint, free floating parallel manipulator, radio telescope, rotary crane, etc.

The same idea of eliminating explicitly the Lagrange multipliers associated with the internal zero-strain constraints is used also by Krenk and Nielsen, in order to reduce the size of the algebro-differential system to be solved. More precisely, the number of variables is reduced by six for each rigid body. “The Lagrange multipliers are eliminated by use of the set of orthogonality conditions between the generalized displacements and the momentum vector, leaving a set of differential equations without additional algebraic constraints on the base vectors” [12].

Using the same “rotationless formulation” of SO(3) special orthogonal Lie group, Gros *et al*. [19] obtain similar model equations in the form of index-3 DAEs of reduced nonlinearity. In the context of an optimal control problem, the most challenging in multibody dynamics, they propose a projection of the resulting Lagrange equations so as to reduce the number of states that need to be integrated by the Nonlinear Model Predictive Control. More precisely, Gros et al. are “using index-reduction techniques, where the constraints are differentiated with respect to time so as to obtain index-1 DAEs with associated consistency conditions” [19]. The re-increase of the symbolic complexity associated to this index-reduction technique is reasonable, thus the authors report worst case execution times less than 10 ms for their real-time optimal control of a tethered airplane [19].

At the University of Poitiers in France, Prof. Claude Vallée has guided several of his PhD students to work using this “rotationless formulation”, i.e., to represent a 3D rotation in multibody dynamics by preserving all 9 elements of the rotation matrix and by imposing the orthogonality constraints [20]. Firstly, Isnard and Vallée *et al*. [20–22] formulated the multibody dynamics equations when using full 3×3 matrices to parameterize 3D rotations. Lagrange multipliers are introduced in order to take into account the rigidity/orthogonality constraints, as well as other constraints characterizing the articulations between the linked solids of the multi-body system. To numerically solve the formulation, two methods are used: a shooting method applied at each incremental time step in order to fulfill the constraints, or the Lagrange multipliers elimination method, where explicit dynamic equations are generated, ready to be directly integrated and thus to obtain the time evolution of the dynamic system. Isnard *et al*. applied the proposed formulation to virtual reality realistic simulations [21, 22].

Dumitriu *et al*. [23–27] continued this work, trying to take more advantage from the algebraic simplicity of the "rotationless formulation" of the motion equations, by an efficient use of tensor calculus. Thus, for a rigid body rotating about its centre of mass, Vallée and Dumitriu proved that "the negative of the Lagrange multipliers matrix Λ associated to the rigidity/orthogonality constraint $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathrm{I}_3$ is a positive matrix and, at each instant of time, an orthonormalized basis exists in which new components of the matrix Λ are constant, which gives *six first integrals of the equations of motion*. It is proved that the three eigenvalues of the matrix Λ do not change with time and, moreover, they can be found in explicit form" [23, 24]. Atchonouglo [28–30] and Monnet [30–32] applied this redundant parameterization method to the identification of mechanical parameters, more precisely to the identification of kinematic and body segment inertial parameters in biomechanics [30–32]. In their case studies, the use of the matricial/tensor formulation using rotation matrices elements as rotational parameters, brought some advantages (numerical precision) with respect to traditional (bio)mechanical parameters identification methods.

Other authors using the "rotationless formulation" are Seguy [33] and Samin and Fisette [34], both focused on modular and symbolic modeling of multibody systems. In fact, the "rotationless formulation" is perfectly suited for object-oriented programming of multibody system dynamics, since the dynamic equations can be generated in a systematic/automatic/symbolic way. In this framework of tensor symbolic modeling, Samin and Fisette propose a variational approach based on Jourdain's principle of virtual powers [34], while Seguy [33] uses the "rotationless formulation" equations of motion both in Lagrange formulation and in Newton-Euler formulation, achieving an object-oriented programming applied to virtual reality purposes. Similarly, Ruf and Horaud [35] use a "projective approach to close the loop between articulated motion and stereo vision". Also in 3D vision, for camera calibration and for the modelization of moving scenes in the three dimensional Euclidean space, Ma and Soatto *et al*. [36] are other authors who preferred to represent the orientation of a moving frame relative to a fixed frame using the coordinates the three orthonormal vectors, then using the singular value decomposition and the least-square estimation and filtering as complementary tools.

In Romania, the matricial systematic modeling in multibody dynamics was developed mainly by Prof. Staicu. Using the virtual powers principle, he established the intrinsic recursive matricial expressions describing the dynamics of mutibody systems. This matricial formulation is validated on numerous serial or parallel robotic systems, performing direct as well as inverse dynamics, with the calculation of active forces and torques needed to realize the desired motion. Only a few papers of Staicu *et al*. are cited here [37–42].

3. ROTATIONAL DYNAMICS FORMULATION USING AS PARAMETERS THE 9 ELEMENTS OF THE ROTATION MATRIX

The multibody dynamics formulation preferred in this paper uses as 3D rotational parameters the 9 elements of the 3×3 rotation matrix $\mathbf{R}_i$, while for translations a pseudo-translation vector $\mathbf{T}_i^*$ is used instead of the classical translation vector $\mathbf{T}_i$ [20–32]:

$$\mathbf{R}_i = \begin{bmatrix} R_{i,11} & R_{i,12} & R_{i,13} \\ R_{i,21} & R_{i,22} & R_{i,23} \\ R_{i,31} & R_{i,32} & R_{i,33} \end{bmatrix} \text{ and } \mathbf{T}_i^* = \begin{bmatrix} T_{i,1}^* \\ T_{i,2}^* \\ T_{i,3}^* \end{bmatrix} = \overrightarrow{\mathrm{OG}_i} - \mathbf{R}_i \overrightarrow{\mathrm{OG}_{i0}} = \mathbf{T}_i - \mathbf{R}_i \overrightarrow{\mathrm{OG}_{i0}}. \quad (1)$$

Everything in this formulation is expressed with respect to an orthonormal inertial reference frame $(\mathrm{O}; \vec{x}_0, \vec{y}_0, \vec{z}_0)$, having its origin in O and $\vec{x}_0, \vec{y}_0, \vec{z}_0$ as axis. G_{i0} is the initial position of the center of mass G_i of the solid S_i.

Besides preserving all 9 elements of the 3×3 rotation matrix $\mathbf{R}_i$, the use of this pseudo-translation vector $\mathbf{T}_i^*$ given by (1) instead of the classical translation vector $\mathbf{T}_i$ represents another particularity of our formulation, which brings further simplicity in writing the dynamical equations.

This assignment of a total of 12 parameters concerns each solid S_i $(i=1,\ldots,N)$ of a multibody system composed of N solids. In order to preserve the rigidity of each solid S_i, the rotation matrix $\mathbf{R}_i$ has to be an orthogonal matrix:

$$\mathbf{R}_i^{\mathrm{T}} \mathbf{R}_i = \mathbf{I}_3. \quad (2)$$

Due to its symmetry, the orthogonality constraint (2) involves in fact only 6 scalar conditions to be fulfilled by the 9 elements of the 3×3 rotation matrix, leaving $3=9-6$ degrees of freedom for the rotational motion of solid S_i. Thus, the degree of redundancy of this parameterization of rotations is 6.

Let M_i be some point of the solid S_i and M_{i0} the initial position of this point. Then the position of point M_i at time t is given by:

$$\begin{aligned} \overrightarrow{\mathrm{OM}_i} &= \overrightarrow{\mathrm{OG}_i} + \overrightarrow{\mathrm{G}_i\mathrm{M}_i} = \overrightarrow{\mathrm{OG}_i} + \mathbf{R}_i \overrightarrow{\mathrm{G}_{i0}\mathrm{M}_{i0}} = \overrightarrow{\mathrm{OG}_i} - \mathbf{R}_i \overrightarrow{\mathrm{OG}_{i0}} + \mathbf{R}_i \overrightarrow{\mathrm{OM}_{i0}} \\ &= (\mathbf{T}_i - \mathbf{R}_i \overrightarrow{\mathrm{OG}_{i0}}) + \mathbf{R}_i \overrightarrow{\mathrm{OM}_{i0}} \overset{\text{definition (1)}}{=} \mathbf{T}_i^* + \mathbf{R}_i \overrightarrow{\mathrm{OM}_{i0}}. \end{aligned} \quad (3)$$

As shown by definition (3), the proposed method consists in characterizing the motion of each rigid body of an articulated system by using a pseudo-translation vector $\mathbf{T}_i^*$ and a rotation matrix $\mathbf{R}_i$, which fully describe the position at time t of S_i. At the initial time $t=0$, it follows from (3) that:

$$\mathbf{R}_i(0) = \mathbf{I}_3 \quad \text{and} \quad \mathbf{T}_i^*(0) = 0 . \tag{4}$$

