ELASTO-PLASTIC FINITE DEFORMATION MODELS FOR ANISOTROPIC DAMAGE

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Abstract. The paper deals with finite deformation elasto-plastic models with anisotropic damage. The models are mathematically coherent and consistent with free energy imbalance principle. One model is based on the existence of deformation-like damage tensor, $\mathbf{F}^d$, which characterizes the passage from the undamaged, stress free configuration to the damaged and stress free configuration. The other one involves a symmetric defect density tensor, which is a measure of non-metricity of the so-called plastic connection, there is a gradient-like model. The constitutive and evolution equations are derived to be compatible with the free energy imbalance principle, which has been reformulated to be applicable to the elasto-plastic materials with damaged structure. We assumed that the plastic flow and the development of damage, i.e. the micro voids and micro cracks, are distinct irreversible mechanisms during the deformation process.

Key words: tensorial damage variables, elastic, plastic, free energy imbalance, evolution equations.

1. INTRODUCTION

The paper deals with the mathematical description of macroscopic behaviour of materials deteriorated by the defects existing at the microscopic level. Two types of problems arise when the state of damaged material is discussed. The first problem is related to the physical nature and the mathematical description of the damage variables, while the second type concerns the elaboration of the constitutive framework. The scalar damage variables were extensively used in continuum isotropic damage, while various tensorial variables allowed to elaborate elasto-plastic models for anisotropic damage in materials with structural defects, see for instance Cleja-Ţigoiu [8]. We emphasize some specification of anisotropic damage measures which are closely related to our approaches to damage. Murakami [21] described the anisotropic damage

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by a second order symmetric tensor, $\mathbf{D}$,

$$\mathbf{D} = \sum_{i=1}^{i=3} D_i \mathbf{n}_i \otimes \mathbf{n}_i, \tag{1}$$

D_i were interpreted as the void area density in the plane perpendicular to direction of the damage $\mathbf{n}_i$.

Menzel at al. [20] developed a framework of continuum damage based on the fictitious configuration and the equivalence principle of the free energy in the fictitious and intermediate configurations, respectively. The intermediate configuration (which is called the local relaxed configuration in the description given by Cleja-Ţigoiu and Soós [4]) is associated with the multiplicative decomposition of the deformation gradient, $\mathbf{F}$, into its plastic, $\mathbf{F}^p$, and elastic, $\mathbf{F}^e$, parts, namely

$$\mathbf{F}(\mathbf{X},t) = \mathbf{F}^e(\mathbf{X},t)\mathbf{F}^p(\mathbf{X},t), \quad \mathbf{F}(\mathbf{X},t) = \nabla\, \chi(\mathbf{X},t), \tag{2}$$

where $\chi(\cdot,t)$ is the motion function at time t and $\mathbf{X}$ is a material point of the body.

Brünig [1] and Brünig and Ricci [2] provided a finite strain framework, using the *multiple undamaged (fictitious) configurations* and specific metric coefficients to describe measures of damage. In the damage-coupled elasto-plastic models, the plastic flow and damage processes, which are dissipative, are treated by the constitutive models, as different in their nature and effects on the mechanical properties of the materials and structures. The free energies involved in plastic flow and damage processes are postulated to be independent and have been introduced separately with respect to the *fictitious undamaged configurations*.

Menzel at al. [20] introduced a second order tensor $\mathbf{F}^d$, called the *integrity tensor*, which characterizes the passage from an *undamaged (fictitious) configuration* to the intermediate configuration. This tensor field has not been involved in the multiplicative decomposition of the deformation gradient. The damage model proposed by Ekh et al. [14] appeals to the crystal plasticity model and the evolution rule for the damage is formulated with respect to the crystalline slip systems.

Voyiadjis and Kattan [22] introduced a *fourth-order anisotropic damage effect tensor* as a key point in describing the anisotropic damage.

In this paper we propose two damage models for elasto-plastic materials involving second-order tensorial damage variables. The physical motivations for these models as well as the mathematical description of the elasto-plastic models are completely different. In the first model an invertible second order tensor $\mathbf{F}^d$ describes the anisotropic damage state of elasto-plastic materials and has been introduced in the paper by Cleja-Ţigoiu [7]. The anisotropic damage variable $\mathbf{F}^d$ is involved in the multiplicative decomposition of deformation gradient into its components, and is a different concept from the tensorial field $\mathbf{F}^d$ which is viewed like a pure internal state variable by Menzel et al. [20], Ekh et al. [14], [15].

We adopted the point of view concerning the different nature of the plastic and damage mechanisms, motivated by the arguments that can be found in the models proposed by Brünig [1], Brünig and Ricci [2].

Another geometrical motivation to introduce tensorial defect measure is given by Kröner [19] and de Wit [13], for instance. The lattice defects, like point defects, micro voids and micro cracks in the damaged zone are modeled in terms of the tensor fields which characterize the *non-metric property* of the plastic connection. We recall here that the defects of lattice structure, like dislocations and disclinations, can be involved through the Cartan torsion of the so-called plastic connection, under the hypothesis that the connection has *metric property*, see Cleja-Ţigoiu [5], [6].

In this paper we propose a second model of anisotropic damage based on the symmetric tensor, called defect density tensor, which is associated with the measure of non-metricity of the plastic connection, see Cleja-Ţigoiu and Ţigoiu [10].

In Section 2.1 the deformation-like damage tensor $\mathbf{F}^d$ and in Section 2.2 the defect density tensor $\mathbf{h}^d$ were introduced from the kinematic point of view. These tensor fields are related to the appropriate configurations. $\mathbf{F}^d$ realizes the passage from the so-called undamaged and stress free configuration to the damaged and stress free configuration. $\mathbf{h}^d$ is a tensorial field which restores the metricity of the plastic connection. Here the pair of the plastic distortion and the so-called plastic connection, $(\mathbf{F}^p, \overset{(p)}{\mathbf{\Gamma}})$, defines the geometrical structure of the plastically deformed configuration, called sometimes configuration with torsion and denoted by $\mathscr{K}$, see Cleja-Ţigoiu [5], [6].

In Section 3 the free energy imbalance principle, formulated within the second order finite elasto-plasticity by Cleja-Ţigoiu [5], [6], is adapted to materials with damaged structure. The free energy imbalance is a key point in developing constitutive theory as can be seen in the paper by Gurtin [17], for instance. We discuss the development of theory based on the imbalance principle, within the constitutive framework of elasto-plastic materials with damaged structure. The elastic type constitutive equations are derived as direct consequence of the free energy imbalance. In Section 4 the evolution equations of the viscoplastic type, are provided for the damage variables, $\mathbf{F}^d$ and $\mathbf{H}$, respectively, coupled with corresponding plastic distortion tensors. Here the symmetric tensor field $\mathbf{H}$ is associated with $\mathbf{h}^d$ by pushed away procedure. $\mathbf{h}^d$ is related to reference configuration, while $\mathbf{H}$ is defined with respect to plastically deformed configurations.

