DYNAMIC STUDY OF AN ELASTIC SYSTEM WITH TWO DEGREES OF FREEDOM

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Abstract. In the paper is presented the dynamic survey of an elastic mechanical system which consists of two rigid solids linked to each other by a cylindrical joint. One of the two rigid solids is linked through an elastic linear spring to another rigid solid which is supposed to be fixed. We aim to study the movement of this elastic mechanical system under the action of forces. In order to do this in this paper is presented a numerical method which requires the writing of the differential equations of motion under matrix form. Finally, a computing program is elaborated with the help of which the differential equations of motion are integrated using numerical integration methods. In this way it is determined the variation with respect to time of the kinematical parameters of the rigid solids which make up the elastic mechanical system.

Keywords: elastic mechanical system, dynamic study, constraint forces, numerical integration methods.

NOMENCLATURE

$\mathbf{S}_{O_i}$ – anti-symmetric matrix associated to the polar static moment vector about the point "O_i" of the rigid solid body "i" of the system

$T_i(O_i x_i y_i z_i)$ – body fixed reference frame of the rigid solid "i" of the system

$T(Oxyz)$ – fixed reference frame

$\mathbf{r}_{C_i}$ – anti-symmetric matrix associated to the position vector of the mass center "C_i" of the rigid solid body "i" of the system

$\tilde{\boldsymbol{\omega}}_i$ – anti-symmetric matrix associated to the angular velocity vector of the rigid solid body "i" of the system

ξ_i, η_i, ζ_i – coordinates of the mass center "C_i" of the solid rigid body "i" of the system

$\mathbf{Q}_i$ – the torque of active forces acting upon the rigid solid body "i" of the system, calculated with respect to the point O_i

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$\boldsymbol{\tau}_{\mathrm{O}_i}$ – the torque of constraint forces acting upon the rigid solid about the point "O$_i$", an arbitrary point that belongs to the rigid solid "i" of the system

$\mathbf{F}_{\mathrm{r}}$ – column matrix associated to the viscous damping force vector in projections on the axis of the fixed reference frame

$\underset{m\times n}{\mathbf{0}}$ – zero-matrix with "m' rows and "n" columns

ψ_i, θ_i, φ_i – angles of precession, nutation and self-rotation (Euler angles) corresponding to the solid rigid body "i" of the system

m_i – the mass of the rigid body "i" of the system

$\mathbf{R}_{20}$ – column matrix associated to constraint external force vector in projections on the axis of the fixed reference frame

Y_{20}, Z_{20} – external constraint force components in projections on the axis of the fixed reference frame

$\mathbf{M}_{20}$ – column matrix associated to constrained external moment vector in projections on the axis of the fixed reference frame

$M_{20,x}, M_{20,y}, M_{20,z}$ – external constrained moment components in projections on the axis of the fixed reference frame

$\mathbf{R}_{12}$ – column matrix associated to internal constrained force vector in projections on the axis of the fixed reference frame

X_{12}, Y_{12}, Z_{12} – constrained internal force components in projections on the axis of the fixed reference frame

$\mathbf{M}_{12}$ – column matrix associated to constrained internal moment vector in projections on the axis of the fixed reference frame

$M_{12,x}, M_{12,y}$ – external constrained moment components in projections on the axis of the fixed reference frame

$\mathbf{I}_3$ – unit matrix of the third order

$\mathbf{I}_2$ – unit matrix of the second order

c – viscous damping factor

1. INTRODUCTION

We consider the mechanical system which is presented in the figure bellow. (Fig. 1) It consists of two rigid solid bodies linked to each other through a cylindrical joint. The solid rigid "2" is linked to a solid rigid body which is supposed to be fixed by a slide and a linear elastic spring. Next we plan to study the dynamics of this mechanical system, namely its movement under the action of forces. In order to do this we will have to consider each rigid solid body as being free and acted by active and constraint forces and we will write under matrix form the differential equations that describe the motion of each rigid solid that compounds the mechanical system. The constraint forces are unknown and therefore they will have to be eliminated from the differential equations of motion. Over the years several methods have been proposed for constrained forces elimination. For instance the constraint forces may be removed from the system of differential equations by its multiplication to the left with a matrix which is called orthogonal complement [6, 7] or natural orthogonal complement [4].

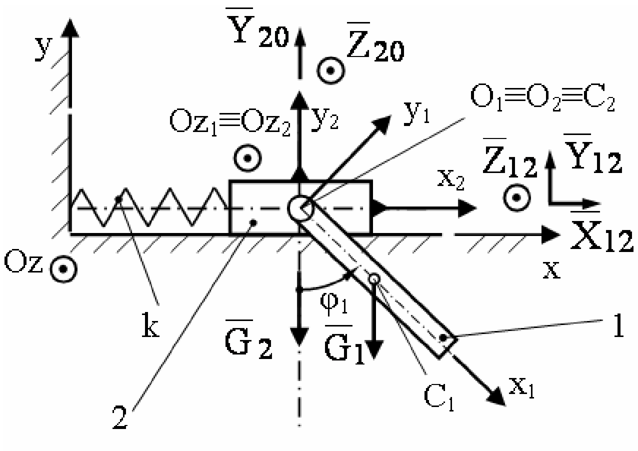

Fig. 1 – Elastic system with two degrees of freedom.

The determination of the orthogonal complement or natural orthogonal complement may be sometimes a quite difficult problem.

The Lagrange equations have also been used to study the dynamics of mechanical systems [1, 2, 3, 8–10].

The equivalence between Euler type equations and Lagrange equations has also been demonstrated [7].