Deriving (3) with respect to time, one obtains the velocity of point M_i in the inertial reference frame $(\mathrm{O}; \vec{x}_0, \vec{y}_0, \vec{z}_0)$:

$$\mathbf{V}(\mathrm{M}_i) = \dot{\mathbf{T}}_i^* + \dot{\mathbf{R}}_i \overrightarrow{\mathrm{OM}_{i0}} . \tag{5}$$

Let $\mathbf{\Omega}_i$ denote the instantaneous rotation vector of solid S_i at time t, expressed in the orthonormal basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$, it is given by:

$$\mathrm{j}(\mathbf{\Omega}_i) = \dot{\mathbf{R}}_i \mathbf{R}_i^{\mathrm{T}} , \tag{6}$$

where $\dot{\mathbf{R}}_i = \dfrac{\mathrm{d}\mathbf{R}_i}{\mathrm{d}t}$ and $\mathrm{j}(\bullet)$ is the skew-symmetric cross-product matrix defined by $\mathrm{j}(u)v = u \wedge v$, for $\forall u, v$ and with $\wedge$ denoting the classical cross-product between two 3×1 vectors. From (6), the derivative $\dot{\mathbf{R}}_i$ with respect to time of $\mathbf{R}_i$ can be expressed as:

$$\dot{\mathbf{R}}_i = \mathrm{j}(\mathbf{\Omega}_i)\mathbf{R}_i . \tag{7}$$

At $t_0 = 0$, the initial conditions for the derivatives $\dot{\mathbf{T}}_i^*$ and $\dot{\mathbf{R}}_i$ are deduced considering (7), (4) and then (5):

$$\begin{cases} \dot{\mathbf{R}}_i(0) = \mathrm{j}(\mathbf{\Omega}_{i0}) \\ \dot{\mathbf{T}}_i^*(0) = \mathbf{V}_0(\mathrm{M}_i) - \mathrm{j}(\mathbf{\Omega}_{i0})\overrightarrow{\mathrm{OM}_{i0}} \end{cases} , \tag{8}$$

where $\mathbf{\Omega}_{i0}$ is the instantaneous rotation vector of the solid S_i at the initial time, being expressed in the orthonormal basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$. The point M_{i0} is the initial position of point M_i of the solid S_i, with $\mathbf{V}_0(\mathrm{M}_i)$ its initial velocity vector. Of course, for the case where $\mathbf{V}_0(\mathrm{M}_i)$ and $\mathbf{\Omega}_{i0}$ are null, the initial conditions in terms of derivatives become: $\begin{cases} \dot{\mathbf{R}}_i(0) = 0 \\ \dot{\mathbf{T}}_i^*(0) = 0 \end{cases}$.

In Lagrangian formulation, the dynamics of the multibody system composed by N solids S_i is described by the following $12\times N$ Lagrange equations [20–23,25–27]:

$$\begin{cases} m_i \ddot{\mathbf{T}}_i^* + m_i \ddot{\mathbf{R}}_i \overrightarrow{\mathrm{OG}_{i0}} = \mathbf{X}_i \\ \ddot{\mathbf{R}}_i \mathbf{K}_{i0} + m_i \ddot{\mathbf{T}}_i^* \otimes \overrightarrow{\mathrm{OG}_{i0}} = \mathbf{Y}_i + \mathbf{R}_i \mathbf{\Lambda}_i \end{cases} , \quad \text{for } i = 1, \ldots, N \tag{9}$$

where $\otimes$ denotes the tensor product, m_i is the mass of the solid S_i, G_{i0} is the initial position of the center of mass of the solid S_i. $\mathbf{K}_{i0}$ is the Poinsot inertia matrix of the solid S_i calculated at the initial time in the point O, with respect to the orthonormal basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$, defined by:

$$\mathbf{K}_{i0} = \int_{S_i} \overrightarrow{\mathrm{OM}_{i0}} \otimes \overrightarrow{\mathrm{OM}_{i0}} \, \mathrm{d}m_i \,. \tag{10}$$

The relationship between this Poinsot inertia matrix and the corresponding classical inertia matrix $\mathbf{J}_{i0} = \int_{S_i} -[\mathrm{j}(\overrightarrow{\mathrm{OM}_{i0}})]^2 \mathrm{d}m_i$ is as follows [21–27]:

$$\mathbf{K}_{i0} = \frac{\mathrm{tr}(\mathbf{J}_{i0})}{2}\mathbf{I}_3 - \mathbf{J}_{i0}, \tag{11}$$

where tr(•) is the matrix trace operator.

Vector $\mathbf{X}_i$ and matrix $\mathbf{Y}_i$ are the generalized efforts, composed of the external efforts acting on the solid S_i and of the internal efforts dues to the joints between S_i and other solids or between S_i and the ground (such as the internal efforts (14) in the case of a spherical joint). $\mathbf{\Lambda}_i$ is the symmetric Lagrange multipliers 3×3 matrix, introduced in order to take into account the orthogonality condition (2) of rotation matrix $\mathbf{R}_i$.

For numerical integration purposes, the ordinary differential equations (9) can be written in the following explicit form of $\ddot{\mathbf{T}}_i^*$ and $\ddot{\mathbf{R}}_i$:

$$\begin{cases} \ddot{\mathbf{T}}_i^* = \dfrac{1}{m_i}\mathbf{X}_i - [(\mathbf{Y}_i + \mathbf{R}_i\mathbf{\Lambda}_i)\mathbf{K}_{\mathrm{G}i0}^{-1}]\overrightarrow{\mathrm{OG}_{i0}} + \langle \mathbf{K}_{\mathrm{G}i0}^{-1}\overrightarrow{\mathrm{OG}_{i0}}, \overrightarrow{\mathrm{OG}_{i0}} \rangle \mathbf{X}_i \\ \ddot{\mathbf{R}}_i = (\mathbf{Y}_i + \mathbf{R}_i\mathbf{\Lambda}_i - \mathbf{X}_i \otimes \overrightarrow{\mathrm{OG}_{i0}})\mathbf{K}_{\mathrm{G}i0}^{-1}, \qquad \text{for } i = 1,\ldots,N \end{cases} \tag{12}$$

where $\mathbf{K}_{\mathrm{G}i0} = \frac{\mathrm{tr}(\mathbf{J}_{\mathrm{G}i0})}{2}\mathbf{I}_3 - \mathbf{J}_{\mathrm{G}i0}$ is the Poinsot inertia matrix of the solid S_i calculated at the initial time in its center of mass G_i, with respect to the orthonormal basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$. This is only the differential part of the algebro-differential equations governing the motion of the multibody system. The algebraic part is composed by all orthogonality constraints (2) imposing the rigidity of solids S_i, plus the constraint equations characterizing the mechanical joints between solids and with the ground. For example, for a spherical joint between two solids S_i and S_j, the following constraint equation (13) has to be fulfilled imposing that the center A_{ij} of the spherical joint belongs to both solids:

$$\mathbf{T}_i^* + \mathbf{R}_i \overrightarrow{\mathrm{OA}_{ij,0}} = \mathbf{T}_j^* + \mathbf{R}_j \overrightarrow{\mathrm{OA}_{ij,0}} \,. \tag{13}$$

Denoting by $\mathbf{\Lambda}_{ij}^{\text{sph}}$ the Lagrange multipliers tridimensional vector associated to the constraint equation (13) characterizing the spherical joint, then the efforts $\mathbf{X}_{i\leftarrow j}^{\text{sph}}$ and $\mathbf{Y}_{i\leftarrow j}^{\text{sph}}$ representing the actions of the solid S_j on the solid S_i, as well as the reactions $\mathbf{X}_{j\leftarrow i}^{\text{sph}}$ and $\mathbf{Y}_{j\leftarrow i}^{\text{sph}}$ of S_i on S_j are given by [21–23,25–27]:

$$\begin{cases} \mathbf{X}_{i\leftarrow j}^{\text{sph}} = \mathbf{\Lambda}_{ij}^{\text{sph}} \\ \mathbf{Y}_{i\leftarrow j}^{\text{sph}} = \mathbf{\Lambda}_{ij}^{\text{sph}} \otimes \overrightarrow{\text{OA}}_{ij,0} \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{X}_{j\leftarrow i}^{\text{sph}} = -\mathbf{\Lambda}_{ij}^{\text{sph}} \\ \mathbf{Y}_{j\leftarrow i}^{\text{sph}} = -\mathbf{\Lambda}_{ij}^{\text{sph}} \otimes \overrightarrow{\text{OA}}_{ij,0} \end{cases}. \tag{14}$$

Thus, the algebro-differential dynamics system is composed by a differential part (12), having the same form for each solid of the multibody system, plus the algebraic constraints formed by the orthogonality constraints (2) and the constraint equations characterizing the mechanical joints (representing the links between solids and thus between the parameters $\mathbf{T}_i^*$, $\mathbf{R}_i$, $\dot{\mathbf{T}}_i^*$ and $\dot{\mathbf{R}}_i$ of each solid). This algebro-differential system is numerically integrated in time starting from initial conditions (4) and (8).