1.1. List of notations

Further the following notations will be used:
$\mathscr{V}-$ the three dimensional vector space; $Lin-$ the set of the linear mappings from $\mathscr{V}$

to $\mathscr{V}$, i.e the set of second order tensor, $Sym \subset Lin$ is the set of all symmetric tensors;

$\mathbf{u}\cdot\mathbf{v}, \mathbf{u}\times\mathbf{v}, \mathbf{u}\otimes\mathbf{v}$ denote scalar, cross and tensorial products of vectors;
$\mathbf{a}\otimes\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}$ are defined to be a second order tensor and a third order tensor by
$(\mathbf{a}\otimes\mathbf{b})\mathbf{u} = \mathbf{a}(\mathbf{b}\cdot\mathbf{u}), \quad (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\mathbf{u} = (\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\cdot\mathbf{u}), \quad$ for all vectors $\mathbf{u}$.
$\{\mathbf{A}\}^S$ and $\mathbf{A}^T$ denote the symmetric part and the transpose of the tensor$\mathbf{A}$
$\mathbf{I}$ is the identity tensor in Lin.

$\partial_{\mathbf{A}}\phi(x)$ denotes the partial differential of the function ϕ with respect to the field $\mathbf{A}$.
Let $\chi : \mathscr{B}\times\mathbf{R}\to\mathscr{V}$ defines the motion of the body $\mathscr{B}$. The deformation gradient and its gradient are expressed in coordinate systems by

$$\mathbf{F}(\mathbf{X},t) = \nabla\chi(\mathbf{X},t) = \frac{\partial x^i}{\partial X^j}\mathbf{g}_i\otimes\mathbf{G}^j, \quad \nabla\mathbf{F}(\mathbf{X},t) = \frac{\partial^2 x^i}{\partial X^j\partial X^k}\mathbf{g}_i\otimes\mathbf{G}^j\otimes\mathbf{G}^k.$$

Here $\{\mathbf{g}_i\}_{i=1,2,3}$ and $\{\mathbf{G}_i\}_{i=1,2,3}$ are local bases in the reference and actual configurations, respectively, and $\mathbf{G}^j$ are vectors of reciprocal basis.

In what follows the anholonomic basis vectors, in the so-called plastically deformed configuration or the configuration with torsion, generically denoted by $\mathscr{K}$, are related with *the crystal* and defined by $\mathbf{e}_j = \mathbf{F}^p\mathbf{G}_j$. The plastic connection is represented by the its coefficients in a component representation given by

$$\overset{(\mathrm{p})}{\mathbf{\Gamma}} = \Gamma^{\alpha}_{\beta\,\gamma}\mathbf{G}_\alpha\otimes\mathbf{G}^\beta\otimes\mathbf{G}^\gamma.$$

The differential of smooth field $\mathbf{A}$, with respect to the anholonomic configuration $\mathscr{K}$, obeys the rule

$$\nabla_{\mathscr{K}}\mathbf{A} = (\nabla\mathbf{A})(\mathbf{F}^p)^{-1}.$$

Curl of a second order tensor field $\mathbf{A}$ is defined by the second order tensor field, curl$\mathbf{A}$,

$$(\mathrm{curl}\mathbf{A})(\mathbf{u}\times\mathbf{v}) := (\nabla\mathbf{A}(\mathbf{u}))\mathbf{v} - (\nabla\mathbf{A}(\mathbf{v}))\mathbf{u} \quad \forall\mathbf{u},\mathbf{v}\in\mathscr{V} \quad \text{and}(curl\mathbf{A})_{pi} = \varepsilon_{ijk}\frac{\partial A_{pk}}{\partial x^j}$$

are the component of *curl*$\mathbf{A}$ given in a Cartesian basis.
The scalar product of second order tensors $\mathbf{A}$ and $\mathbf{B}$ is defined in terms of their Cartesian components by $\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}, \quad \forall\,\mathbf{A},\mathbf{B}\in Lin$.
The third order tensor, denoted by $\mathscr{A}[\mathbf{F}_1,\mathbf{F}_2]$, is associated to the set of third order tensor $\mathscr{A}$, and $\mathbf{F}_1,\mathbf{F}_2\in Lin$, and is defined by

$$((\mathscr{A}[\mathbf{F}_1,\mathbf{F}_2])\mathbf{u})\mathbf{v} = (\mathscr{A}(\mathbf{F}_1\mathbf{u}))\mathbf{F}_2\mathbf{v}, \quad \forall\,\mathbf{u},\mathbf{v}\in\mathscr{V}.$$

2. TENSORIAL DAMAGE VARIABLES

The physical motivations for these models as well as the mathematical description of the elasto-plastic models are completely different. The first model involves a damage tensorial variable $\mathbf{F}^d$ which is an invertible second order tensorial variable, and realizes the passage from an undamaged stress free configuration to a damaged stress free configuration. The second model introduces a non-metricity measure, in terms of the symmetric tensor fields, either $\mathbf{h}^d$ or $\mathbf{H}$, associated with the the so-called plastic connection.

2.1. Deformation-like damage tensor

In Cleja-Ţigoiu (2011) $\mathbf{F}^d$ has been introduced as a measure of anisotropic damage, which enters the multiplicative decomposition of the deformation gradient.

Let us consider k the reference configuration and the actual (deformed) configuration $\chi(\cdot,t)$ of the body $\mathscr{B}$, where χ represents a motion of the body.

We **assume** that at any time t, for any $\mathbf{X} \in \mathscr{B}$ there exist:

• $\tilde{\mathscr{K}}$ a *stress free, damaged configuration* and

• $\mathscr{K}$ a stress free, undamaged configuration.

Starting from these assumptions, we define three local deformations: the elastic component, $\mathbf{F}^e$, which characterizes the passage from $\tilde{\mathscr{K}}$ to $\chi(\cdot,t)$, the plastic component, $\mathbf{F}^p$, which characterizes the passage from the reference configuration to $\mathscr{K}$ and the damage deformation tensor, $\mathbf{F}^d$, which characterizes the passage from the stress free, undamaged (fictitious) configuration $\mathscr{K}$ to the damaged one, $\tilde{\mathscr{K}}$. In the Fig.1 these configurations were sketched.