2. WRITING THE EQUATIONS OF MOTION FOR THE FIRST RIGID SOLID BODY

The differential equations of motion for the rigid solid "1" may be written under the following matrix form [2, 4, 6–8, 12–17]:

$$\mathbf{M}_{O_1} \cdot \dot{\mathbf{v}}_1 = -\mathbf{\Omega}_1 \cdot \mathbf{v}_1 + \mathbf{Q}_1 + \boldsymbol{\tau}_{O_1} . \tag{1}$$

The relationship (1) represents in fact the algebraic sum of three torques: the torque of inertia forces, the torque of applied forces $\mathbf{Q}_1$ and the torque of constraint forces. All the three torques are calculated with respect to the origin of the body fixed reference frame $T_1(O_1x_1y_1z_1)$ and are expressed in projections on its axis.

In the mathematical relation (1) the quantities involved have the following mathematical expressions:

$$\mathbf{M}_{\mathrm{O}_1} = \left[\begin{array}{c|c} \mathbf{M}_1 & -\mathbf{S}_{\mathrm{O}_1} \\ \hline \mathbf{S}_{\mathrm{O}_1} & \mathbf{J}_{\mathrm{O}_1} \end{array}\right], \quad \mathbf{M}_1 = \left[\begin{array}{c|c|c} m_1 & 0 & 0 \\ \hline 0 & m_1 & 0 \\ \hline 0 & 0 & m_1 \end{array}\right] \tag{2}$$

$$\mathbf{S}_{\mathrm{O}_1} = \mathbf{M}_1 \cdot \mathbf{r}_{\mathrm{C}_1}, \quad \mathbf{r}_{\mathrm{C}_1} = \left[\begin{array}{c|c|c} 0 & -\zeta_1 & \eta_1 \\ \hline \zeta_1 & 0 & -\xi_1 \\ \hline -\eta_1 & \xi_1 & 0 \end{array}\right], \quad \xi_1 = l_1/2. \tag{3}$$

In the relations (2) "m_1" represents the mass of the rigid solid "1".

$$\mathbf{J}_{\mathrm{O}_1} = \left[\begin{array}{c|c|c} J_{x_1} & -J_{x_1 y_1} & -J_{x_1 z_1} \\ \hline -J_{x_1 y_1} & J_{y_1} & -J_{y_1 z_1} \\ \hline -J_{x_1 z_1} & -J_{y_1 z_1} & J_{z_1} \end{array}\right] \tag{4}$$

In the relation (3) ξ_1, η_1 and ς_1 represent the coordinates of the central mass "C_1" relatively to the body fixed reference frame $\mathrm{T}_1(\mathrm{O}_1 x_1 y_1 z_1)$.

In the relation (4) the scalar quantities J_{x_1}, J_{y_1} and J_{z_1} represent the axial moments of inertia relatively to the axis of the body fixed reference frame $\mathrm{T}_1(\mathrm{O}_1 x_1 y_1 z_1)$ and $J_{x_1 y_1}$, $J_{x_1 z_1}$, $J_{y_1 z_1}$ represent the centrifugal moments of inertia relatively to the pairs of planes which belong to the same reference frame $\mathrm{T}_1(\mathrm{O}_1 x_1 y_1 z_1)$.

$$\dot{\mathbf{v}}_1 = \left[\begin{array}{c|c} \dot{\mathbf{v}}_{\mathrm{O}_1} & \dot{\boldsymbol{\omega}}_1 \end{array}\right]^{\mathrm{T}}, \qquad \mathbf{v}_1 = \left[\begin{array}{c|c} \mathbf{v}_{\mathrm{O}_1} & \boldsymbol{\omega}_1 \end{array}\right]^{\mathrm{T}} \tag{5}$$

$$\dot{\mathbf{v}}_{\mathrm{O}_1} = \left[\begin{array}{c|c|c} \dot{v}_{\mathrm{O}_1 x_1} & \dot{v}_{\mathrm{O}_1 y_1} & \dot{v}_{\mathrm{O}_1 z_1} \end{array}\right]^{\mathrm{T}}, \quad \dot{\boldsymbol{\omega}}_1 = \left[\begin{array}{c|c|c} \dot{\omega}_{x_1} & \dot{\omega}_{y_1} & \dot{\omega}_{z_1} \end{array}\right]^{\mathrm{T}} \tag{6}$$

$$\mathbf{v}_{\mathrm{O}_1} = \left[\begin{array}{c|c|c} v_{\mathrm{O}_1 x_1} & v_{\mathrm{O}_1 y_1} & v_{\mathrm{O}_1 z_1} \end{array}\right]^{\mathrm{T}}, \quad \boldsymbol{\omega}_1 = \left[\begin{array}{c|c|c} \omega_{x_1} & \omega_{y_1} & \omega_{z_1} \end{array}\right]^{\mathrm{T}} \tag{7}$$

$$\boldsymbol{\Omega}_1 = \left[\begin{array}{c|c} \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{M}_1 & -\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{S}_{\mathrm{O}_1} \\ \hline \mathbf{S}_{\mathrm{O}_1} \cdot \tilde{\boldsymbol{\omega}}_1 & \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{J}_{\mathrm{O}_1} \end{array}\right], \quad \tilde{\boldsymbol{\omega}}_1 = \left[\begin{array}{c|c|c} 0 & -\omega_{z_1} & \omega_{y_1} \\ \hline \omega_{z_1} & 0 & -\omega_{x_1} \\ \hline -\omega_{y_1} & \omega_{x_1} & 0 \end{array}\right] \tag{8}$$

The torque of active forces (resultant force vector and resultant moment vector) which are acting upon the rigid solid "1", with respect to the point "O_1", will be written in matrix form as follows:

$$\mathbf{Q}_1 = \left[\mathbf{R}_1^{\mathrm{aT}} \mid \mathbf{M}_{\mathrm{O}_1}^{\mathrm{aT}} \right]^{\mathrm{T}}. \tag{9}$$

The resultant force vector in projections on the axis of the body fixed reference frame $T_1(O_1x_1y_1z_1)$ will be written under the following matrix form:

$$\mathbf{R}_1^{\mathrm{a}} = \mathbf{G}_1. \tag{10}$$

In the relation above, $\mathbf{G}_1$ represents the force of gravity which acts upon the rigid solid “1”. The mathematical expression of this force is the following:

$$\mathbf{G}_1 = \mathbf{R}_1^{\mathrm{T}} \cdot \left[0 \mid -m_1 \cdot g \mid 0 \right]^{\mathrm{T}}. \tag{11}$$