In other words, one can say that dynamic equations can be generated in a systematic/automatic way: in what concerns the dynamic part, the equations of motion in explicit form (12) are written separately for each solid. They are coupled/linked only by means of the algebraic constraint equations characterizing the joints/links of the multibody system. The differential equations (12) are linear in $\ddot{\mathbf{T}}_i^*$ and $\ddot{\mathbf{R}}_i$, this low complexity is a considerable advantage from the point of view of numerical integration. The differential nonlinearity is thus “eliminated” from the differential equations of motion for the formulation using as rotational parameters the 9 elements of the 3×3 rotation matrix. The price to pay is an increased number of algebro-differential equations, for which the computational time might be increased. But this problem is solved with nowadays real-time computational speed and using size-reductions techniques performed by eliminating the Lagrange multipliers [15, 17, 19, 21, 23, 25–27]. Another advantage is that the proposed redundant parameterization approach avoids any singularity in the representation of rotations.

As observed in the above discussion, the dynamic equations formulation will contain differential equations more or less linear depending on the choice of the 3D rotations parameterization technique. The more one approaches a linear differential dynamic equations system to be solved, the more will be facilitated the numerical resolution of this system (as a tradeoff, an algebraic part must also be dealt with).

4. SIMPLE CASE STUDY FOR VALIDATION: THE DOUBLE PENDULUM

The proposed formulation using as rotational parameters the 9 elements of the 3×3 rotation matrix is applied here on a simple example: the plane double pendulum. This classical example is aimed to validate the proposed method, our further goal being to solve the direct dynamics of a Stewart platform.

The plane double pendulum in Figure 1 is formed by two identical homogenous bars S_1 and S_2, each of mass m, length l and rectangular transversal section with dimensions a and b (with b the dimension in $\vec{z}_0$ direction). The applied external forces are the weights of the two bars.

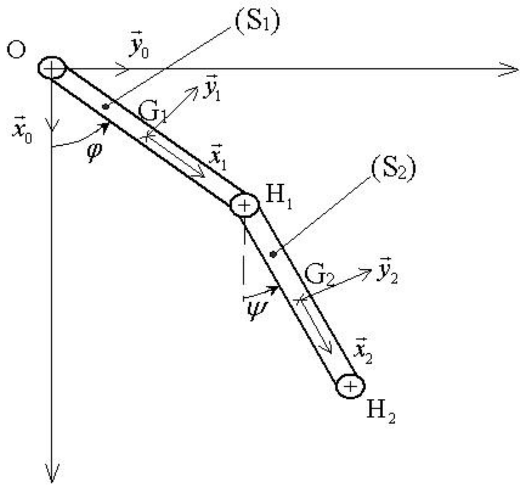

Fig. 1 – Double pendulum case study [23, 25–27].

The orthonormal inertial reference frame $(\mathrm{O}; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ is chosen with $\vec{y}_0$ horizontal axis and $\vec{x}_0$ as vertical unit vector orientated downwards. Due to the initial conditions and to the fact that all external forces act in the vertical plane, the motion happens only in the vertical plane of $\vec{x}_0$ and $\vec{y}_0$. Let us attach to solid S_1 the right-hand orthonormal reference frame $(\mathrm{G}_1; \vec{x}_1, \vec{y}_1, \vec{z}_0)$, with $\vec{x}_1$ following the axis of symmetry of the bar. Similarly, to solid S_2 is attached the right-hand orthonormal reference frame $(\mathrm{G}_2; \vec{x}_2, \vec{y}_2, \vec{z}_0)$. The bars being homogenous, the two centers of mass G_1 and G_2 are located in the middle of the bars. The bar S_1 is linked in point O to a fixed support, by means of a spherical joint without friction, while S_2 is linked to S_1 in point H_1 by means of another spherical joint without friction. In fact, since the motion happens only in the vertical plane, one could have considered cylindrical/revolute joints instead of spherical joints. But our intention is to use spherical joints and to verify that the motion remains in the vertical plane, as a validation of our general formulation.

The motion of S_1 is fully characterized by the evolution in time of its pseudo-translation vector $\mathbf{T}_1^*$ and its rotation matrix $\mathbf{R}_1$, while the motion of S_2 is fully characterized by its pseudo-translation vector $\mathbf{T}_2^*$ and its rotation matrix $\mathbf{R}_2$.

Let us denote by φ the angle between $\vec{x}_1$ and $\vec{x}_0$ and by ψ the angle between $\vec{x}_2$ and $\vec{x}_0$, with initial values $\varphi(0)=\varphi_0$ and $\psi(0)=\psi_0$. The expressions $\mathbf{K}_{G1,0}$ and $\mathbf{K}_{G2,0}$ of the Poinsot inertia matrices of the bars S_1 and S_2 at the initial time are [21, 23, 25–27]:

$$\mathbf{K}_{G_1,0}=\frac{m}{12}\begin{bmatrix} l^2\cos^2\varphi_0+a^2\sin^2\varphi_0 & (l^2-a^2)\sin\varphi_0\cos\varphi_0 & 0\\ (l^2-a^2)\sin\varphi_0\cos\varphi_0 & l^2\sin^2\varphi_0+a^2\cos^2\varphi_0 & 0\\ 0 & 0 & b^2 \end{bmatrix},$$

$$\mathbf{K}_{G_2,0}=\frac{m}{12}\begin{bmatrix} l^2\cos^2\psi_0+a^2\sin^2\psi_0 & (l^2-a^2)\sin\psi_0\cos\psi_0 & 0\\ (l^2-a^2)\sin\psi_0\cos\psi_0 & l^2\sin^2\psi_0+a^2\cos^2\psi_0 & 0\\ 0 & 0 & b^2 \end{bmatrix}. \tag{15}$$

The planar double pendulum is moving in the vertical plan defined by $\vec{x}_0$ et $\vec{y}_0$, let us consider the case of null initial velocities. From (4) and (8), it comes:

$$\begin{cases} \mathbf{T}_1^*(0)=0\ ,\ \mathbf{R}_1(0)=\mathbf{I}_3\ ,\ \dot{\mathbf{T}}_1^*(0)=0\ ,\ \dot{\mathbf{R}}_1(0)=0\ ;\\ \mathbf{T}_2^*(0)=0\ ,\ \mathbf{R}_2(0)=\mathbf{I}_3\ ,\ \dot{\mathbf{T}}_2^*(0)=0\ ,\ \dot{\mathbf{R}}_2(0)=0\ . \end{cases} \tag{16}$$

For the double pendulum, the differential part (12) of the equations of motion for bar S_1 are absolutely similar with the ones for bar S_2:

$$\begin{aligned} &\text{for } S_1 \begin{cases} \ddot{\mathbf{T}}_1^* = \dfrac{1}{m}\mathbf{X}_1 - [(\mathbf{Y}_1+\mathbf{R}_1\mathbf{\Lambda}_1)\mathbf{K}_{G_1,0}^{-1}]\overrightarrow{\mathrm{OG}_{1,0}} + \langle \mathbf{K}_{G_1,0}^{-1}\overrightarrow{\mathrm{OG}_{1,0}}\,,\,\overrightarrow{\mathrm{OG}_{1,0}}\rangle \mathbf{X}_1\\ \ddot{\mathbf{R}}_1 = (\mathbf{Y}_1+\mathbf{R}_1\mathbf{\Lambda}_1-\mathbf{X}_1\otimes\overrightarrow{\mathrm{OG}_{1,0}})\mathbf{K}_{G_1,0}^{-1} \end{cases}\\ &\text{for } S_2 \begin{cases} \ddot{\mathbf{T}}_2^* = \dfrac{1}{m}\mathbf{X}_2 - [(\mathbf{Y}_2+\mathbf{R}_2\mathbf{\Lambda}_2)\mathbf{K}_{G_2,0}^{-1}]\overrightarrow{\mathrm{OG}_{2,0}} + \langle \mathbf{K}_{G_2,0}^{-1}\overrightarrow{\mathrm{OG}_{2,0}}\,,\,\overrightarrow{\mathrm{OG}_{2,0}}\rangle \mathbf{X}_2\\ \ddot{\mathbf{R}}_2 = (\mathbf{Y}_2+\mathbf{R}_2\mathbf{\Lambda}_2-\mathbf{X}_2\otimes\overrightarrow{\mathrm{OG}_{2,0}})\mathbf{K}_{G_2,0}^{-1}. \end{cases} \end{aligned} \tag{17}$$