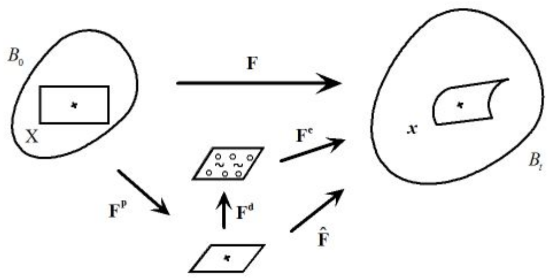

Figure 1: Elastic, plastic and damage tensors as parts of the deformation gradient $\mathbf{F}$, $\mathbf{F} = \mathbf{F}^e\mathbf{F}^d, \mathbf{F}^p$, with $\mathbf{F}^d$ the transformation from the undamaged and stress free configuration, $\mathscr{K}$, to the damaged and stress free configuration, $\tilde{\mathscr{K}}$, following [10].

Assumption M.1 For any motion χ, $\forall \mathbf{X}, \forall t$, the deformation gradient $\mathbf{F} := \nabla\chi(\cdot,t)$ is multiplicatively decomposed into its plastic, $\mathbf{F}^p$, damage, $\mathbf{F}^d$, and elastic, $\mathbf{F}^e$, parts, namely

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^d\mathbf{F}^p, \quad \hat{\mathbf{F}} = \mathbf{F}^e\mathbf{F}^d. \tag{3}$$

All the tensor fields are invertible.

Mass densities ρ^d, ρ^p, ρ and ρ_0 are written in *stress free damaged and undamaged configurations, respectively, and in actual and reference configurations, respectively*

$$\rho \ \det\mathbf{F}^e = \rho^d, \quad \rho^d \ \det\mathbf{F}^d = \rho^p, \quad \rho \ \det\mathbf{F} = \rho_0. \tag{4}$$

The elastic strain tensor associated with the elastic deformation component $\mathbf{F}^e$ is defined by

$$\boldsymbol{\Delta}^e(\mathbf{X},t) = \frac{1}{2}(\mathbf{C}^e - \mathbf{I}), \quad \mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e. \tag{5}$$

The other strain tensors generated by the local deformations which enter the multiplicative decomposition (3) can be introduced by

$$\hat{\mathbf{F}} = \mathbf{F}^e\mathbf{F}^d, \quad \hat{\mathbf{C}} := \hat{\mathbf{F}}^T\hat{\mathbf{F}}, \quad \mathbf{C}^d = (\mathbf{F}^d)^T\mathbf{F}^d. \tag{6}$$

Let us remark that

$$\mathbf{F} = \hat{\mathbf{F}}\mathbf{F}^p, \quad \hat{\mathbf{C}} := (\mathbf{F}^p)^{-T}\mathbf{C}(\mathbf{F}^p)^{-1}, \quad \text{where} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \tag{7}$$

as a consequence of (6).

Let us calculate the elastic strain measure which has been introduced in (5), via the relationship (6)

$$\begin{aligned} \boldsymbol{\Delta}^e &= \frac{1}{2}(\mathbf{F}^d)^{-T}\big(\hat{\mathbf{C}} - (\mathbf{F}^d)^T\mathbf{F}^d\big)(\mathbf{F}^d)^{-1} \\ \text{or} \quad \hat{\mathbf{C}} - \mathbf{C}^d &= 2(\mathbf{F}^d)^T(\boldsymbol{\Delta}^e)\mathbf{F}^d. \end{aligned} \tag{8}$$

Consequently, the elastic strain with respect to the stress free, damaged configuration, $\boldsymbol{\Delta}^e$, can be viewed in the stress free, undamaged configuration as $\hat{\mathbf{C}} - \mathbf{C}^d$ by pulled back procedure. $\mathbf{F}^d$ describes the passage from the stress free, undamaged configuration $\mathscr{K}$ to the the stress free, damaged configuration $\tilde{\mathscr{K}}$.

The elastic type constitutive equation which characterizes the Cauchy stress, with respect to the stress free and undamaged configuration, is stipulated to be given by

$$\mathbf{T} = \rho\hat{\mathbf{F}}\hat{\mathbf{h}}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha})(\hat{\mathbf{F}})^T. \tag{9}$$

Remark. In describing the elastic type behaviour of the elasto-plastic body with damaged structure the following stress tensors with respect to the appropriate configurations can be defined

$\mathbf{T}(\mathbf{x},t)-$ the Cauchy stress in the actual configuration $\chi(\cdot,t)$, where $\mathbf{x}=\chi(\mathbf{X},t)$;

$\bar{\mathbf{T}}(\mathbf{x},t)-$ the Piola-Kirchhoff stress in the stress free and undamaged configuration, denoted by $\mathscr{K}$, the so-called *effective stress*.

These stress measures are related by the following relationship

$$\bar{\mathbf{T}} = (\det \hat{\mathbf{F}})(\hat{\mathbf{F}})^{-1}\mathbf{T}(\hat{\mathbf{F}})^{-T}, \tag{10}$$

which suggested the constitutive assumption (9). Moreover, (9) holds if and only if the elastic type constitutive equation in terms of the effective stress $\bar{\mathbf{T}}$ can be written by

$$\bar{\mathbf{T}} = \tilde{\rho}\ \hat{\mathbf{h}}_{\mathscr{K}}(\hat{\mathbf{C}} - \mathbf{C}^d, \boldsymbol{\alpha}). \tag{11}$$

We remark that $\mathbf{F}^d$ is involved in the elastic constitutive relation (11) like an internal variable and $\boldsymbol{\alpha}$ denotes the set of internal state variables.

In [4] the stress relaxation (or stress free) restriction has been formulated in order to formalize the fact that the stress with respect to certain configuration is vanishing if and only if the body is undeformed, i.e. *pure elastic deformation* is the identity tensor. Following the development given by Cleja-Ţigoiu and Soós [4] the stress free restriction will be given by

$$\hat{\mathbf{h}}_{\mathscr{K}}(\boldsymbol{\Delta}, \boldsymbol{\alpha}) = 0 \quad \text{for} \quad \boldsymbol{\Delta} \in Sym, \quad \text{if and only if} \quad \boldsymbol{\Delta} = 0. \tag{12}$$

Here in the considered case, the relaxation restriction takes place if and only if

$$\hat{\mathbf{C}} := (\mathbf{F}^d)^T\mathbf{F}^d \equiv \mathbf{C}^d, \quad \text{or if and only if} \quad \mathbf{C}^e := (\mathbf{F}^e)^T\mathbf{F}^e = \mathbf{I}.$$

Remark As a generalization of the two-term decomposition of the deformation gradient a three-term multiplicative decomposition

$$\mathbf{F} = \mathbf{F}^{\mathscr{L}}\mathbf{F}^p, \quad \mathbf{F}^{\mathscr{L}} = \mathbf{F}^e\mathbf{F}^i. \tag{13}$$

has been considered by Clayton et al. [3], where

- $\mathbf{F}^{\mathscr{L}}$ is named the total lattice deformation and
- $\mathbf{F}^i$ characterizes the residual deformation due to micro-heterogeneity in the presence of lattice defects.