In the relation (11) the matrix $\mathbf{R}_1$ has the following expression:

$$\mathbf{R}_1 = \mathbf{\Psi}_1 \cdot \mathbf{\Theta}_1 \cdot \mathbf{\Phi}_1 \tag{12}$$

$$\mathbf{\Psi}_1 = \begin{bmatrix} \cos(\psi_1) & -\sin(\psi_1) & 0 \\ \sin(\psi_1) & \cos(\psi_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Theta}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \tag{13}$$

$$\mathbf{\Phi}_1 = \begin{bmatrix} \sin(\varphi_1) & \cos(\varphi_1) & 0 \\ -\cos(\varphi_1) & \sin(\varphi_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

The resultant moment vector of active forces may be written in matrix form as follows:

$$\mathbf{M}_{\mathrm{O}_1}^{\mathrm{a}} = \mathbf{r}_{\mathrm{C}_1} \cdot \mathbf{R}_1^{\mathrm{T}} \cdot \left[0 \mid -m_1 \cdot g \mid 0 \right]^{\mathrm{T}}. \tag{15}$$

In the relations (13) and (14) ψ_1, θ_1 and φ_1 represent Euler angles corresponding to the rigid solid “1”.

The torque of constraint forces about the point “O_1” which are acting upon the rigid solid “1” of the system has the following matrix expression:

$$\boldsymbol{\tau}_{\mathrm{O}_1} = \left[\mathbf{R}_1^{\mathrm{cT}} \mid \mathbf{M}_{\mathrm{O}_1}^{\mathrm{cT}} \right]^{\mathrm{T}} \tag{16}$$

$$\mathbf{R}_1^{\mathrm{c}} = \mathbf{R}_1^{\mathrm{T}} \cdot \left[X_{12} \mid Y_{12} \mid Z_{12} \right]^{\mathrm{T}}, \quad \mathbf{M}_{\mathrm{O}_1}^{\mathrm{c}} = \mathbf{R}_1^{\mathrm{T}} \cdot \left[M_{12,x} \mid M_{12,y} \mid 0 \right]^{\mathrm{T}} \tag{17}$$

The torque of constraint forces may be written under matrix form as followings:

$$\boldsymbol{\tau}_{O_1} = \mathbf{L}_{\lambda_1} \cdot \boldsymbol{\lambda}. \tag{18}$$

In the relation (18) the quantities involved have the followings expressions:

$$\mathbf{L}_{\lambda_1} = \mathbf{R}_{1,\text{ext}}^{\text{T}} \cdot [\mathbf{A}_1 \,\vdots\, \mathbf{A}_2] \tag{19}$$

$$\mathbf{R}_{1,\text{ext}}^{\text{T}} = \begin{bmatrix} \mathbf{R}_1^{\text{T}} & \underset{3\times3}{\mathbf{0}} \\ \underset{3\times3}{\mathbf{0}} & \mathbf{R}_1^{\text{T}} \end{bmatrix} \tag{20}$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{I}_3 & \underset{3\times2}{\mathbf{0}} \\ \underset{3\times3}{\mathbf{0}} & \mathbf{N}_1 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{\text{T}}, \quad \mathbf{A}_2 = \underset{6\times5}{\mathbf{0}} \tag{21}$$

$$\boldsymbol{\lambda} = \left[\boldsymbol{\lambda}_1^{\text{T}} \,\vdots\, \boldsymbol{\lambda}_2^{\text{T}} \right]^{\text{T}} \tag{22}$$

$$\boldsymbol{\lambda}_1 = \left[X_{12} \,\vdots\, Y_{12} \,\vdots\, Z_{12} \,\vdots\, M_{12,x} \,\vdots\, M_{12,y} \right]^{\text{T}} \tag{23}$$

$$\boldsymbol{\lambda}_2 = \left[Y_{20} \,\vdots\, Z_{20} \,\vdots\, M_{20,x} \,\vdots\, M_{20,y} \,\vdots\, M_{20,z} \right]^{\text{T}}. \tag{24}$$

3. WRITING THE EQUATIONS OF MOTION FOR THE SECOND RIGID SOLID BODY

The differential equations of motion for the rigid solid “2” (Fig. 2) which is supposed to be free may be written under the following matrix form [2, 4, 6–8, 12–17]:

$$\mathbf{M}_2 \cdot \dot{\mathbf{v}}_2 = -\boldsymbol{\Omega}_2 \cdot \mathbf{v}_2 + \mathbf{Q}_2 + \boldsymbol{\tau}_{O_2}. \tag{25}$$

In the mathematical relation (25) the quantities involved have the following expressions:

$$\mathbf{M}_{O_2} = \begin{bmatrix} \mathbf{M}_2 & \underset{3\times3}{\mathbf{0}} \\ \underset{3\times3}{\mathbf{0}} & \mathbf{J}_{O_2} \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} m_2 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \tag{26}$$

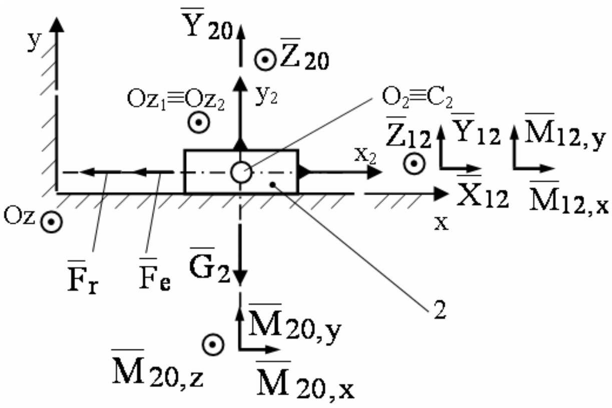

Fig. 2 – Forces acting on the second solid rigid of the mechanical system.