As supposed by our formulation, all measures in equations (18) are expressed with respect to the orthonormal inertial reference frame $(\mathrm{O}; \vec{x}_0, \vec{y}_0, \vec{z}_0)$. Thus:

$$\overrightarrow{\mathrm{OG}_{1,0}} = \frac{l}{2}\cos\varphi_0\,\vec{x}_0 + \frac{l}{2}\sin\varphi_0\,\vec{y}_0\,,\quad \overrightarrow{\mathrm{OH}_{1,0}} = l\cos\varphi_0\vec{x}_0 + l\sin\varphi_0\,\vec{y}_0\,,$$

$$\overrightarrow{\mathrm{OG}_{2,0}} = (l\cos\varphi_0 + \frac{l}{2}\cos\psi_0)\vec{x}_0 + (l\sin\varphi_0 + \frac{l}{2}\sin\psi_0)\vec{y}_0\,. \tag{18}$$

Still in equations (17), $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are the 3×3 symmetrical Lagrange multipliers matrices, associated to the orthogonality conditions (2):

$$\begin{cases} \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_1 = \mathbf{I}_3 \\ \mathbf{R}_2^{\mathrm{T}}\mathbf{R}_2 = \mathbf{I}_3 \end{cases}. \tag{19}$$

Other Lagrange multipliers are gathered in vectors $\mathbf{\Lambda}_{01}^{\mathrm{sph}}$ and $\mathbf{\Lambda}_{12}^{\mathrm{sph}}$, being associated to the constraint equations of type (13), characterizing the two spherical joints O and H_1 of the double pendulum:

$$\begin{cases} 0 = \mathbf{T}_1^* + \mathbf{R}_1\,\overrightarrow{\mathrm{OO}} \\ \mathbf{T}_1^* + \mathbf{R}_1\,\overrightarrow{\mathrm{OH}_{1,0}} = \mathbf{T}_2^* + \mathbf{R}_2\,\overrightarrow{\mathrm{OH}_{1,0}} \end{cases} \Rightarrow \begin{cases} \mathbf{T}_1^* = 0 \\ \mathbf{T}_2^* + (\mathbf{R}_2 - \mathbf{R}_1)\overrightarrow{\mathrm{OH}_{1,0}} = 0 \end{cases}. \tag{20}$$

Still in equations (17), the efforts $\mathbf{X}_1$, $\mathbf{Y}_1$ and $\mathbf{X}_2$, $\mathbf{Y}_2$ represent the sum of the external efforts dues to the weights of S_1 and S_2, plus the internal efforts of type (14) dues to the two spherical joints:

$$\begin{cases} \mathbf{X}_1 = \mathbf{X}_1^{\mathrm{ext}} + \mathbf{X}_{1\leftarrow 0}^{\mathrm{sph}} + \mathbf{X}_{1\leftarrow 2}^{\mathrm{sph}} = mg\,\vec{x}_0 - \mathbf{\Lambda}_{01}^{\mathrm{sph}} + \mathbf{\Lambda}_{12}^{\mathrm{sph}} \\ \mathbf{Y}_1 = \mathbf{Y}_1^{\mathrm{ext}} + \mathbf{Y}_{1\leftarrow 0}^{\mathrm{sph}} + \mathbf{Y}_{1\leftarrow 2}^{\mathrm{sph}} = mg\,\vec{x}_0 \otimes \overrightarrow{\mathrm{OG}_{1,0}} + \mathbf{\Lambda}_{12}^{\mathrm{sph}} \otimes \overrightarrow{\mathrm{OH}_{1,0}} \\ \mathbf{X}_2 = \mathbf{X}_2^{\mathrm{ext}} + \mathbf{X}_{2\leftarrow 1}^{\mathrm{sph}} = mg\,\vec{x}_0 - \mathbf{\Lambda}_{12}^{\mathrm{sph}} \\ \mathbf{Y}_2 = \mathbf{Y}_2^{\mathrm{ext}} + \mathbf{Y}_{2\leftarrow 1}^{\mathrm{sph}} = mg\,\vec{x}_0 \otimes \overrightarrow{\mathrm{OG}_{2,0}} - \mathbf{\Lambda}_{12}^{\mathrm{sph}} \otimes \overrightarrow{\mathrm{OH}_{1,0}} \end{cases}. \tag{21}$$

By replacing the expressions (21) of the generalized efforts in the Lagrange equations of motion (17), one obtains the following differential part of the dynamics algebro-differential equations system:

$$\text{for } S_1 \begin{cases} \ddot{\mathbf{T}}_1^* = g\vec{x}_0 - \left(\dfrac{1}{m} + \langle \mathbf{K}_{\mathrm{G}_1,0}^{-1} \overrightarrow{\mathrm{OG}_{1,0}}, \overrightarrow{\mathrm{OG}_{1,0}} \rangle \right) \mathbf{\Lambda}_{01}^{\mathrm{sph}} + \\ \qquad + \left(\dfrac{1}{m} - \langle \mathbf{K}_{\mathrm{G}_1,0}^{-1} \overrightarrow{\mathrm{OG}_{1,0}}, \overrightarrow{\mathrm{G}_{1,0}\mathrm{H}_{1,0}} \rangle \right) \mathbf{\Lambda}_{12}^{\mathrm{sph}} - \mathbf{R}_1 \mathbf{\Lambda}_1 \mathbf{K}_{\mathrm{G}_1,0}^{-1} \overrightarrow{\mathrm{OG}_{1,0}} \\ \ddot{\mathbf{R}}_1 = \left(\mathbf{\Lambda}_{01}^{\mathrm{sph}} \otimes \overrightarrow{\mathrm{OG}_{1,0}} + \mathbf{\Lambda}_{12}^{\mathrm{sph}} \otimes \overrightarrow{\mathrm{G}_{1,0}\mathrm{H}_{1,0}} + \mathbf{R}_1 \mathbf{\Lambda}_1 \right) \mathbf{K}_{\mathrm{G}_1,0}^{-1} \end{cases} . \quad (22)$$

$$\text{for } S_2 \begin{cases} \ddot{\mathbf{T}}_2^* = g\vec{x}_0 - \left(\dfrac{1}{m} + \langle \mathbf{K}_{\mathrm{G}_2,0}^{-1} \overrightarrow{\mathrm{OG}_{2,0}}, \overrightarrow{\mathrm{H}_{1,0}\mathrm{G}_{2,0}} \rangle \right) \mathbf{\Lambda}_{12}^{\mathrm{sph}} - \mathbf{R}_2 \mathbf{\Lambda}_2 \mathbf{K}_{\mathrm{G}_2,0}^{-1} \overrightarrow{\mathrm{OG}_{2,0}} \\ \ddot{\mathbf{R}}_2 = \left(\mathbf{\Lambda}_{12}^{\mathrm{sph}} \otimes \overrightarrow{\mathrm{H}_{1,0}\mathrm{G}_{2,0}} + \mathbf{R}_2 \mathbf{\Lambda}_2 \right) \mathbf{K}_{\mathrm{G}_2,0}^{-1} \end{cases}$$

So, the algebro-differential system describing the dynamics of the double pendulum is composed by 2×(3+9)=24 scalar second order differential equations (22) and by 2×6+2×3=18 scalar algebraic constraints (19) and (20). It must be numerically integrated in time starting from initial conditions (16). Due to the symmetry, each orthogonality condition in (19) counts only for 6 instead of 9 scalar constraint equations. The unknows of the problem are: $\mathbf{T}_1^*$, $\mathbf{R}_1$, $\mathbf{T}_2^*$, $\mathbf{R}_2$ and their derivatives, as well as the Lagrange multipliers introduced by our formulation (symmetric matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ and vectors $\mathbf{\Lambda}_{01}^{\mathrm{sph}}$ and $\mathbf{\Lambda}_{12}^{\mathrm{sph}}$). The size of this algebro-differential system can be reduced using a Lagrange multipliers elimination method in two steps, presented in next section.