We can say that $\mathbf{F}^i$ and $\mathbf{F}^d$ are somehow equivalent. Our point of view, expressed by Cleja-Ţigoiu and Soós [4], emphasizes that the introduction of various types of decomposition of the deformation gradient is justified or become possible if and only if they were introduced simultaneously with the material laws. For instance Clayton et al. [3] state that $\mathbf{F}^i$ "includes effects arising from defects contained within the volume element at time t, residual thermoelastic strains, and internal boundaries and stacking faults left by moving effects such as partial dislocations and partial disclination dipoles." The distinction between $\mathbf{F}^i$ and $\mathbf{F}^p$ has been made in [3] by saying that $\mathbf{F}^p$ characterizes "lattice preserving contributions from the mobile defects which traversed the volume." The physical attributed specification to different fields, as being declared by the authors, has to be reflected in the constitutive relationships, otherwise they remain simple words.

2.2. The defect density tensor $\mathbf{h}^d$

In general the damage deformation tensor, denoted here by $\mathbf{F}^d$, characterizes the passage from an undamaged (fictitious) configuration to a certain plastically deformed configuration, as a measure of anisotropic damage. Concerning *the defect density tensor* $\mathbf{h}^d$ we adopted here the geometrical motivation to introduce tensorial defect measure given by Kröner [19] and de Wit [13], for instance. The behaviour of elasto-plastic materials with damaged microstructure is described in terms of specific differential geometry elements which characterize the internal mechanical state, following [19] and [13]. In what follows we shortly present basic hypotheses related to the elasto-plastic model proposed by Cleja-Ţigoiu and Ţigoiu [10], but we develop the description of the model within the plastically deformed configuration.

We pay attention to the defects of lattice structure, such as point defects, micro voids and micro cracks. These defects can be associated with dislocations and disclinations, being different in their physical and mathematical nature. The lattice defects are involved in the Cartan torsion of the so-called plastic connection, see Cleja-Ţigoiu [6], [8]. The point defects, micro voids and micro cracks in the damaged zone are modeled specifically in terms of the tensor field which characterizes the *non-metric property* of the plastic connection, see [10], while the dislocations and disclinations can be modeled by the geometrical concepts, *torsion and curvature of the plastic connection with metric property*.

We introduce the **hypothesis:** The plastic behaviour is characterized in terms of the pair $(\mathbf{F}^p, \overset{(p)}{\mathbf{\Gamma}})$. The second order tensor field $\mathbf{F}^p$, which is called plastic distortion, or the plastic part of the deformation gradient given by (3), and $\overset{(p)}{\mathbf{\Gamma}}$ is characterized by a third order field in a curvilinear coordinate system and represents the Christoffel-Riemann coefficient of a connection, called here plastic connection.

We assume that the plastic distortion can not be expressed in terms of a vector valued potential, which means that *the plastic distortion* $\mathbf{F}^p$ *is incompatible.* The plastic connection $\overset{(p)}{\mathbf{\Gamma}}$ is not compatible with the plastic distortion $\mathbf{F}^p$, which means that it can not be represented as $\overset{(p)}{\mathbf{\Gamma}} = (\mathbf{F}^p)^{-1}\nabla\mathbf{F}^p$.

Axiom 1. The plastic connection $\overset{(p)}{\mathbf{\Gamma}}$ has *non-metric property with respect to the metric tensor* $\mathbf{C}^p = (\mathbf{F}^p)^T\mathbf{F}^p$. Consequently *there exists* a third order tensor $\mathbf{Q}$, such that $\mathbf{Qu} \in Sym$ and the following equality

$$-(\nabla\mathbf{C}^p)\mathbf{u} + (\mathbf{C}^p \overset{(p)}{\mathbf{\Gamma}} \mathbf{u})^T + \mathbf{C}^p(\overset{(p)}{\mathbf{\Gamma}} \mathbf{u}) = \mathbf{Qu}, \tag{14}$$

holds for all vectors $\mathbf{u}$.

The following representation for the plastic connection can be derived, see [6] and [10],

$$\overset{(p)}{\mathbf{\Gamma}} = \overset{(p)}{\mathscr{A}} + (\mathbf{C}^p)^{-1}\big(\mathbf{\Lambda}\times\mathbf{I} + \frac{1}{2}\mathbf{Q}\big), \quad \overset{(p)}{\mathscr{A}} = (\mathbf{F}^p)^{-1}\nabla\mathbf{F}^p, \tag{15}$$

Here the third order tensor field $\mathbf{\Lambda}\times\mathbf{I}$ is defined in such a way to have the equality $((\mathbf{\Lambda}\times\mathbf{I})\mathbf{u})\mathbf{v} = (\mathbf{\Lambda u})\times\mathbf{v}$, written for all vectors $\mathbf{u},\mathbf{v}$.

The fields which enter the formula (15) have the following meaning:

1. $\overset{(p)}{\mathscr{A}}$ defines the so-called *Bilby type plastic connection*, which is compatible with the plastic distortion, namely it is defined in terms of the *gradient of the plastic distortion*;

2. the second-order tensor $\mathbf{\Lambda}$ is called the disclination tensor;

3. the third-order tensor $\mathbf{Q}$ is called the non-metric or extra-matter tensor $\mathbf{Q}$.

Following Kröner [19] we assume the existence of a second order tensor, $\mathbf{h}^d$, which is a potential for the third-order tensor field $\mathbf{Q}$, namely

$$\mathbf{Q} = \nabla\mathbf{h}^d, \text{ and } \mathbf{h}^d \text{ is called the defect density tensor.}$$

Consequently, the plastic metric tensor $\mathbf{C}^p$ is corrected by $\mathbf{h}^d$, to restore the metric property of the plastic connection, i.e.

$$-\nabla(\mathbf{C}^p + \mathbf{h}^d)\mathbf{u} + (\mathbf{C}^p \overset{(p)}{\mathbf{\Gamma}} \mathbf{u})^T + \mathbf{C}^p(\overset{(p)}{\mathbf{\Gamma}} \mathbf{u}) = 0, \quad \forall \quad \mathbf{u} \in \mathscr{V}, \tag{16}$$

as can be seen from (14).

The physical meaning of the tensor field $\mathbf{h}^d$ is put into evidence by relation (16), which reveals that $\mathbf{h}^d$ is a strain-like variable. Moreover $\mathbf{h}^d$ appears to be a special type of deformation necessary to be added in order to establish the geometry of damaged structure.