$$\mathbf{J}_{O_2} = \left[\begin{array}{c|c|c} J_{x_2} & -J_{x_2y_2} & -J_{x_2z_2} \\ \hline -J_{x_2y_2} & J_{y_2} & -J_{y_2z_2} \\ \hline -J_{x_2z_2} & -J_{y_2z_2} & J_{z_2} \end{array}\right] \tag{27}$$

In the relation (27) the scalar quantities J_{x_2}, J_{y_2} and J_{z_2} represent the axial moments of inertia relatively to the axis of the body fixed reference frame $T_2(O_2x_2y_2z_2)$ and $J_{x_2y_2}$, $J_{x_2z_2}$, $J_{y_2z_2}$ represent the centrifugal moments of inertia relatively to the pairs of planes which belong to the same reference frame $T_2(O_2x_2y_2z_2)$.

$$\dot{\mathbf{v}}_2 = \left[\dot{\mathbf{v}}_{O_2} \mid \dot{\boldsymbol{\omega}}_2\right]^{T}, \qquad \mathbf{v}_2 = \left[\mathbf{v}_{O_2} \mid \boldsymbol{\omega}_2\right]^{T} \tag{28}$$

$$\dot{\mathbf{v}}_{O_2} = \left[\dot{v}_{O_2x_2} \mid \dot{v}_{O_2y_2} \mid \dot{v}_{O_2z_2}\right]^{T}, \quad \dot{\boldsymbol{\omega}}_2 = \left[\dot{\omega}_{x_2} \mid \dot{\omega}_{y_2} \mid \dot{\omega}_{z_2}\right]^{T} \tag{29}$$

$$\mathbf{v}_{O_2} = \left[v_{O_2x_2} \mid v_{O_2y_2} \mid v_{O_2z_2}\right]^{T}, \quad \boldsymbol{\omega}_2 = \left[\omega_{x_2} \mid \omega_{y_2} \mid \omega_{z_2}\right]^{T} \tag{30}$$

$$\boldsymbol{\Omega}_2 = \left[\begin{array}{c|c} \tilde{\boldsymbol{\omega}}_2 \cdot \mathbf{M}_2 & \underset{3\times3}{\mathbf{0}} \\ \hline \underset{3\times3}{\mathbf{0}} & \tilde{\boldsymbol{\omega}}_2 \cdot \mathbf{J}_{O_2} \end{array}\right], \quad \tilde{\boldsymbol{\omega}}_2 = \left[\begin{array}{c|c|c} 0 & -\omega_{z_2} & \omega_{y_2} \\ \hline \omega_{z_2} & 0 & -\omega_{x_2} \\ \hline -\omega_{y_2} & \omega_{x_2} & 0 \end{array}\right] \tag{31}$$

The torque of active forces acting upon rigid solid "2" will be written as follows:

$$\mathbf{Q}_2 = \left[\mathbf{R}_2^{\mathrm{aT}} \mid \mathbf{M}_{\mathrm{O}_2}^{\mathrm{aT}} \right]^{\mathrm{T}}. \tag{32}$$

The resultant force vector will be written under the following matrix form:

$$\mathbf{R}_2^{\mathrm{a}} = \mathbf{G}_2 + \mathbf{F}_{\mathrm{e}} + \mathbf{F}_{\mathrm{r}}. \tag{33}$$

In the relation (33) the gravity force vector in projections on the body fixed reference frame T_2 has the following matrix expression:

$$\mathbf{G}_2 = \mathbf{R}_2^{\mathrm{T}} \cdot \left[0 \mid -m_2 \cdot g \mid 0 \right]^{\mathrm{T}}. \tag{34}$$

The elastic force generate by the linear spring in projections on the body fixed reference frame T_2 has the following matrix expression:

$$\mathbf{F}_{\mathrm{e}} = -\mathbf{R}_2^{\mathrm{T}} \cdot (\Delta L/L) \cdot \mathbf{K} \cdot \mathbf{p}, \quad \mathbf{K} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \tag{35}$$

$$\mathbf{p} = \left[p_x \mid p_y \mid p_z \right]^{\mathrm{T}} \tag{36}$$

$$L = \sqrt{p_x^2 + p_y^2 + p_z^2} \tag{37}$$

$$\Delta L = L - L_0 \tag{38}$$

$$p_x = \left(x_{\mathrm{O}_2} + L_0 \right) - x_{\mathrm{O}}, \quad p_y = y_{\mathrm{O}_2} - y_{\mathrm{O}}, \quad p_z = z_{\mathrm{O}_2} - z_{\mathrm{O}}. \tag{39}$$

In the relations (35), (37), (38) and (39) the quantity "L_0" represents the initial length of the linear spring as it is shown in Fig.2 and the quantity "L" represents the length of the spring at a certain moment "t". In the relation (35) "k" represents the elastic constant of the spring.

The viscous damping force acting on the rigid "2" in the system is written under its matrix form in projections on the axes of the reference system $\mathrm{T}_2(\mathrm{O}_2 x_2 y_2 z_2)$ as follows:

$$\mathbf{F}_{\mathrm{r}} = -\mathbf{R}_2^{\mathrm{T}} \cdot \mathbf{C} \cdot \mathbf{R}_2 \cdot \mathbf{v}_{\mathrm{O}_2}, \quad \mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \mathbf{R}_2 = \mathbf{\Psi}_2 \cdot \mathbf{\Theta}_2 \cdot \mathbf{\Phi}_2. \tag{40}$$

In the mathematical relation (40) the terms involved have the following expressions:

$$\mathbf{\Psi}_2 = \left[\begin{array}{c|c|c} \cos(\psi_2) & -\sin(\psi_2) & 0 \\ \hline \sin(\psi_2) & \cos(\psi_2) & 0 \\ \hline 0 & 0 & 1 \end{array}\right], \quad \mathbf{\Theta}_2 = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \cos(\theta_2) & -\sin(\theta_2) \\ \hline 0 & \sin(\theta_2) & \cos(\theta_2) \end{array}\right] \tag{41}$$

$$\mathbf{\Phi}_2 = \left[\begin{array}{c|c|c} \cos(\varphi_2) & -\sin(\varphi_2) & 0 \\ \hline \sin(\varphi_2) & \cos(\varphi_2) & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{42}$$

In the relations (41) and (42) ψ_2, θ_2 and φ_2 represent Euler angles corresponding to the rigid solid body “2”.