5. TWO-STEP METHOD OF LAGRANGE MULTIPLIERS ELIMINATION

In the context of a quite important size of the differential-algebraic equations system corresponding to the dynamics formulation using as rotational parameters the 9 elements of the 3×3 rotation matrix (very redundant parameterization), several authors have proposed size-reductions techniques performed by eliminating the Lagrange multipliers [15, 17, 19]. Our previous work [21, 23, 25–27] performed also Lagrange multipliers eliminations, the method consisting in replacing the expressions of $\ddot{\mathbf{T}}_1^*, \ddot{\mathbf{R}}_1, \ddot{\mathbf{T}}_2^*$ and $\ddot{\mathbf{R}}_2$ given by (22) into the following differentiated forms obtained by differentiating twice with respect to time the original orthogonality constraints (19):

$$\begin{cases} \ddot{\mathbf{R}}_1^{\mathrm{T}} \mathbf{R}_1 + \mathbf{R}_1^{\mathrm{T}} \ddot{\mathbf{R}}_1 + 2\dot{\mathbf{R}}_1^{\mathrm{T}} \dot{\mathbf{R}}_1 = 0 \\ \ddot{\mathbf{R}}_2^{\mathrm{T}} \mathbf{R}_2 + \mathbf{R}_2^{\mathrm{T}} \ddot{\mathbf{R}}_2 + 2\dot{\mathbf{R}}_2^{\mathrm{T}} \dot{\mathbf{R}}_2 = 0 \end{cases} \quad (23)$$

as well as by differentiating twice with respect to time the constraint equations (20) characterizing the two spherical joints O and H_1 of the double pendulum:

$$\begin{cases} \ddot{\mathbf{T}}_1^* = 0 \\ \ddot{\mathbf{T}}_2^* + (\ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1)\overrightarrow{OH_{1,0}} = 0 \end{cases}. \tag{24}$$

As already proved in previous work [21, 23], it is possible to use the differential forms (23) and (24) instead of the original constraint equations (19) and (20), with no major negative influence on the accuracy of the numerical integration of the algebro-differential system governing the motion of a multibody system when using the formulation described in this paper.

Thus, by replacing expressions of $\ddot{\mathbf{T}}_1^*$, $\ddot{\mathbf{R}}_1$, $\ddot{\mathbf{T}}_2^*$ and $\ddot{\mathbf{R}}_2$ given by (22) into (23) and (24), one obtains 2×6+2×3=18 scalar linear equations having as unknowns the elements of $\mathbf{\Lambda}_1$ (only 6 of them, the matrix being symmetric), the elements of $\mathbf{\Lambda}_2$ and the components of vectors $\mathbf{\Lambda}_{01}^{\mathrm{sph}}$ and $\mathbf{\Lambda}_{12}^{\mathrm{sph}}$ (3 components per vector). This linear equations system with 18 scalar equations and 18 unknowns can be solved by classical methods (using numerical methods or a symbolic package such as Mathematica), the only possible problem being the size of this linear equations system. Obviously, this size would increase for multibody systems composed of several solids, such as a Stewart platform.

In order to reduce the size of the linear equations system to be solved for eliminating the Lagrange multipliers, a two-step method is proposed in this paper. More precisely, in a first step of the elimination method $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ can be separately eliminated, leaving to be solved in the second step a simpler linear equations system formed only by 18−12=6 equations, with 6 unknowns ($\mathbf{\Lambda}_{01}^{\mathrm{sph}}$ and $\mathbf{\Lambda}_{12}^{\mathrm{sph}}$).

The first step of the Lagrange multipliers elimination consists in replacing into (23) the expressions $\ddot{\mathbf{R}}_1$ and $\ddot{\mathbf{R}}_2$ given by (22). Taking into account the symmetry of matrices $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$, $\mathbf{K}_{G_1,0}^{-1}$ and $\mathbf{K}_{G_2,0}^{-1}$ and some usual properties of tensor calculus, one obtains:

$$\begin{cases} \mathbf{\Lambda}_1\mathbf{K}_{G_1,0}^{-1} + \mathbf{K}_{G_1,0}^{-1}\mathbf{\Lambda}_1 = -2\dot{\mathbf{R}}_1^{\mathrm{T}}\dot{\mathbf{R}}_1 - \mathbf{K}_{G_1,0}^{-1}\left(\overrightarrow{OG_{1,0}}\otimes\mathbf{\Lambda}_{01}^{\mathrm{sph}} + \overrightarrow{G_{1,0}H_{1,0}}\otimes\mathbf{\Lambda}_{12}^{\mathrm{sph}}\right)\mathbf{R}_1 - \\ \qquad - \mathbf{R}_1^{\mathrm{T}}\left(\mathbf{\Lambda}_{01}^{\mathrm{sph}}\otimes\overrightarrow{OG_{1,0}} + \mathbf{\Lambda}_{12}^{\mathrm{sph}}\otimes\overrightarrow{G_{1,0}H_{1,0}}\right)\mathbf{K}_{G_1,0}^{-1} \\ \mathbf{\Lambda}_2\mathbf{K}_{G_2,0}^{-1} + \mathbf{K}_{G_2,0}^{-1}\mathbf{\Lambda}_2 = -2\dot{\mathbf{R}}_2^{\mathrm{T}}\dot{\mathbf{R}}_2 - \mathbf{K}_{G_2,0}^{-1}\left(\overrightarrow{H_{1,0}G_{2,0}}\otimes\mathbf{\Lambda}_{12}^{\mathrm{sph}}\right)\mathbf{R}_2 - \\ \qquad - \mathbf{R}_2^{\mathrm{T}}\left(\mathbf{\Lambda}_{12}^{\mathrm{sph}}\otimes\overrightarrow{H_{1,0}G_{2,0}}\right)\mathbf{K}_{G_2,0}^{-1} \end{cases}. \tag{25}$$

The unknowns being $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, the algebraic equations (25) are decoupled, each of them being in fact a discrete Lyapunov-like equation of type $\mathbf{AX}+\mathbf{XA}=\mathbf{B}$. Analytical and numerical solutions are available for these Lyapunov-like equations, providing without difficulty symbolic analytical or numerical solutions $\mathbf{\Lambda}_1=\mathbf{\Lambda}_1(\mathbf{\Lambda}_{01}^{\mathrm{sph}},\mathbf{\Lambda}_{12}^{\mathrm{sph}},\mathbf{R}_1,\dot{\mathbf{R}}_1)$ and $\mathbf{\Lambda}_2=\mathbf{\Lambda}_2(\mathbf{\Lambda}_{12}^{\mathrm{sph}},\mathbf{R}_2,\dot{\mathbf{R}}_2)$. At this point of the two-step method (after the first step), $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are now computed/eliminated.

In the second step of the Lagrange multipliers elimination, after replacing the computed $\mathbf{\Lambda}_1=\mathbf{\Lambda}_1(\mathbf{\Lambda}_{01}^{\mathrm{sph}},\mathbf{\Lambda}_{12}^{\mathrm{sph}},\mathbf{R}_1,\dot{\mathbf{R}}_1)$ and $\mathbf{\Lambda}_2=\mathbf{\Lambda}_2(\mathbf{\Lambda}_{12}^{\mathrm{sph}},\mathbf{R}_2,\dot{\mathbf{R}}_2)$ into (24), the remaining algebraic part consists only in 6 equations (24) which are linear with respect to the 6 unknowns $\mathbf{\Lambda}_{01}^{\mathrm{sph}}$ and $\mathbf{\Lambda}_{12}^{\mathrm{sph}}$. The size of the linear equations system is thus reduced compared with the initial size, thus the numerical solution is even more trivial. Due to its matricial formulation, this remained linear equations system can also be solved using a symbolic package such as Mathematica.

In what concerns the integration over each timestep of the differential part (22) of the algebro-differential system, the Lagrange multipliers are considered as piecewise constant during each timestep. Previous work proved that this simplification has no major influence on the accuracy of the numerical solution [21, 23].

The two-step method of Lagrange multipliers elimination proposed here works properly on the case study presented in next section, but is aimed to show all its improvements for multibody systems composed of several solids (such as a Stewart platform), where the size of the algebro-differential system becomes much more important in the context of our very redundant paramaterization.

