OPTIMAL AUXILIARY FUNCTIONS METHOD FOR THIN FILM FLOW OF A FOURTH GRADE FLUID DOWN A VERTICAL CYLINDER

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Abstract. A new approximate analytical procedure namely the Optimal Auxiliary Functions Method (OAFM) is proposed and has been applied to thin film flow of a fourth grade fluid down a vertical cylinder. The main advantage of our approach consists in providing a convenient way to control the convergence of the approximate solution in a very rigorous way. This methods, however, does not depend upon any small or large parameters in comparison with other methods. A very good agreement was found between approximate and numerical solution which reveals that OAFM is more effective, very efficient, accurate and easy to use.

Key words: Optimal Auxiliary Functions Method, thin film flow, fourth grade fluid, nonlinear problem.

1. INTRODUCTION

It is well known that the subject of non-Newtonian fluids is very popular and is an area of active research especially in industry and engineering problems. Examples of non-Newtonian fluids include microchip production, performance of lubricants, food processing, movements of biological fluids, wire and fiber coating, paper production, gaseous diffusion and so on [1,2]. These fluids are described by a nonlinear relationship between stress and the rate of deformation tensors and therefore several models have been proposed. It is very difficult to suggest a single-model which exhibits all properties of non-Newtonian fluids. As a consequence several fluid models have been proposed to predict the non-Newtonian behaviour of various types of materials. Fourth grade fluid is one of the important fluids and its equation is based on strong theoretical foundations, where relation between stress and strain is not linear. Some experiments may be well described by the fluids on the order four. Because the exact solutions of these equations are difficult to achieve, approximate analytical and numerical methods are widely used to solve nonlinear differential equations modelling such physical phenomena. There exists some analytical approaches such as the Lindstedt-Poincare method, the KBM method, the Adomian Decomposition Method, the elliptic perturbation method, the

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harmonic balance method [3-5], or some iteration procedures [6,7]. Most of the perturbation methods unfortunately require the inclusion of a small parameter in the equation, but this parameter is not already specified in the equation and it is artificially introduced and finally equated to unity to obtain the solution of the original problem. Considerable efforts have been made to study strongly nonlinear non-Newtonian fluids for various geometrical configurations via analytical techniques. Some developments in this direction are discussed by different investigators. Among them Khalid and Vafai [8] discussed hydrodynamic squeezed flow and heat transfer over a sensor surface. Miladinova et al [9] studied this film flow over a power law liquid falling from an inclined plate where it was observed that saturation of nonlinear interaction occur in a permanent finite amplitude wave. Sajid at al. [2] investigated the sleep effects of thin film flow grade fluid down a vertical cylinder using the homotopy analysis method. Analytic solution for thin film flow of a fourth grade fluid down a vertical cylinder is presented by Hayat and Sajid [10]. Gul et al. [11] investigated effects of sleep condition on thin film flow of third grade fluids for lifting and drainage problem under the condition of constant viscosity. The homotopy perturbation method and the traditional perturbation method are applied by Siddiqui et al. [12] to the nonlinear equations modelling thin film flow of a fourth grade fluid falling in the outer surface of an infinitely long vertical cylinder. For other studies see [13-17].

The main objective of the present research is to study the fourth grade fluid type applying the optimal auxiliary function method in order to analyse the nonlinear behaviour of thin film flow down a vertical cylinder. The results obtained by means of the OAFM where compared with those obtained by numerical simulation and was observed to be in very good agreement. Our procedure is not valid only for small parameters but also provides us with a convenient way to control and adjust the convergence of approximate analytical solution and demonstrates the validity and great potential to solve a large number of nonlinear problems in science and engineering.

2. GOVERNING EQUATIONS OF THIN FILM FLOW OF A FOURTH GRADE FLUID DOWN A VERTICAL CYLINDER

In what follows we consider a non-Newtonian fluid of fourth grade falling on the outside surface of an infinitely long vertical cylinder of radius *R*. The flow is considered in thin, uniform axisymmetric film with thickness δ, in contact with stationary air. The velocity field is of the form [10,12-17]:

$$v = [0, 0, u(r)] \tag{1}$$

In cylindrical coordinates we have

$$\frac{\partial p}{\partial r} = (2\alpha_1 + \alpha_2)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right] + \frac{4}{r}(\gamma_3 + \gamma_4 + \gamma_5 + \frac{1}{2}\gamma_6)\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^4\right] \tag{2}$$

$$\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) + \frac{2}{r}(\beta_2 + \beta_3)\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3\right] + \rho g \tag{3}$$

$$\frac{\partial p}{\partial \theta} = 0 \tag{4}$$

where $p \neq p(z)$ is the pressure. From Eq. (3) it is clear that

$$r\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{\mathrm{d}u}{\mathrm{d}r} + \frac{2(\beta_2 + \beta_3)}{\mu}\left[3r\left(\frac{\partial u}{\partial r}\right)^2\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^3\right] + \frac{\rho g}{\mu}r = 0 \tag{5}$$