Let us introduce the second order torsion tensor, $\mathscr{N}^p$, related with the third order Cartan torsion $\mathbf{S}^p$, via the following relationships

$$
\begin{aligned}
&(\mathbf{S}^p\tilde{\mathbf{u}})\tilde{\mathbf{v}} = \mathscr{N}^p(\tilde{\mathbf{u}}\times\tilde{\mathbf{v}}), \quad \forall \quad \tilde{\mathbf{u}},\tilde{\mathbf{v}};\\
&\mathscr{N}^p = (\mathbf{F}^p)^{-1}\mathrm{curl}\,\mathbf{F}^p + (\mathbf{C}^p)^{-1}\big(\mathrm{curl}\,\mathbf{h}^d + (\mathrm{tr}\,\mathbf{\Lambda})\mathbf{I} - (\mathbf{\Lambda})^T\big).
\end{aligned}
\tag{17}
$$

Remark. Note that the tensorial damage variable $\mathbf{h}^d$ has a contribution to the Cartan torsion, if and only if *is incompatible*, which means that curl $\mathbf{h}^d$ is not vanishing.

The following *defect fields* have been introduced

$$
\begin{aligned}
&\boldsymbol{\alpha} && := (\mathbf{F}^p)^{-1}\mathrm{curl}\,(\mathbf{F}^p) && \text{dislocation density}\\
&\boldsymbol{\alpha}^d && = (\mathbf{C}^p)^{-1}\mathrm{curl}\,\mathbf{h}^d && \text{damage defect density}\\
&\boldsymbol{\alpha}^\Lambda && := \mathrm{tr}\,\mathbf{\Lambda}\mathbf{I} - (\mathbf{\Lambda})^T && \text{disclination density,}
\end{aligned}
\tag{18}
$$

which characterize the incompatibilities existing in the materials see for instance [18], [19] and [13].

Consequently the lattice defects, previously introduced by (18), are involved in the Cartan torsion of the so-called plastic connection as can be see from the formula $(17)_2$, rewritten as $\mathscr{N}^p = \boldsymbol{\alpha} + \boldsymbol{\alpha}^d + \boldsymbol{\alpha}^\Lambda$.

Remark. The *damage defect density* $\boldsymbol{\alpha}^d = (\mathbf{C}^p)^{-1}\mathrm{curl}\,\mathbf{h}^d$ is a not symmetric tensor, which is defined in terms of the plastic metric tensor, $\mathbf{C}^p$, and damage tensor $\mathbf{h}^d$. Thus $\boldsymbol{\alpha}^d$ is a measure of damage explicitly dependent on the plastic distorsion.

In the model proposed here the damage tensor $\mathbf{h}^d$ and the associated damage tensor $\mathbf{H}$ by pushed away procedure have been considered as measures of the anisotropic damage.

3. ENERGETIC RESTRICTIONS

As basis of the constitutive relations we adopt here the free energy imbalance principle, which generalizes the second principle of thermodynamics and it is adapted to isothermal elasto-plastic processes. The principle is formulated with respect to an appropriate configuration and is strongly dependent on the postulated expression for the free energy density, as well as of the postulated form for the virtual internal power. We adopt here the following formulation:

Axiom 2. The elasto-plastic behaviour of the material with damaged structure is restricted to satisfy in the so called damaged configuration, say $\mathscr{K}$, the dissipation condition

$$-\dot{\varphi}_{\mathscr{K}} + (\mathscr{P}_{int})_{\mathscr{K}} \geq 0 \quad \text{for any virtual (isothermal) processes.} \tag{19}$$

Here we denoted the free energy function with respect to the damaged configuration by $\varphi_{\mathscr{K}}$ and the internal expended power by $(\mathscr{P}_{int})_{\mathscr{K}}$.

It is our purpose here to discuss the development of theory based on the imbalance principle, within the constitutive framework of elasto-plastic materials with damage. As two models have been introduced, the first one involving the the deformation-like damage tensor, $\mathbf{F}^p$, and the other one considering the defect density tensor $\mathbf{h}^d$, the specifications of the ingredients has to be done.

3.1. Free energy principle for the model with deformation-like damage tensor, $\mathbf{F}^p$

In the case analyzed here the free energy imbalance principle has been formulated by Cleja-Ţigoiu [7] with respect to the stress free and undamaged configuration, denoted by $\mathscr{K}$.

Motivated by the principle of the elastic free energy equivalence, see [20], the free energy with respect to $\mathscr{K}$ is postulated here under the form

$$\varphi_{\mathscr{K}} = \hat{\varphi}^e(\hat{\mathbf{C}} - \mathbf{C}^d) + \varphi^{(iv)}(\mathbf{F}^d, (\mathbf{F}^p)^{-1}, \boldsymbol{\alpha}), \tag{20}$$

were the relative strain measure $\hat{\mathbf{C}} - \mathbf{C}^d$ was introduced by (8) and $\boldsymbol{\alpha}$ denotes the internal variables.

The internal power is expressed as in the classical elasto-plastic models in terms of the Cauchy stress tensor and velocity gradient, both of them being related to the deformed configuration, namely

$$\mathscr{P}_{int} = \frac{1}{\rho}\mathbf{T}\cdot\{\mathbf{L}\}^S, \quad \text{with} \quad \mathbf{L} = \nabla\mathbf{v} \equiv \dot{\mathbf{F}}(\mathbf{F})^{-1}, \quad \text{and}$$
$$\{\mathbf{L}\}^S = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T). \tag{21}$$

We provide here the expression of the internal power based on the kinematical relationship (3).

Proposition 1. The internal power is expressed in terms of the elastic, plastic and damage internal powers, represented here by the scalar product of the appropriate rates with the power conjugate stress measures, respectively,

$$\frac{1}{\rho}\mathbf{T}\cdot\mathbf{L} = \frac{1}{\rho^d}\widetilde{\boldsymbol{\Sigma}}\cdot\dot{\mathbf{F}}^d(\mathbf{F}^d)^{-1} + \frac{1}{\rho^p}\bar{\boldsymbol{\Sigma}}\cdot\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} + \frac{\mathbf{T}}{\rho}\cdot\dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1},$$
$$\frac{1}{\rho^d}\widetilde{\boldsymbol{\Sigma}} = \frac{1}{\rho\det\mathbf{F}^e}(\mathbf{F}^e)^T\mathbf{F}^e\widetilde{\mathbf{T}}, \quad \frac{1}{\rho^p}\bar{\boldsymbol{\Sigma}} = \frac{1}{\rho\det\hat{\mathbf{F}}}(\hat{\mathbf{F}})^T\hat{\mathbf{F}}\mathbf{T}. \tag{22}$$

Here $\widetilde{\boldsymbol{\Sigma}}$ and $\bar{\boldsymbol{\Sigma}}$ are Mandel type stresses and are defined with respect to the configurations $\mathscr{K}$ and $\tilde{\mathscr{K}}$.