The resultant moment vector of active forces may be written in matrix form as follows:

$$\mathbf{M}_{O_2}^{a} = \left[0 \mid 0 \mid 0\right]^{T}. \tag{43}$$

The torque of constraint forces about the point “O_2” which are acting upon the rigid solid “2” of the system has the following matrix expression:

$$\tau_{O_2} = \left[\mathbf{R}_2^{cT} \mid \mathbf{M}_{O_2}^{cT}\right]^{T} \tag{44}$$

$$\mathbf{R}_2^{c} = \mathbf{R}_2^{T} \cdot \left(\mathbf{R}_{12} + \mathbf{R}_{20}\right), \quad \mathbf{M}_{O_2}^{c} = \mathbf{R}_2^{T} \cdot (\mathbf{M}_{20} + \mathbf{M}_{12}) \tag{45}$$

$$\mathbf{R}_{12} = \left[X_{12} \mid Y_{12} \mid Z_{12}\right]^{T}, \quad \mathbf{R}_{20} = \left[0 \mid Y_{20} \mid Z_{20}\right]^{T} \tag{46}$$

$$\mathbf{M}_{12} = \left[M_{12,x} \mid M_{12,y} \mid 0\right]^{T}, \quad \mathbf{M}_{20} = \left[M_{20,x} \mid M_{20,y} \mid M_{20,z}\right]^{T} \tag{47}$$

The torque of constraint forces may be written under following matrix form:

$$\tau_{O_2} = \mathbf{L}_{\lambda_2} \cdot \boldsymbol{\lambda}. \tag{48}$$

In the relation (48) the quantities involved have the following expressions:

$$\mathbf{L}_{\lambda_2} = \mathbf{R}_{2,ext}^{T} \cdot \left[\mathbf{A}_3 \mid \mathbf{A}_4\right] \tag{49}$$

$$\mathbf{R}_{2,ext}^{T} = \left[\begin{array}{c|c} \mathbf{R}_2^{T} & \underset{3\times3}{\mathbf{0}} \\ \hline \underset{3\times3}{\mathbf{0}} & \mathbf{R}_2^{T} \end{array}\right] \tag{50}$$

$$\mathbf{A}_3 = -\mathbf{A}_1, \quad \mathbf{A}_4 = \left[\begin{array}{c|c} \mathbf{N}_2 & \underset{3\times3}{\mathbf{0}} \\ \hline \underset{3\times3}{\mathbf{0}} & \mathbf{I}_3 \end{array}\right], \quad \mathbf{N}_2 = \left[\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]^{\mathrm{T}}. \tag{51}$$

The expression of "$\boldsymbol{\lambda}$" is given by the relations (22), (23) and (24).

4. WRITING THE EQUATIONS OF MOTION FOR THE MECHANICAL SYSTEM

The differential equations of motion which describe the motion of the elastic mechanical system may be written under the following matrix form:

$$\mathbf{M}\cdot\dot{\mathbf{v}} = -\boldsymbol{\Omega}\cdot\mathbf{v} + \mathbf{Q} + \boldsymbol{\tau} \tag{52}$$

$$\dot{\mathbf{x}} = \mathbf{D}\cdot\mathbf{v}. \tag{53}$$

In the relations (52) and (53) the quantities involved have the following expressions:

$$\dot{\mathbf{v}} = \left[\dot{\mathbf{v}}_1^{\mathrm{T}} \mid \dot{\mathbf{v}}_2^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \mathbf{v} = \left[\mathbf{v}_1^{\mathrm{T}} \mid \mathbf{v}_2^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \mathbf{Q} = \left[\mathbf{Q}_1^{\mathrm{T}} \mid \mathbf{Q}_2^{\mathrm{T}}\right]^{\mathrm{T}} \tag{54}$$

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{M}_{\mathrm{O}_1} & \underset{6\times6}{\mathbf{0}} \\ \hline \underset{6\times6}{\mathbf{0}} & \mathbf{M}_{\mathrm{O}_2} \end{array}\right], \quad \boldsymbol{\Omega} = \left[\begin{array}{c|c} \boldsymbol{\Omega}_1 & \underset{6\times6}{\mathbf{0}} \\ \hline \underset{6\times6}{\mathbf{0}} & \boldsymbol{\Omega}_2 \end{array}\right] \tag{55}$$

$$\boldsymbol{\tau} = \left[\boldsymbol{\tau}_{\mathrm{O}_1}^{\mathrm{T}} \mid \boldsymbol{\tau}_{\mathrm{O}_2}^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{56}$$

The expression of the constraint forces torque given by the relation (56) may be written under the following matrix form:

$$\boldsymbol{\tau} = \mathbf{L}_\lambda\cdot\lambda, \quad \mathbf{L}_\lambda = \left[\mathbf{L}_{\lambda_1}^{\mathrm{T}} \mid \mathbf{L}_{\lambda_2}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{57}$$

$$\dot{\mathbf{x}} = \left[\dot{\mathbf{x}}_1^{\mathrm{T}} \mid \dot{\mathbf{x}}_2^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \dot{\mathbf{x}}_1 = \mathbf{D}_1\cdot\mathbf{v}_1, \quad \dot{\mathbf{x}}_2 = \mathbf{D}_2\cdot\mathbf{v}_2 \tag{58}$$

$$\dot{\mathbf{x}}_1 = \left[\dot{x}_{\mathrm{O}_1} \mid \dot{y}_{\mathrm{O}_1} \mid \dot{z}_{\mathrm{O}_1} \mid \dot{\psi}_1 \mid \dot{\theta}_1 \mid \dot{\varphi}_1\right]^{\mathrm{T}} \tag{59}$$