The corresponding boundary conditions are

$$u(R) = 0, \quad \frac{\mathrm{d}u}{\mathrm{d}r}(R + \delta) = 0 \tag{6}$$

Defining

$$\eta = \frac{r}{R}, \quad f = \frac{R}{\nu}u, \quad k = \frac{gR^3}{\nu^2}, \quad \beta = \frac{\mu(\beta_2 + \beta_3)}{R^4\rho^2}, \quad d = 1 + \frac{\delta}{R}, \tag{7}$$

Eq. (5) and (6) reduces to

$$\eta\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \frac{\mathrm{d}f}{\mathrm{d}\eta} + k\eta + 2\beta\left[\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^3 + 3\eta\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right] = 0 \tag{8}$$

$$f(1) = 0, \quad f'(d) = 0 \tag{9}$$

The dynamical system described by Eqs. (8) and (9) will be solved using the optimal auxiliary functions method [18].

2. BASIC IDEAS OF THE OPTIMAL AUXILIARY FUNCTIONS METHOD

We consider the most general form of a nonlinear differential equation as

$$L\left[f(\eta)\right]+N\left[f(\eta)\right]=0 \tag{10}$$

in which L is a linear operator, $f(\eta)$ is an unknown function, N is a nonlinear operator and D is the domain of interest. The initial/boundary conditions are known as

$$B\left(f(\eta),\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\right)=0 \tag{11}$$

For Eqs. (10) and (11) we demand an approximate solution $\tilde{f}(\eta)$ which contains only two components:

$$\tilde{f}(\eta)=f_0(\eta)+f_1(\eta,C_i),\quad i=1,2...,n \tag{12}$$

where C_i, $i=1,2,\ldots,n$ are unknown parameters at this moment.

Substituting Eq. (12) into Eq. (10) we obtain

$$L\left[f_0(\eta)\right]+L\left[f_1(\eta,C_i)\right]+N\left[f_0(\eta)+f_1(\eta,C_i)\right]=0 \tag{13}$$

The initial approximation $f_0(\eta)$ can be determined from the linear equation

$$L\left[f_0(\eta)\right]=0,\quad B\left(f_0(\eta),\frac{\mathrm{d}f_0(\eta)}{\mathrm{d}\eta}\right)=0 \tag{14}$$

and the first approximation is obtained from the remaining equation

$$L\left[f_1(\eta,C_i)\right]+N\left[f_0(\eta)+f_1(\eta,C_i)\right]=0,\quad B\left(f_1(\eta),\frac{\mathrm{d}f_1(\eta)}{\mathrm{d}\eta}\right)=0 \tag{15}$$

In general Eq. (15) is a nonlinear differential equation which is often very difficult to solve. At this stage, the nonlinear term from Eq. (15) is expanded in the form

$$N\left[f_0(\eta)+f_1(\eta,C_i)\right]=N[f_0(\eta)]+\sum_{k\geq 1}\frac{F_1^k(\eta,C_i)}{k!}N^{(k)}[f_0(\eta)] \tag{16}$$

where $N^{(k)}=\dfrac{\mathrm{d}^k N}{\mathrm{d}\eta^k}$. In order to avoid the difficulties that appear in solving the nonlinear differential equation (15) and to accelerate the convergence of the first approximation and implicitly of the approximate solution $\tilde{f}(\eta, C_i)$, instead of the last term arising in Eq. (15) we propose another expression, such that Eq. (15) can be written in a new form

$$L[f_1(\eta, C_i)] + A_1\left(f_0(\eta, C_j)\right)P[N(f_0(\eta))] + A_2\left(f_0(\eta), C_k\right) = 0,$$
$$B\left(f_1(\eta), \frac{df_1(\eta)}{d\eta}\right) = 0 \qquad (17)$$

where A_1 and A_2 are arbitrary auxiliary functions depending on the initial approximation $f_0(\eta)$ and several unknown parameters C_j and C_k, $j=1,2,\ldots,p$, $k=p+1,\ p+2,\ldots,n$, $i=j+k$. $P[N(f_0(\eta))]$ means a part of the operator $N(f_0(\eta))$. The auxiliary functions A_1 and A_2 called optimal auxiliary functions are not unique and can be chosen of the same form like $f_0(\eta)$ or of the form of $N(f_0(\eta))$ or combinations of $f_0(\eta)$ and $N(f_0(\eta))$.