Assumption M2. The elasto-plastic behaviour of the material with damage variable $\mathbf{F}^d$ structure is restricted to satisfy in $\mathscr{K}$ the dissipation condition

$$-\dot{\varphi}_{\mathscr{K}} + \frac{1}{\rho}\mathbf{T}\cdot\mathbf{L} \geq 0 \tag{23}$$

for any virtual (isothermal) processes, where the free energy function in given by (20).

Proposition 2. Within the constitutive framework formulated in Section 2.1 the free energy imbalance (19) is derived under the form

$$\{\frac{\mathbf{T}}{\rho} - 2\,\hat{\mathbf{F}}\partial_{\bar{\mathbf{C}}}\varphi^{(e)}(\hat{\mathbf{F}})^T\}\cdot\{\mathbf{L}\}^S + \{2\,\mathbf{F}^d\partial_{\bar{\mathbf{C}}}\varphi^{(e)}(\mathbf{F}^d)^T - \partial_{\mathbf{F}^d}\varphi^{(iv)}(\mathbf{F}^d)^T\}\cdot\mathbf{L}^d +$$
$$+\{2\hat{\mathbf{C}}\partial_{\bar{\mathbf{C}}}\varphi^{(e)} + \mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}\}\cdot\mathbf{L}^p + \partial_{\boldsymbol{\alpha}}\varphi^{(iv)}\cdot\dot{\boldsymbol{\alpha}} \geq 0, \tag{24}$$

with the notation $\quad \bar{\mathbf{C}} \equiv \hat{\mathbf{C}} - \mathbf{C}^d,$

which has been written for all the velocity gradient $\mathbf{L}$, plastic rate $\mathbf{L}^p$, rate of the damage $\mathbf{L}^d$, and involving the variation in time of the internal variable, i.e. $\dot{\boldsymbol{\alpha}}$.

The detailed proof of the above proposition can be found in [8]. We recall here that the appropriate rates at a fixed time moment t, which are involved in the dissipation formula (24), are not independent, they being related by the kinematic relationship derived from (3). The relationship between these rates is written as

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e\mathbf{L}^d(\mathbf{F}^e)^{-1} + \hat{\mathbf{F}}\mathbf{L}^p(\hat{\mathbf{F}})^{-1}, \tag{25}$$

where the current values of the deformation components are given.

We derive the thermodynamic restriction imposed by the free energy principle on the constitutive equation. Two steps have to be analyzed.

- First we assume that the rates of plastic distortion $\mathbf{L}^p$ and of the damage $\mathbf{L}^d$ are vanishing, while $\mathbf{L}$ is arbitrarily given. The so-called elastic restrictions has been found, as $\mathbf{L} = \mathbf{L}^e$ in this case.

- Second we return to the expression of the free energy principle (24) by recovering the elastic type constitutive restrictions and the residual dissipation inequality follows at once.

Theorem 1. *The following thermodynamic restrictions are provided from the free energy imbalance (24):*

I.The free energy density is potential for the Cauchy stress tensor

$$\frac{\mathbf{T}}{\rho} = 2\hat{\mathbf{F}}\partial_{\bar{\mathbf{C}}}\varphi^{(e)}(\hat{\mathbf{F}})^T \quad \text{or} \quad \frac{\bar{\mathbf{T}}}{\rho^p} = 2\,\partial_{\bar{\mathbf{C}}}\hat{\varphi}^e, \text{with the notation} \quad \bar{\mathbf{C}} \equiv \hat{\mathbf{C}} - \mathbf{C}^d, \tag{26}$$

if the free energy density is written under the form (20).

II. The residual dissipation inequality becomes

$$\begin{aligned} &\{\mathbf{C}^d\frac{\bar{\mathbf{T}}}{\rho^p} - (\mathbf{F}^d)^T\partial_{\mathbf{F}^d}\varphi^{(iv)}\}\cdot\mathbf{l}^d + \\ &+\{\hat{\mathbf{C}}\frac{\bar{\mathbf{T}}}{\rho^p} + \mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}\}\cdot\mathbf{L}^p - \partial_{\boldsymbol{\alpha}}\varphi^{(iv)}\cdot\dot{\boldsymbol{\alpha}} \geq 0. \end{aligned} \tag{27}$$

Here $\mathbf{l}^d = (\mathbf{F}^d)^{-1}\dot{\mathbf{F}}^d$ *is the rate of damage tensor relative to the stress free and undamaged configuration.*

3.2. Free energy principle for the model with the defect density tensor $\mathbf{h}^d$

We adapted the free energy principle developed by Cleja-Ţigoiu [5] and [6] to the constitutive framework of elasto-plastic materials with the damage variable, $\mathbf{h}^d$, the so-called defect density tensor, introduced in the Section 2.2. In the model proposed here we do not study the influence of the defects such as disclinations on the damage, which means that the plastic connection is not dependent on the $\boldsymbol{\Lambda}$, namely $\boldsymbol{\Lambda} = 0$, in the formulae.

In the model the damage variable $\mathbf{h}^d$ was defined in the reference configuration. Let us introduce the defect density tensor, $\mathbf{H}$, with respect to the plastically deformed configuration $\mathscr{K}$, which has been introduced in the Section 2.2 by the pair plastic distortion and plastic connection, $(\mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}})$. The tensorial damage variable $\mathbf{H}$ is the pushed forward to the damaged configuration of the tensor $\mathbf{h}^d$, while the passage from the reference configuration to the plastically deformed configuration is characterized by plastic distortion. The relationships between the tensor fields $\mathbf{H}$ and $\mathbf{h}^{\mathbf{d}}$, and between the appropriate gradients of $\mathbf{H}$, respectively, are defined by

$$\mathbf{H} = (\mathbf{F}^p)^{-T}\mathbf{h}^d(\mathbf{F}^p)^{-1}, \qquad \nabla_{\mathscr{K}}\mathbf{H} = (\nabla\mathbf{H})(\mathbf{F}^p)^{-1}. \tag{28}$$

We introduce now the free energy density function and the internal power in $\mathscr{K}$ expanded during an elasto-plastic process.

Axiom 3. The *free energy density* in $\mathscr{K}$ is postulated to be dependent on the elastic strain, $\mathbf{C}^e$, where $\mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e$. The free energy density function is dependent on the damaged configuration, through the second order plastic deformation $((\mathbf{F}^p)^{-1}, \overset{(p)}{\mathscr{A}}_{\mathscr{K}})$, tensorial damage variable $\mathbf{H}$ and its gradient $\nabla_{\mathscr{K}}\mathbf{H}$, namely

$$\psi = \psi_{\mathscr{K}}(\mathbf{C}^e - \mathbf{I}, (\mathbf{F}^p)^{-1}, \overset{(p)}{\mathscr{A}}_{\mathscr{K}}, \mathbf{H}, \nabla_{\mathscr{K}}\mathbf{H}), \qquad \text{where} \quad \overset{(p)}{\mathscr{A}}_{\mathscr{K}} = \mathbf{F}^p\nabla_{\mathscr{K}}(\mathbf{F}^p)^{-1}. \tag{29}$$

The third-order field $\overset{(p)}{\mathscr{A}}_{\mathscr{K}}$ defines the so-called Bilby's type plastic connection with respect to plastically deformed configuration, $\mathscr{K}$.