$$\dot{\mathbf{x}}_2 = \left[\dot{x}_{\mathrm{O}_2} \mid \dot{y}_{\mathrm{O}_2} \mid \dot{z}_{\mathrm{O}_2} \mid \dot{\psi}_2 \mid \dot{\theta}_2 \mid \dot{\varphi}_2\right]^{\mathrm{T}} \tag{60}$$

$$\mathbf{D}_1 = \left[\begin{array}{c|c} \mathbf{R}_1 & \underset{3\times3}{\mathbf{0}} \\ \hline \underset{3\times3}{\mathbf{0}} & \mathbf{T}_1 \end{array}\right], \qquad \mathbf{T}_1 = \mathbf{P}_2^{-1} \cdot \mathbf{P}_1^{-1} \tag{61}$$

$$\mathbf{P}_1 = \left[\begin{array}{c|c|c} -\cos(\varphi_1) & \sin(\varphi_1) & 0 \\ \hline \sin(\varphi_1) & \cos(\varphi_1) & 0 \\ \hline 0 & 0 & 1 \end{array}\right], \qquad \mathbf{P}_2 = \left[\begin{array}{c|c|c} \sin(\theta_1) & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline \cos(\theta_1) & 0 & 1 \end{array}\right] \tag{62}$$

$$\mathbf{D}_2 = \left[\begin{array}{c|c} \mathbf{R}_2 & \underset{3\times3}{\mathbf{0}} \\ \hline \underset{3\times3}{\mathbf{0}} & \mathbf{T}_2 \end{array}\right], \qquad \mathbf{T}_2 = \mathbf{P}_4^{-1} \cdot \mathbf{P}_3^{-1} \tag{63}$$

$$\mathbf{P}_3 = \left[\begin{array}{c|c|c} \sin(\varphi_2) & \cos(\varphi_2) & 0 \\ \hline \cos(\varphi_2) & -\sin(\varphi_2) & 0 \\ \hline 0 & 0 & 1 \end{array}\right], \qquad \mathbf{P}_4 = \left[\begin{array}{c|c|c} \sin(\theta_2) & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline \cos(\theta_2) & 0 & 1 \end{array}\right]. \tag{64}$$

5. CONSTRAINT FORCES ELIMINITATION

Between the kinematical parameters of the rigid solids that make up the system the following relations may be written:

$$\mathbf{L}_\lambda^{\mathrm{T}} \cdot \mathbf{v} = \underset{1\times10}{\mathbf{0}} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}}_{1\times10}{}^{\mathrm{T}}. \tag{65}$$

If we derive the relation (65) with respect to time we will obtain:

$$\mathbf{L}_\lambda^{\mathrm{T}} \cdot \dot{\mathbf{v}} + \dot{\mathbf{L}}_\lambda^{\mathrm{T}} \cdot \mathrm{v} = \underset{1\times10}{\mathbf{0}}. \tag{66}$$

By pre-multiplication to the left of the relation (52) with the matrix $\mathbf{M}^{-1}$ we will obtain:

$$\dot{\mathbf{v}} = -\mathbf{M}^{-1} \cdot \mathbf{\Omega} \cdot \mathbf{v} + \mathbf{M}^{-1} \cdot \mathbf{Q} + \mathbf{M}^{-1} \cdot \mathbf{L}_\lambda \cdot \lambda. \tag{67}$$

We will multiply to the left the relation (67) by the matrix $\mathbf{L}_\lambda^{\mathrm{T}}$, we will take into account the relation (66) and we will obtain:

$$\boldsymbol{\lambda} = \mathbf{B}_1^{-1} \cdot (\mathbf{B}_3 \cdot \mathbf{v} - \mathbf{B}_2 \cdot \mathbf{Q}). \tag{68}$$

In the relation (68) the expressions of the matrices $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ are the followings:

$$\mathbf{B}_1 = \mathbf{L}_\lambda^{\mathrm{T}} \cdot \mathbf{M}^{-1} \cdot \mathbf{L}_\lambda \tag{69}$$

$$\mathbf{B}_2 = \mathbf{L}_\lambda^{\mathrm{T}} \cdot \mathbf{M}^{-1} \tag{70}$$

$$\mathbf{B}_3 = \mathbf{L}_\lambda^{\mathrm{T}} \cdot \mathbf{M}^{-1} \cdot \boldsymbol{\Omega} - \dot{\mathbf{L}}_\lambda^{\mathrm{T}}. \tag{71}$$

The derivation of the matrix $\mathbf{L}_\lambda$ with respect to time may be performed as follows:

$$\dot{\mathbf{L}}_\lambda = \left[\dot{\mathbf{L}}_{\lambda_1}^{\mathrm{T}} \;\vdots\; \dot{\mathbf{L}}_{\lambda_2}^{\mathrm{T}} \right]^{\mathrm{T}} \tag{72}$$

$$\dot{\mathbf{L}}_{\lambda_1} = \tilde{\boldsymbol{\omega}}_{1,\mathrm{ext}}^{\mathrm{T}} \cdot \mathbf{L}_{\lambda_1} \tag{73}$$

$$\tilde{\boldsymbol{\omega}}_{1,\mathrm{ext}} = \begin{bmatrix} \tilde{\boldsymbol{\omega}}_1 & \underset{3\times 3}{\mathbf{0}} \\ \underset{3\times 3}{\mathbf{0}} & \tilde{\boldsymbol{\omega}}_1 \end{bmatrix} \tag{74}$$

$$\dot{\mathbf{L}}_{\lambda_2} = \tilde{\boldsymbol{\omega}}_{2,\mathrm{ext}}^{\mathrm{T}} \cdot \mathbf{L}_{\lambda_2} \tag{75}$$

$$\tilde{\boldsymbol{\omega}}_{2,\mathrm{ext}} = \begin{bmatrix} \tilde{\boldsymbol{\omega}}_2 & \underset{3\times 3}{\mathbf{0}} \\ \underset{3\times 3}{\mathbf{0}} & \tilde{\boldsymbol{\omega}}_2 \end{bmatrix}. \tag{76}$$

Substituting the relation (68) in (52) and taking into account the relation (53) we will obtain a system of twenty-four first order differential equations with twenty four unknowns which may be solved by using numerical integration methods and in this way we will determine the position kinematical parameters and the speeds of the rigid solids which compound the elastic mechanical system.