The unknown parameters C_j and C_k can be optimally identified by means of different methods. Among them is minimizing the square residual error

$$J(C_i, C_k) = \int_{(D)} R^2(\eta, C_i, C_k)\,\mathrm{d}\eta \qquad (18)$$

where $R(\tau, C_j, C_k) = L[\tilde{f}(\eta, C_i)] + N[\tilde{f}(\eta, C_i)]$, $i=j+k$, $j=1,2,\ldots,p$, $k=p+1,\ldots,n$. The condition of the minimization of the residual are

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots \frac{\partial J}{\partial C_n} = 0 \qquad (19)$$

By this novel approach the approximate solution (12) is well determined. Also, the parameters C_i (called convergence-control parameters) can be obtained by means of Ritz method, Galerkin method, Kantorowich method, the collocation method, and so on [16, 17].

Our novel procedure proves to be a powerful tool for solving nonlinear problems not depending on small or large parameters. It should be emphasized that our method contains the optimal auxiliary functions A_1 and A_2 which provides provide us with a simple way to adjust and control the convergence of the approximate solutions after only one iteration.

4. APPROXIMATE SOLUTION OF THE EQS (8) AND (9)

To apply our procedure to obtain an approximate solution of Eqs. (8) and (9) we consider the linear operator for the Eq. (8) in the form

$$L[f(\eta)] = \eta f''(\eta) + f'(\eta) + k\eta \tag{20}$$

where $f'(\eta) = \mathrm{d}f / \mathrm{d}\eta$ and the nonlinear operator

$$N[f(\eta)] = 2\beta\left[f'^3(\eta) + 3\eta f'^2(\eta) f''(\eta)\right] \tag{21}$$

The initial approximation $f_0(\eta)$ is determined from Eq. (14) which becomes

$$\eta f_0'' + f_0' + k\eta = 0, \quad f_0(1) = 0, \quad f_0'(d) = 0 \tag{22}$$

The solution of Eq. (22) is

$$f_0'(\eta) = \frac{k}{2}\left(\frac{d^2}{\eta} - \eta\right) \tag{23}$$

The nonlinear operator (21) for the initial approximation (23) becomes

$$N[f_0'(\eta)] = \frac{1}{4}\beta k^3\left(\frac{d^6}{\eta^2} - 3d^4 + 3d^2\eta^2 - \eta^4\right) \tag{24}$$

or

$$N[f_0'(\eta)] = 2\beta(\eta f_0'^3)' \tag{25}$$

Eq. (17) can be written in the form

$$\eta f_1''(\eta) + f_1'(\eta) + k\eta + A_1 P\left[2\beta(\eta f_0'^3)'\right] + A_2 = 0, \quad f_1(1) = 0, \quad f_1'(d) = 0 \tag{26}$$

The Eq. (26) can be rewritten as, if we consider $P\left[2\beta(\eta f_0'^3)'\right] = 2\beta(\eta f_0'^3)'$

$$\left(\eta f_1'\right)' + \left(\frac{1}{2}k\eta^2\right)' + A_1 2\beta\left(\eta f_0'^3\right)' + A_2 = 0 \tag{27}$$

For the functions A_1 and A_2 we propose the following forms

$$A_1 = A_1(\eta, C_j) = -1, \quad A_2 = A_2(\eta, C_k) = -\left[\eta g(\eta, C_i)\right]' - \left(\frac{k}{2}\eta^2\right)' \tag{28}$$

where $g(\eta, C_i)$ is an unknown function at this moment. Taking into account Eq. (28), from Eq. (27) by integrating we obtain

$$\eta f_1' = 2\beta(\eta f_0'^3) + \eta g(\eta, C_i) + C \tag{29}$$

From Eqs. (22), (26) and (29) we obtain for $\eta = d$

$$C = 0, \quad g(d, C_i) = 0 \tag{30}$$

From Eqs. (29) and (30) results

$$f_1'(\eta) = 2\beta f_0'^3 + g(\eta), \quad g(d) = 0 \tag{31}$$

Having the freedom to choose

$$g(\eta) = C_1\left(\frac{d^2}{\eta} - \eta\right) + C_2\left(\frac{d^2}{\eta} - \eta\right)^2 + C_3\left(\frac{d^2}{\eta} - \eta\right)^3 + \\ + C_4\left(\frac{d^2}{\eta} - \eta\right)^4 + C_5\left(\frac{d^2}{\eta} - \eta\right)^5 \tag{32}$$

from Eqs. (23), (32) and (31) we have

$$f'(\eta) = C_1\left(\frac{d^2}{\eta} - \eta\right) + C_2\left(\frac{d^2}{\eta} - \eta\right)^2 + \left(\frac{\beta k^3}{4} + C_3\right)\left(\frac{d^2}{\eta} - \eta\right)^3 + \\ + C_4\left(\frac{d^2}{\eta} - \eta\right)^4 + C_5\left(\frac{d^2}{\eta} - \eta\right)^5 \tag{33}$$