6. VALIDATION RESULTS

The case study has already been described in §4. The simulation results below concern the planar double pendulum from Figure 1, described by the following inertial and geometrical characteristics of the two bars S_1 and S_2:

- mass of each bar $m=0{,}108\ \mathrm{kg}$;
- dimensions of each bar: $l=20\ \mathrm{cm}$, $a=2\ \mathrm{cm}$, $b=1\ \mathrm{cm}$ (with b the dimension in $\vec{z}_0$ direction).

The only external forces applied are the weights of the two bars. The planar double pendulum motion starts from the following initial values of angles φ and ψ:

$$\varphi_0=\frac{\pi}{9}\ \mathrm{rad}=20^\circ \quad \text{and} \quad \psi_0=\frac{\pi}{18}\ \mathrm{rad}=10^\circ,$$

with null initial velocities:

$$\dot{\varphi}_0 = 0 \quad \text{and} \quad \dot{\psi}_0 = 0 .$$

The planar double pendulum was considered for validation case study, since an alternate numerical solution is easily available as reference for comparing our results. This reference is obtained by solving the following classical Lagrange formulation based on the non-redundant parameterization of planar rotations by the angles φ_{ref} and ψ_{ref}:

$$\begin{cases} \left(C+\frac{5}{4}m\,l^2\right)\ddot{\varphi} + \frac{m\,l^2}{2}\left[\cos(\varphi-\psi)\,\ddot{\psi} + \sin(\varphi-\psi)\,\dot{\psi}^2\right] + \frac{3}{2}mg\,l\sin\varphi = 0 \\ \left(C+\frac{1}{4}m\,l^2\right)\ddot{\psi} + \frac{m\,l^2}{2}\left[\cos(\varphi-\psi)\,\ddot{\varphi} - \sin(\varphi-\psi)\,\dot{\varphi}^2\right] + \frac{1}{2}mg\,l\sin\psi = 0, \end{cases} \tag{26}$$

where $C = m\dfrac{l^2+a^2}{12} = 3{,}636\ [\text{kg}\cdot\text{cm}^2]$ is the moment of inertia with respect to the $\vec{z}_0$ symmetry axis. By simple substitution in equations (26), one obtains the explicit expressions of $\ddot{\varphi}_{ref}$ and $\ddot{\psi}_{ref}$ (as functions of φ_{ref}, ψ_{ref}, $\dot{\varphi}_{ref}$ and $\dot{\psi}_{ref}$), which can be easily integrated by means of a 4th order Runge-Kutta method providing the reference solution φ_{ref} and ψ_{ref}.

Figure 2 compares φ_{ref} and ψ_{ref} with φ_{RT} and ψ_{RT} obtained by the two-step method of the Lagrange multipliers elimination (see §5) for the dynamics formulation using as rotational parameters the 9 elements of the 3×3 rotation matrix. φ_{RT} and ψ_{RT} are deduced from $\mathbf{T}_1^*$, $\mathbf{R}_1$, $\mathbf{T}_2^*$ and $\mathbf{R}_2$ as follows:

$$\begin{cases} \varphi_{RT} = \text{arctg}\dfrac{\langle \overrightarrow{\text{OG}_1}, \vec{y}_0\rangle}{\langle \overrightarrow{\text{OG}_1}, \vec{x}_0\rangle} = \text{arctg}\dfrac{\langle \mathbf{R}_1 \overrightarrow{\text{OG}_{1,0}}, \vec{y}_0\rangle}{\langle \mathbf{R}_1 \overrightarrow{\text{OG}_{1,0}}, \vec{x}_0\rangle} = \text{arctg}\dfrac{\langle \mathbf{R}_1(\cos\varphi_0\,\vec{x}_0 + \sin\varphi_0\,\vec{y}_0),\ \vec{y}_0\rangle}{\langle \mathbf{R}_1(\cos\varphi_0\,\vec{x}_0 + \sin\varphi_0\,\vec{y}_0),\ \vec{x}_0\rangle} \\ \psi_{RT} = \text{arctg}\dfrac{\langle \overrightarrow{\text{H}_1\text{G}_2}, \vec{y}_0\rangle}{\langle \overrightarrow{\text{H}_1\text{G}_2}, \vec{x}_0\rangle} = \text{arctg}\dfrac{\langle \mathbf{T}_2^* + \mathbf{R}_2\,\overrightarrow{\text{H}_1\text{G}_{2,0}}, \vec{y}_0\rangle}{\langle \mathbf{T}_2^* + \mathbf{R}_2\,\overrightarrow{\text{H}_1\text{G}_{2,0}}, \vec{x}_0\rangle} = \\ \qquad = \text{arctg}\dfrac{\langle \mathbf{T}_2^* + \mathbf{R}_2(\cos\psi_0\,\vec{x}_0 + \sin\psi_0\,\vec{y}_0),\ \vec{y}_0\rangle}{\langle \mathbf{T}_2^* + \mathbf{R}_2(\cos\psi_0\,\vec{x}_0 + \sin\psi_0\,\vec{y}_0),\ \vec{x}_0\rangle} \end{cases} ,(27)$$

where $\overrightarrow{\text{H}_1\text{G}_{2,0}} = \frac{l}{2}\cos\psi_0\vec{x}_0 + \frac{l}{2}\sin\psi_0\vec{y}_0$.

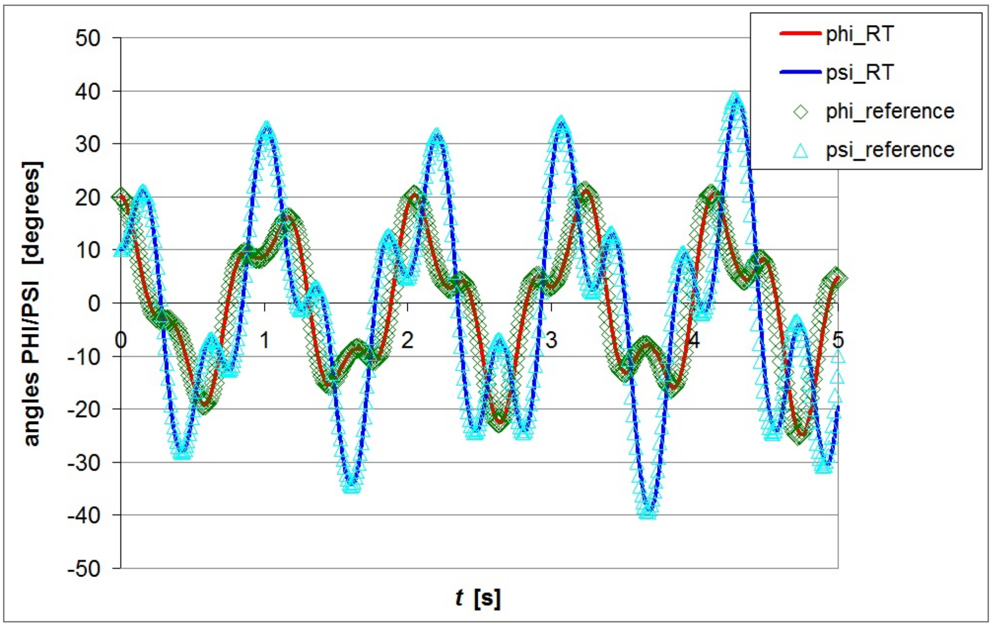

Fig. 2 – Results φ_{RT} and ψ_{RT} obtained using the two-step method of Lagrange multipliers elimination, compared with φ_{ref} and ψ_{ref}.

As shown in Fig. 2, the results φ_{RT} and ψ_{RT} obtained using the two-step method of Lagrange multipliers elimination are similar with φ_{ref} and ψ_{ref}. This validates our method based on the dynamics formulation using as rotational parameters the 9 elements of the 3×3 rotation matrix. Since φ_{ref} and ψ_{ref} do not represent an exact solution of the double pendulum motion, being another numerical solution, this paper will not develop an error analysis. Further work will be devoted to that issue. But, a partial proof of the performance of our method could be the fact that $\overrightarrow{OG_1}$ computed from $\mathbf{T}_1^*$ and $\mathbf{R}_1$ as $\overrightarrow{OG_1} = \mathbf{T}_1^* + \mathbf{R}_1 \overrightarrow{OG_{1,0}}$ and $\overrightarrow{OG_2}$ computed from $\mathbf{T}_2^*$ and $\mathbf{R}_2$ as $\overrightarrow{OG_2} = \mathbf{T}_2^* + \mathbf{R}_2 \overrightarrow{OG_{2,0}}$ do not develop in time any component in the $\vec{z}_0$ direction, which is an indirect proof of nonpropagation of numerical errors.