The final differential equations system which describe the motion of the elastic mechanical system in the presence of constraints may be written under matrix form as follows:

$$\mathbf{M} \cdot \dot{\mathbf{v}} = -\boldsymbol{\Omega} \cdot \mathbf{v} + \mathbf{Q} + \mathbf{L}_\lambda \cdot \lambda \tag{77}$$

$$\dot{\mathbf{x}} = \mathbf{D} \cdot \mathbf{v}. \tag{78}$$

In the relation (77) “$\boldsymbol{\lambda}$” is given buy the relation (68).

6. EQUATIONS OF MOTION IN PARTICULAR CASES

Let’s suppose that the first solid rigid body describes a plane motion and the second one is in translation. The differential equations that describe the motion of the mechanical system may be written as followings:

$$\widetilde{\mathbf{M}} \cdot \dot{\mathbf{q}} = \widetilde{\mathbf{Q}} + \widetilde{\boldsymbol{\tau}} \tag{79}$$

$$\dot{\widetilde{\mathbf{x}}} = \widetilde{\mathbf{R}} \cdot \mathbf{q} \tag{80}$$

$$\widetilde{\mathbf{M}} = \left[\begin{array}{c|c} \widetilde{\mathbf{M}}_{O_1} & \underset{3\times 2}{\mathbf{0}} \\ \hline \underset{2\times 3}{\mathbf{0}} & \widetilde{\mathbf{M}}_{O_2} \end{array}\right] \tag{81}$$

$$\widetilde{\mathbf{M}}_{O_1} = \left[\begin{array}{c|c|c} m_1 & 0 & 0 \\ \hline 0 & m_1 & m_1 \cdot \xi_1 \\ \hline 0 & m_1 \cdot \xi_1 & J_{z_1} \end{array}\right], \quad \widetilde{\mathbf{M}}_{O_2} = m_2 \cdot \mathbf{I}_2 \tag{82}$$

$$\dot{\mathbf{q}} = \left[\dot{v}_{O_1 x_1} \mid \dot{v}_{O_1 y_1} \mid \dot{\omega}_{z_1} \mid \dot{v}_{O_2 x_2} \mid \dot{v}_{O_2 y_2} \right]^{\mathrm{T}} \tag{83}$$

$$\tilde{\mathbf{Q}} = \left[\tilde{\mathbf{Q}}_1^{\mathrm{T}} \mid \tilde{\mathbf{Q}}_2^{\mathrm{T}} \right]^{\mathrm{T}} \tag{84}$$

$$\tilde{\mathbf{Q}}_1 = \left[m_1 \cdot g \cdot \cos(\varphi_1) \mid -m_1 \cdot g \cdot \cos(\varphi_1) \mid -m_1 \cdot g \cdot \xi_1 \cdot \sin(\varphi_1) \right]^{\mathrm{T}} \tag{85}$$

$$\tilde{\mathbf{Q}}_2 = \left[-k \cdot x_{O_2} - c \cdot \dot{x}_{O_2} \mid -m_2 \cdot g \right]^{\mathrm{T}} \tag{86}$$

$$\widetilde{\boldsymbol{\tau}} = \widetilde{\mathbf{L}}_{\lambda} \cdot \widetilde{\boldsymbol{\lambda}} \tag{87}$$

$$\tilde{\boldsymbol{\lambda}} = \left[X_{12} \mid Y_{12} \mid Y_{20} \right]^{\mathrm{T}} \tag{88}$$

$$\widetilde{\boldsymbol{\lambda}} = \widetilde{\mathbf{B}}_1^{-1} \cdot \left(\widetilde{\mathbf{B}}_3 \cdot \widetilde{\mathbf{q}} - \widetilde{\mathbf{B}}_2 \cdot \widetilde{\mathbf{Q}} \right) \tag{89}$$

$$\tilde{\mathbf{B}}_1 = \tilde{\mathbf{L}}_{\lambda}^{\mathrm{T}} \cdot \tilde{\mathbf{M}}^{-1} \cdot \tilde{\mathbf{L}}_{\lambda} \tag{90}$$

$$\tilde{\mathbf{B}}_2 = \tilde{\mathbf{L}}_{\lambda}^{\mathrm{T}} \cdot \tilde{\mathbf{M}}^{-1} \tag{91}$$

$$\tilde{\mathbf{B}}_3 = -\dot{\tilde{\mathbf{L}}}_{\lambda}^{\mathrm{T}} \cdot \mathbf{q} \tag{92}$$

$$\mathbf{q} = \left[v_{O_1 x_1} \mid v_{O_1 y_1} \mid \omega_{z_1} \mid v_{O_2 x_2} \mid v_{O_2 y_2} \right]^{\mathrm{T}} \tag{93}$$

$$\tilde{\mathbf{L}}_{\lambda} = \left[\begin{array}{c|c|c} \sin(\varphi_1) & -\cos(\varphi_1) & 0 \\ \hline \cos(\varphi_1) & \sin(\varphi_1) & 0 \\ \hline 0 & 0 & 0 \\ \hline -1 & 0 & 0 \\ \hline 0 & -1 & 1 \end{array}\right] \tag{94}$$

$$\dot{\tilde{\mathbf{L}}}_{\lambda} = \left[\begin{array}{c|c|c} \cos(\varphi_1)\cdot\omega_{z_1} & \sin(\varphi_1)\cdot\omega_{z_1} & 0 \\ \hline -\sin(\varphi_1)\cdot\omega_{z_1} & \cos(\varphi_1)\cdot\omega_{z_1} & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{95}$$