Finally from Eqs. (23), (33) and (12) we can obtain the differentiation of the approximate solution of Eqs. (8) and (9) in the form

$$\tilde{f}'(\eta) = \left(\frac{k}{2} + C_1\right)\left(\frac{d^2}{\eta} - \eta\right) + C_2\left(\frac{d^2}{\eta} - \eta\right)^2 + \left(\frac{\beta k^3}{4} + C_3\right)\left(\frac{d^2}{\eta} - \eta\right)^3 + \\ + C_4\left(\frac{d^2}{\eta} - \eta\right)^4 + C_5\left(\frac{d^2}{\eta} - \eta\right)^5 \tag{34}$$

From Eq. (34) it is easy to obtain the approximate solution of Eqs. (8) and (9) by integration and having in view that $\tilde{f}(1) = f(1) = 0$.

5. NUMERICAL EXAMPLE

For the case when $k=1$, $\beta=8$, $d=1.5$, we obtain

$$C_1 = 0.0446457909117; \qquad C_2 = -0.58304317787714;$$
$$C_3 = -1.5722449276332089; \quad C_4 = -0.160737450874938; \qquad (34)$$
$$C_5 = 0.0214437762867209.$$

In [10] $\beta \geq 0.3$ is considered a parameter corresponding to strong non-linearity. Therefore the explicit analytic expression given by Eq. (34) for the convergence-control parameters given by (35), the first order approximate solution becomes

$$f'(\eta) = 0.5446457909117669\left(\frac{2.25}{\eta}-\eta\right) - 0.583043177877\left(\frac{2.25}{\eta}-\eta\right)^2 +$$
$$+0.4277550723667911\left(\frac{2.25}{\eta}-\eta\right)^3 - 0.160737450874938\left(\frac{2.25}{\eta}-\eta\right)^4 + \qquad (36)$$
$$+0.0214437762867209\left(\frac{2.25}{\eta}-\eta\right)^5$$

In Table 1 is presented a comparison between the present solution obtained from formula (36) and the numerical solution of Eqs. (8) and (9).

It can be seen that the solution obtained through OAFM is near identical with that given by the numerical results, demonstrating a very good accuracy.

Table 1

Comparison of analytical and numerical results

η	$\tilde{f}'(\eta)$, Eq. (36)	$f'(\eta)$, numerical	$\varepsilon = \left\lvert f'(\eta) - \tilde{f}'(\eta)\right\rvert$
1.00	0.2782771936982422	0.27868677728446	$4.09\ 10^{-4}$
1.05	0.2613391820243754	0.2619592020457328	$6.20\ 10^{-4}$
1.10	0.243037748916253	0.243037749162644	$2.46\ 10^{-10}$
1.15	0.2237293973885208	0.223832705241852	$1.03\ 10^{-4}$
1.20	0.2031782753168658	0.203219171567154	$4.08\ 10^{-5}$
1.25	0.180723124261417	0.180659142650312	$6.39\ 10^{-5}$
1.30	0.1554016332342926	0.155372275453134	$2.93\ 10^{-5}$
1.35	0.126040063073303	0.126185641229246	$1.45\ 10^{-4}$
1.40	0.091316354263098	0.091367589247079	$5.12\ 10^{-5}$
1.45	0.0498036954997723	0.048981783573852	$8.21\ 10^{-4}$
1.50	0	0	0

6. CONCLUSIONS

In the present paper a new technique was proposed to solve the nonlinear problem of thin film flow of a fourth grade fluid down a vertical cylinder. Our approach is very effective and has a distinct advantage over usual approximation methods in that it proves to be valid not only for weekly nonlinear equations, but also for highly complex nonlinear ones. It was observed that we need only one iteration to obtain a remarkable accuracy. The results obtained by means of a OAFM reveal very good agreement with the numerical results. Convergence and errors are remarkable and this procedure provides a convenient way to control the convergence of approximate solution. This is realized using the auxiliary functions $A_1(\eta,C_j)$ and $A_2(\eta,C_j)$ (not unique) used for adjusting and controlling the convergence of solutions. The convergence-control parameters C_i are determined by minimizing the residual square errors, which is a very rigorous and effective procedure.

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