Figures 3–6 present the evolutions in time of the Lagrange multipliers introduced by the formulation. Only the non-null elements of $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are shown in Figs. 3 and 4, more precisely $\mathbf{\Lambda}_{1,11}, \mathbf{\Lambda}_{1,12} \equiv \mathbf{\Lambda}_{1,21}, \mathbf{\Lambda}_{1,22}$ and $\mathbf{\Lambda}_{2,11}, \mathbf{\Lambda}_{2,12} \equiv \mathbf{\Lambda}_{2,21}, \mathbf{\Lambda}_{2,22}$, respectively.

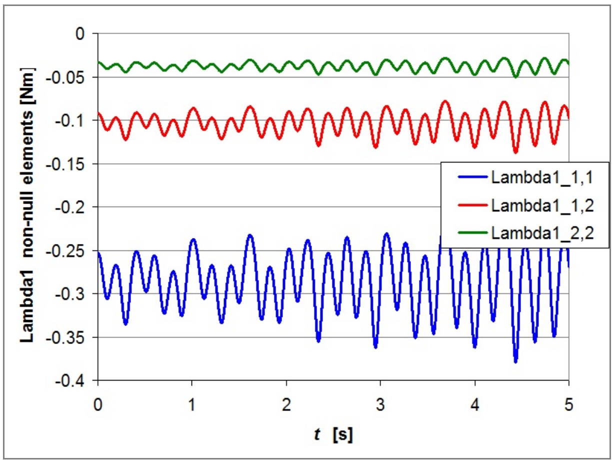

Fig. 3 – Evolution in time of the non-null elements $\mathbf{\Lambda}_{1,11}$, $\mathbf{\Lambda}_{1,12}$, $\mathbf{\Lambda}_{1,22}$ of the symmetric Lagrange multipliers matrix $\mathbf{\Lambda}_1$.

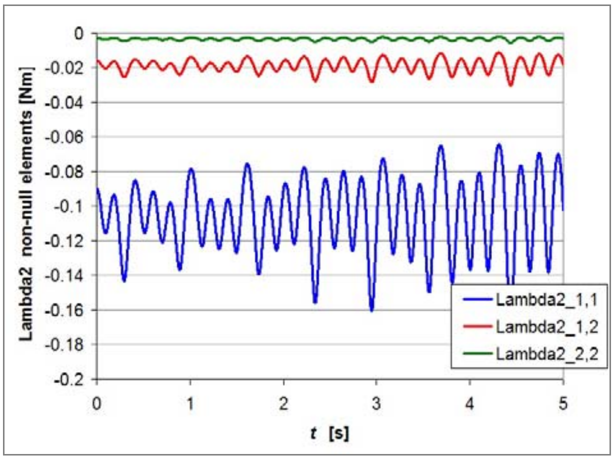

Fig. 4 – Evolution in time of the non-null elements of the symmetric Lagrange multipliers matrix $\mathbf{\Lambda}_2$.

Figures 5 and 6 show the evolutions of the components of the Lagrange multipliers vectors $\mathbf{\Lambda}_{01}^{\text{sph}}$ and $\mathbf{\Lambda}_{12}^{\text{sph}}$ introduced in association with the constraint equations characterizing the two spherical joints O and H_1 of the double pendulum.

It is very likely that all these Lagrange multipliers have a regulation role in the integration scheme of the algebro-differential system corresponding to our very redundant parameterization. A further error analysis will try to study this aspect of numerical autoregulation in the context of this matricial formulation.

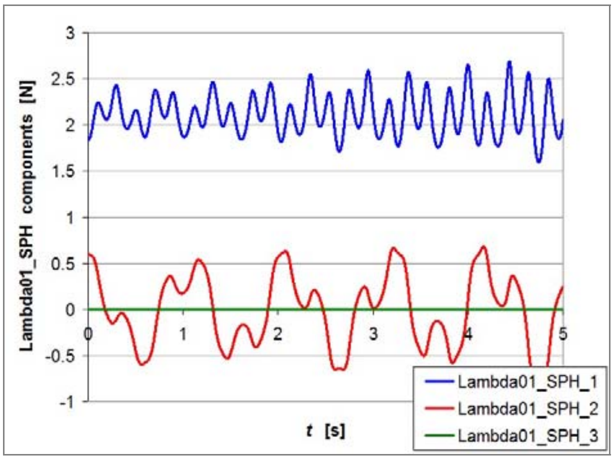

Fig. 5 – Evolution in time of the three components of the Lagrange multipliers vector $\mathbf{\Lambda}_{01}^{\text{sph}}$.

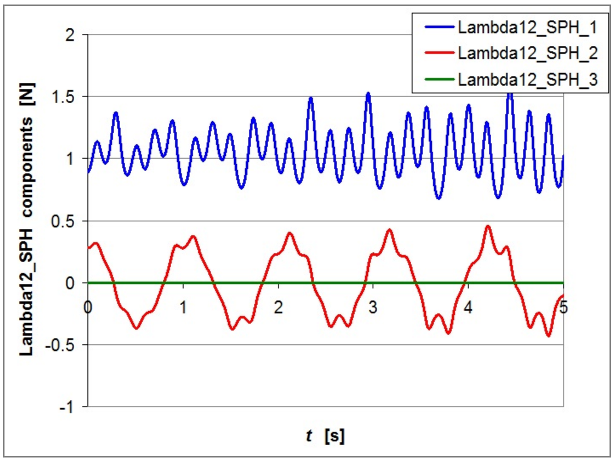

Fig. 6 – Evolution in time of the three components of the Lagrange multipliers vector $\mathbf{\Lambda}_{12}^{\text{sph}}$.

7. CONCLUSION AND FURTHER WORK

The proposed parameterization of 3D rotations by preserving the 9 elements of the 3×3 rotation matrix is a very redundant one. Using 9 parameters for 3 degrees of freedom means that 9–3 = 6 orthogonality scalar equations per solid are necessary. Lagrange multipliers are introduced in order to take into account these rigidity constraints, as well as other constraints characterizing the articulations between the linked solids S_i of the multi-body system. The matricial formulation based on this very redundant parameterization is very systematic, the dynamics equations for each solid being generated in an automatic way. The differential part of the dynamics equations is linear in $\ddot{\mathbf{T}}_i^*$ and $\ddot{\mathbf{R}}_i$ for each solid S_i, which is a considerable advantage from the point of view of numerical investigation. This is the main advantage of the method: it is systematic, having an increased degree of generality and facilitating the computer simulation of multibody system dynamics. For example, for a parallel robot the formulations based on traditional rotational parameters are difficult to implement due to nonlinearities, while our formulation is written without difficulty, almost automatically, the difficulty consisting only in the important size of the algebro-differential system to be solved.

With respect to our previous work, in this sense of improving the numerical solving of such algebro-differential system, this paper proposes a Lagrange multipliers elimination method in two-steps. In the first step of the method are eliminated the Lagrange multipliers $\mathbf{\Lambda}_i$ associated to the orthogonality conditions, which take the form of discrete Lyapunov-like equations that are easily solved by specific methods. In the second step of the elimination method, only the constraints characterizing the articulations between the linked solids of the multi-body system remain to be solved, forming a linear equations system of smaller size than the one for the case where the first step would not have been performed. So, this two-step method of Lagrange multipliers elimination clearly improves the numerical resolution of the algebraic part.

The simple case study of a planar double pendulum is presented here, its purpose being to show that the formulation works correctly on an example were alternate solution is easily available for validation. Further work intends to apply the formulation on a Stewart platform and compare the results with experimental ones. In such case of a more complicated parallel robot, the two-step method of Lagrange multipliers elimination will have a good opportunity to show its full numerical performance.

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REFERENCES

1. STUELPNAGEL, J., *On the parametrization of the three-dimensional rotation group*, SIAM Review, **6**, *4*, pp. 422–430, 1964.
2. GARCÍA DE JALÓN, J., BAYO, E., *Kinematic and Dynamic Simulation of Multibody Systems – The Real-Time Challenge*, Springer-Verlag, New York, 1994.