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{R}} \cdot \mathbf{q} \tag{96}$$

$$\dot{\tilde{\mathbf{x}}} = \left[\dot{x}_{O_1} \mid \dot{y}_{O_1} \mid \dot{\varphi}_1 \mid \dot{x}_{O_2} \mid \dot{y}_{O_2}\right]^{T} \tag{97}$$

$$\tilde{\mathbf{R}} = \left[\begin{array}{c|c} \tilde{\mathbf{R}}_1 & \underbrace{\mathbf{0}}_{3\times 2} \\ \hline \underbrace{\mathbf{0}}_{2\times 3} & \mathbf{I}_2 \end{array}\right] \tag{98}$$

$$\tilde{\mathbf{R}}_1 = \left[\begin{array}{c|c|c} \sin(\varphi_1) & \cos(\varphi_1) & 0 \\ \hline -\cos(\varphi_1) & \sin(\varphi_1) & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{99}$$

The initial conditions of the problem are the following:

$$v^0_{O_1x_1} = v^0_{O_1y_1} = v^0_{O_2x_2} = v^0_{O_2y_2} = 0 \;\; [\text{meter/second}] \tag{100}$$

$$\omega^0_{z_1} = 0 \;\; [\text{radian/second}] \tag{101}$$

$$x^0_{O_1} = y^0_{O_1} = x^0_{O_2} = y^0_{O_2} = 0 \;\; [\text{meter}] \tag{102}$$

$$\varphi_{1,0} = \pi/2 \;\; [\text{radian}]. \tag{103}$$

Using MATLAB software it was designed a computing program which solves the differential equations system. The computing program contains the following input data:

$$m_1 = 1 \text{ kg}, \quad m_2 = 5 \text{ kg} \tag{104}$$

$$J_{x_1} = J_{y_1} = J_{z_1} = J_{x_2} = J_{y_2} = J_{z_2} = 1\text{kg} \cdot \text{m}^2 \tag{105}$$

$$k = 250 \text{ N/m} \tag{106}$$

$$\varphi_{1,0} = \pi/2 \,\text{kg/second} \;. \tag{107}$$

The results are presented in the Figs. 3, 4 and 5.

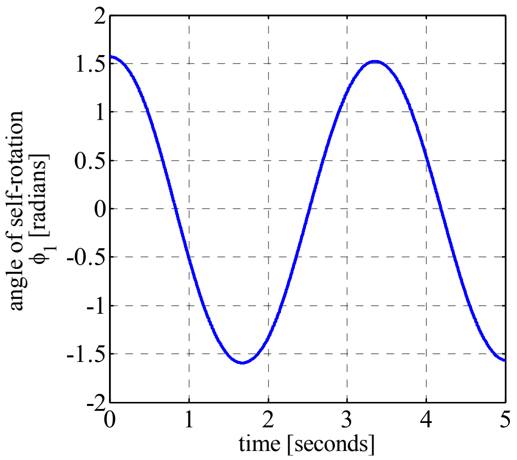

Fig. 3 – Variation of the self-rotation angle φ_1.

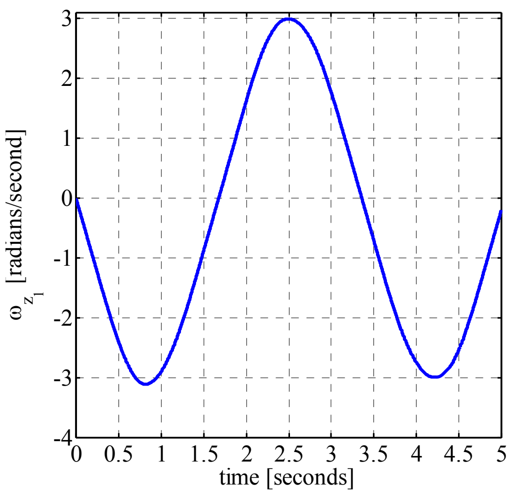

Fig. 4 – Variation of the angular velocity ω_{z_1} .

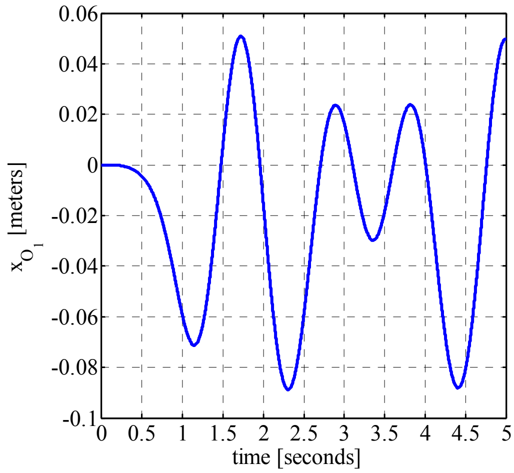

Fig. 5 – Variation of the linear displacement.

7. CONCLUSIONS

The numerical method presented in the paper presents a high degree of generality and for this reason it may be used to perform the dynamic study of any mechanical system no matter how complex it might be. The mechanical system whose dynamics has been presented in this paper is only an example so that the numerical method shown in the paper could be understood easily.

Using the numerical method presented in the paper we can determine for each moment of the movement both kinematical parameters values and the values of the constraint forces.

The constraint forces elimination from the differential equations system may be performed without being necessary to determine the expression of the orthogonal complement [6, 7] or natural orthogonal complement [4].

Analyzing the relations (1) and (25) it may be observed that they represent the algebraic sum of three torques, namely: inertial forces torque, active forces torque and constraint forces torque.

The paper also presents a method for inferring linking relations between kinematical parameters of the rigid solids that make up the system and which are expressed by the matrix relation (65). Expressing the torque of the constraint forces under matrix form we can obtain very easily the link relation between kinematical parameters of the rigid solids that make up the system.

A follow-up of the dynamic study presented in this work could be the determination of the constraint forces.

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