As the tensorial damage variable and its gradient have been introduced among the independent variables in the expression of the free energy density, the power conjugate variables with their rates ought to be introduced in the expression postulated for the virtual internal power.

The imbalance free energy principle formalized by (19) has to be considered by taking into account the free energy expression (29) and the internal power relative to the configuration $\mathscr{K}$.

Axiom 4. The internal power expended during the elasto-plastic damaged material is characterized by

$$\begin{aligned}(\mathscr{P}_{int})_{\mathscr{K}} = &\frac{1}{2}\frac{\boldsymbol{\pi}}{\tilde{\rho}}\cdot\frac{d}{dt}\mathbf{C}^e + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\cdot((\mathbf{F}^e)^{-1}(\nabla_{\chi}\mathbf{L}[\mathbf{F}^e,\mathbf{F}^e]) - \nabla_{\mathscr{K}}\mathbf{L}^p) + \\ &\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p_{\mathscr{K}}\cdot\mathbf{L}^p + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p_{\mathscr{K}}\cdot\nabla_{\mathscr{K}}\mathbf{L}^p + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^h\cdot\frac{d}{dt}\mathbf{H} + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^h_{\mathscr{K}}\cdot\nabla_{\mathscr{K}}\frac{d}{dt}\mathbf{H}.\end{aligned} \tag{30}$$

Here $\boldsymbol{\pi}$ represents the Piola-Kirchhoff stress tensor with respect to the plastically deformed configuration,

$\boldsymbol{\Upsilon}^p_{\mathscr{K}}$ and $\boldsymbol{\Upsilon}^d_{\mathscr{K}}$ are micro stresses related to the plastic and damage mechanisms,

$\boldsymbol{\mu}^p_{\mathscr{K}}, \boldsymbol{\mu}^h_{\mathscr{K}}$ are *micro stress momenta* which are conjugated with the gradients of the rate of plastic distortion $\mathbf{L}^p$ and of $\frac{d}{dt}\mathbf{H}$, respectively.

We recall here the relationship between the Piola-Kirchoff stress tensor $\boldsymbol{\pi}$ in the configuration $\mathscr{K}$ and the Cauchy stress $\mathbf{T}$ defined by

$$\begin{aligned} \boldsymbol{\pi} &= \det(\mathbf{F}^e)(\mathbf{F}^e)^{-1}\mathbf{T}(\mathbf{F}^e)^{-T}, \\ \text{where} \quad & \rho \det(\mathbf{F}^e) = \tilde{\rho}, \end{aligned} \tag{31}$$

ρ and $\tilde{\rho}$ denote the mass densities in the actual and plastically deformed configurations, respectively.

In order to calculate the time derivative of the free energy function (29) the rate of the appropriate fields which enter its expression have to be calculated in terms of $\mathbf{L}^p$ and $\nabla_{\mathscr{K}}\, \mathbf{L}^p$ as follows.

- The time derivative of the inverted plastic distortion tensor is given by

$$\frac{d}{dt}(\mathbf{F}^p)^{-1} = -(\mathbf{F}^p)^{-1}\mathbf{L}^p \tag{32}$$

- The rate of Cauchy-Green elastic tensor is expressed in terms of $\mathbf{L}$ and $\mathbf{L}^p$ as follows

$$\frac{d}{dt}(\mathbf{C}^e) = 2(\mathbf{F}^e)^T\{\mathbf{L}\}^s\mathbf{F}^e - 2\{\mathbf{C}^e\mathbf{L}^p\}^s. \tag{33}$$

- The rate of gradient with respect to plastically deformed configuration of the damage tensor $\mathbf{H}$ can be expressed as

$$\frac{d}{dt}(\nabla_{\mathscr{K}}\mathbf{H}) = \nabla_{\mathscr{K}}(\frac{d}{dt}\mathbf{H}) - (\nabla_{\mathscr{K}}\mathbf{H})\mathbf{L}^p. \tag{34}$$

- The rate of $\overset{(p)}{\mathscr{A}}_{\mathscr{K}}$ leads to

$$\frac{d}{dt}(\overset{(p)}{\mathscr{A}}_{\mathscr{K}}) = -\nabla_{\mathscr{K}}\, \mathbf{L}^p + \mathbf{L}^p\, \overset{(p)}{\mathscr{A}}_{\mathscr{K}} - \overset{(p)}{\mathscr{A}}_{\mathscr{K}}\, [\mathbf{I}, \mathbf{L}^p] - \overset{(p)}{\mathscr{A}}_{\mathscr{K}}\, \mathbf{L}^p. \tag{35}$$

  by taking the time derivative of the tensorial field introduced by (29).

Finally we get the formula which express the variation in time of the free energy during the considered elasto-plastic process

$$
\begin{aligned}
&\dot{\varphi}_{\mathscr{K}} = \partial_{\mathbf{C}^e-\mathbf{I}}\varphi \cdot \frac{d}{dt}\mathbf{C}^e - \partial_{(\mathbf{F}^p)^{-1}}\varphi \cdot \big((\mathbf{F}^p)^{-1}\mathbf{L}^p\big) + \\
&+\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi \cdot (-\nabla_{\mathscr{K}}\, \mathbf{L}^p) + \partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi \cdot \{\mathbf{L}^p \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}} - \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}[\mathbf{I},\mathbf{L}^p] - \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}\mathbf{L}^p\} + \\
&+\partial_{\mathbf{H}}\varphi \cdot \frac{d}{dt}\mathbf{H} + \partial_{\nabla_{\mathscr{K}}\mathbf{H}}\varphi \cdot \big(\nabla_{\mathscr{K}}(\frac{d}{dt}\mathbf{H}) - (\nabla_{\mathscr{K}}\mathbf{H})\mathbf{L}^p\big)
\end{aligned}
\tag{36}
$$