3. DA LIO, M., COSSALTER, V., LOT, R., *On the use of natural coordinates in optimal synthesis of mechanisms*, Mechanism and Machine Theory, **35**, *5*, pp. 669–679, 2000.
4. KRAUS, C., WINCKLER, M., BOCK, H.G., *Modeling mechanical DAE using natural coordinates*, Mathemathical and Computer Modelling of Dynamical Systems, **7**, *2*, pp. 145–158, 2001.
5. SHABANA, A.A., *Dynamics of Multibody Systems*, Cambridge University Press, 3rd edition, 2005.
6. BLACK, H.D., *A passive system for determining the attitude of a satellite*, AIAA Journal, **2**, *7*, pp. 1350–1351, 1964.
7. WAHBA, G., *A least squares estimate of satellite attitude, Problem 65–1*, SIAM Review, **7**, *5*, p. 409, 1965.
8. FARRELL, J.L., STUELPNAGEL, J.C., WESSNER, R.H., VELMAN, J.R., BROOK, J.E., *A least squares estimate of satellite attitude, Solution 65–1*, SIAM Review, **8**, *3*, pp. 384–386, 1966.
9. MARKLEY, F.L., *Attitude determination using vector observations and the singular value decomposition*, The Journal of the Astronautical Sciences, **36**, *3*, pp. 245–258, 1988.
10. IZADI, M., SANYAL, A.K., SAMIEI, E., VISWANATHAN, S.P., *Discrete-time rigid body attitude state estimation based on the discrete Lagrange-d'Alembert principle*, American Control Conference (ACC), Chicago, IL, July 1–3, 2015, pp. 3392–3397.
11. HORN, B.K.P., HILDEN, H.M., NEGAHDARIPOUR, S., *Closed-form solution of absolute orientation using orthonormal matrices*, Journal of the Optical Society A, **5**, *7*, pp. 1127–1135, 1988.
12. KRENK, S., NIELSEN M.B., *Conservative rigid body dynamics by convected base vectors with implicit constraints*, Computer Methods in Applied Mechanics and Engineering, **269**, pp. 437–453, 2014.
13. SIMO, J.C., WONG, K.K., *Unconditionally stable algorithms for rigid body dynamics that exactly preserve energy and momentum*, International Journal for Numerical Methods in Engineering, **31**, *1*, pp. 19–52, 1991.
14. BETSCH, P., STEINMANN, P., *Constrained integration of rigid body dynamics*, Comput. Methods Appl. Mech. Engrg., **191**, *3–5*, pp. 467–488, 2001.
15. BETSCH, P., *The discrete null space method for the energy consistent integration of constrained mechanical systems Part I: Holonomic constraints*, Comput. Methods Appl. Mech. Engrg., **194**, *50–52*, pp. 5159–5190, 2005.
16. BETSCH, P., LEYENDECKER, S., *The discrete null space method for the energy consistent integration of constrained mechanical systems. Part II: Multibody dynamics*, International Journal for Numerical Methods in Engineering, **67**, *4*, 2006, pp. 499–552.
17. BETSCH, P., UHLAR, S., *Energy-momentum conserving integration of multibody dynamics*, Multibody System Dynamics, **17**, *4*, pp. 243–289, 2007.
18. UHLAR, S., BETSCH, P., *A rotationless formulation of multibody dynamics: Modeling of screw joints and incorporation of control constraints*, Multibody System Dynamics, **22**, *1*, pp. 69–95, 2009.
19. GROS, S., ZANON, M., VUKOV, M., DIEHL, M., *Nonlinear MPC and MHE for mechanical multi-body systems with application to fast tethered airplanes*, 4th IFAC Nonlinear Model Predictive Control Conference, Noordwijkerhout, NL, August 23–27, 2012, pp. 86–93.
20. VALLÉE, C., HAMDOUNI, A., ISNARD, F., FORTUNÉ, D., *The equations of motion of a rigid body without parametrization of rotations*, Journal of applied mathematics and mechanics, **63**, *1*, pp. 25–30, 1999.
21. ISNARD, F., *Génération des équations du mouvement de systèmes polyarticulés avec prise en compte des rigidités par des multiplicateurs de Lagrange*, PhD Thesis (in French, advisor: C. Vallée), University of Poitiers, France, 1997.
22. ISNARD, F., DODDS, G., VALLÉE, C., FORTUNÉ, D, *Real-time dynamics simulation of a closed-chain robot within a virtual reality environment*, Proceedings of the Institution of Mechanical Engineers, Part K: Journal of Multi-body Dynamics, **214**, *4*, pp. 219–232, 2000.
23. DUMITRIU, D., *Modélisation dynamique des systèmes articulés par des vecteurs translation et des matrices rotation. Prise en compte des rigidités par des multiplicateurs de Lagrange.*

Simulations du mouvement à l'aide d'un code en C++, PhD Thesis (in French, advisor: C. Vallée), University of Poitiers, France, 2003.

24. VALLÉE, C., DUMITRIU, D., *The Lagrange multipliers associated with the rotation matrix characterizing the motion of a rigid body about its centre of mass*, Journal of Applied Mathematics and Mechanics, **65**, *5*, pp. 731–739, 2001.
25. DUMITRIU, D., VALLÉE, C., *Rigid body dynamics using the parameterization of rotations by full rotation matrices*, XXX-th National Conference on Solid Mechanics, Constanța, Romania, September 15–16, 2006, pp. 57–62.
26. DUMITRIU, D., VALLÉE, C., *A new approach of rotations in multibody system dynamics* (Ch. 3), in: *Research Trends in Mechanics* – Vol. 1 (eds. D. Popa, V. Chiroiu, I. Toma), Edit. Academiei Române, Bucharest, 2007, pp. 55-81.
27. DUMITRIU, D., VALLÉE, C., *Abordare matriceală a dinamicii sistemelor de solide rigide articulate*, Editura Bren, Bucharest, 2011.
28. ATCHONOUGLO, K., *Identification des paramètres caractéristiques d'un phénomène mécanique ou thermique régi par une équation différentielle ou aux dérivées partielles*, PhD Thesis (Advisor: C. Vallée), University of Poitiers, France, 2007.
29. ATCHONOUGLO, K., VALLÉE, C. *Matrix formulation of the motion equations of a rigid body* (Ch. 1), in: *Research Trends in Mechanics* – Vol. 3 (eds. D. Popa, V. Chiroiu, L. Munteanu), Edit. Academiei Române, Bucharest, 2009, pp. 9–29.
30. ATCHONOUGLO, K., VALLÉE, C., MONNET, T., FORTUNÉ, D., *Identification of the ten inertia parameters of a rigid body*, Journal of Applied Mathematics and Mechanics, **72**, *1*, pp. 22–25, 2008.
31. MONNET, T., *Contribution à l'identification des paramètres inertiels des segments du corps humain*, PhD Thesis (Advisor: C. Vallée), University of Poitiers, France, 2007.
32. MONNET, T., BEGON, M., VALLÉE, C., LACOUTURE, P., *Improvement of the input data in biomechanics: kinematic and body segment inertial parameters* (Ch. 15), in: *Biomechanics: principles, trends and applications* (ed. J.H. Levy), Nova Science Publishers, 2010, pp. 353–385,.
33. SEGUY, N., *Modélisation modulaire de systèmes articulés: Conception orientée-objet de plates-formes de simulation*, PhD Thesis (Advisor: M. Pascal), University of Evry Val d'Essonne, France, 2003.
34. SAMIN, J.-C., FISETTE, P., *Symbolic modeling of multibody systems*, Kluwer Academic Publishers, 2003.
35. RUF, A., *Closing the loop between articulated motion and stereo vision: a projective approach*, PhD Thesis (Advisor: R. Horaud), Grenoble Institute of Technology, France, 2001.
36. MA, Y., SOATTO, S., KOŠECKÁ, J., SASTRY, S.S., *An invitation to 3-D vision: from images to geometric models*, Springer, 2003.
37. STAICU, S., *Aplicații ale calculului matriceal în mecanica solidelor*, Edit. Academiei Române, Bucharest, 1986.
38. STAICU, S., LERBET, J., *A comparative analysis in dynamics of parallel manipulators*, Politehnica University of Bucharest Scientific Bulletin – Series D: Mechanical Engineering, **67**, *2*, pp. 3–18, 2005.
39. STAICU, S., ZHANG, D., *A novel dynamic modelling approach for parallel mechanisms analysis*, Robotics and Computer-Integrated Manufacturing, **24**, *1*, pp. 167–172, 2008.
40. STAICU, S., LIU, X.-J., LI, J., *Explicit dynamics equations of the constrained robotic systems*, Nonlinear Dynamics, **58**, *1–2*, pp. 217–235, 2009.
41. STAICU, S., *Matrix modeling of inverse dynamics of spatial and planar parallel robots*, Multibody System Dynamics, **27**, *2*, pp. 239–265, 2012.
42. STAICU, S., *Dynamics modelling of a Stewart-based hybrid parallel robot*, Advanced Robotics, **29**, *14*, pp. 929–938, 2015.