The free energy imbalance (19) can be written as

$$
\begin{aligned}
&(\frac{1}{2\tilde{\rho}}\boldsymbol{\pi} - \partial_{(\mathbf{C}^e-\mathbf{I})}\psi) \cdot \frac{d}{dt}(\mathbf{C}^e) + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}} \cdot ((\mathbf{F}^e)^{-1}(\nabla_{\chi}\mathbf{L}[\mathbf{F}^e,\mathbf{F}^e]) + \\
&+\big(\frac{1}{\tilde{\rho}}(\boldsymbol{\mu}^p_{\mathscr{K}} - \boldsymbol{\mu}_{\mathscr{K}}\big) - \partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi) \cdot (\nabla_{\mathscr{K}}\mathbf{L}^p) + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p_{\mathscr{K}} \cdot \mathbf{L}^p + \\
&+\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi \cdot \{\mathbf{L}^p \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}} - \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}[\mathbf{I},\mathbf{L}^p] - \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}\mathbf{L}^p\} + \partial_{\nabla_{\mathscr{K}}\mathbf{H}}\varphi \cdot ((\nabla_{\mathscr{K}}\mathbf{H})\mathbf{L}^p) + \\
&+(\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^h - \partial_{\mathbf{H}}\varphi) \cdot \frac{d}{dt}\mathbf{H} + \big(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^h_{\mathscr{K}} - \partial_{\nabla_{\mathscr{K}}\mathbf{H}}\varphi\big) \cdot \nabla_{\mathscr{K}}\frac{d}{dt}\mathbf{H} \geq 0.
\end{aligned}
\tag{37}
$$

The elastic type constitutive equation can be provided as a direct consequence of the free energy imbalance, if $\mathbf{L}^p = 0$ and no variation of damage occurs during the considered process, i.e $\dot{\mathbf{H}} = 0$. Using also (33) as well as the fact that $\mathbf{L}$ and $\nabla_{\chi}\mathbf{L}$ can be arbitrary given, the elastic response is derived. Coming back to the free energy inequality (37) the reduced dissipation inequality follows.

Theorem 2. *The following thermodynamic restrictions are provided from the free energy imbalance:*

I.The free energy density is potential the stress

$$
\boldsymbol{\pi} = 2\tilde{\rho}\,\partial_{(\mathbf{C}^e-\mathbf{I})}\psi, \quad \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}} = 0,
\tag{38}
$$

where $\boldsymbol{\pi}$ *denotes the symmetric Piola-Kirchhoff and* $\boldsymbol{\mu}_{\mathscr{K}}$ *represents the macro momentum, both of then being related to the plastically deformed configuration.*

II. The residual dissipation inequality becomes

$$
\begin{aligned}
&(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p_{\mathscr{K}} - \partial_{\overset{(p)}{\mathscr{A}}_{\mathscr{K}}}\varphi)\cdot(\nabla_{\mathscr{K}}\mathbf{L}^p) + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p_{\mathscr{K}}\cdot\mathbf{L}^p + \\
&+\partial_{\overset{(p)}{\mathscr{A}}_{\mathscr{K}}}\varphi\cdot\{\mathbf{L}^p \overset{(p)}{\mathscr{A}}_{\mathscr{K}} - \overset{(p)}{\mathscr{A}}_{\mathscr{K}}[\mathbf{I},\mathbf{L}^p] - \overset{(p)}{\mathscr{A}}_{\mathscr{K}}\mathbf{L}^p\} + \partial_{\nabla_{\mathscr{K}}\mathbf{H}}\varphi\cdot((\nabla_{\mathscr{K}}\mathbf{H})\mathbf{L}^p) + \\
&+(\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^h - \partial_{\mathbf{H}}\varphi)\cdot\frac{d}{dt}\mathbf{H} + \big(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^h_{\mathscr{K}} - \partial_{\nabla_{\mathscr{K}}\mathbf{H}}\varphi\big)\cdot\nabla_{\mathscr{K}}\frac{d}{dt}\mathbf{H} \geq 0.
\end{aligned}
\tag{39}
$$

4. EVOLUTION OF PLASTIC DEFORMATION AND DAMAGE

The models considered here are dissipative and the elastic type constitutive equations were derived as direct consequences of the free energy imbalance. We introduce now assumptions concerning the irreversible behaviour of elasto-plastic materials coupled with damage, which are compatible with the reduced dissipative inequalities.

The expressions of viscoplastic constitutive equations are suggested by the reduced dissipation inequalities (27) for the model with tensorial damage $\mathbf{F}^d$ and (39), respectively, for the model based on $\mathbf{H}$, damage tensorial measure.

4.1. Evolution of damage tensorial variable $\mathbf{F}^d$

Assumption M3. *The evolution equations for plastic part of deformation, damage* and internal variables are postulated to be given by

$$
\begin{aligned}
\lambda_d\,\mathbf{l}^d &= \mathbf{C}^d\frac{\bar{\mathbf{T}}}{\rho^p} - (\mathbf{F}^d)^T\partial_{\mathbf{F}^d}\varphi^{(iv)}, \quad \text{where} \quad \mathbf{l}^d = (\mathbf{F}^d)^{-1}\dot{\mathbf{F}}^d, \\
\lambda_p\,\mathbf{L}^p &= \hat{\mathbf{C}}\frac{\bar{\mathbf{T}}}{\rho^p} + \mathbf{F}^{p-T}\partial_{(\mathbf{F}^p)^{-1}}\varphi^{(iv)}, \\
\lambda_a\,\dot{\boldsymbol{\alpha}} &= -\partial_{\boldsymbol{\alpha}}\varphi^{(iv)}.
\end{aligned}
\tag{40}
$$

The evolution equation for the damage variable $\mathbf{F}^d$ involves the Cauchy -Green tensors $\hat{\mathbf{C}} = (\mathbf{F}^d)^{-T}\mathbf{C}(\mathbf{F}^d)^{-1}$ and $\mathbf{C}^d = (\mathbf{F}^d)^T\mathbf{F}^d$, as well as the stress tensor $\bar{\mathbf{T}}$ which is expressed in terms of the appropriate deformation tensor via the elastic type constitutive equation (26).

Assumption M4. The evolution equations (40) are compatible with the reduced dissipative inequality, namely the constitutive functions λ_d, λ_p and λ_a are given to satisfy the inequality

$$\lambda_d \mathbf{L}^d \cdot \mathbf{L}^d + \lambda_p \mathbf{L}^p \cdot \mathbf{L}^p + \lambda_a \, \dot{\boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} \geq 0. \tag{41}$$

4.2. Evolution of damage tensorial variable H

Let us introduce the energetic type constitutive equations for the micro momenta related to the plastic and damage mechanisms, in order to eliminate the gradient of the rate of plastic distortion and rate of the damage, respectively, following Gudmudson [16], namely

$$\begin{aligned} \frac{1}{\tilde{\rho}} \boldsymbol{\mu}^p_{\mathscr{K}} &= \partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}} \varphi, \\ \frac{1}{\tilde{\rho}} \boldsymbol{\mu}^h_{\mathscr{K}} &= \partial_{\nabla_{\mathscr{K}} \mathbf{H}} \varphi \end{aligned} \tag{42}$$

Consequently, as a direct consequence of (42) the **reduced dissipation inequality** (39) becomes

$$\begin{aligned} &\frac{1}{\tilde{\rho}} \boldsymbol{\Upsilon}^p_{\mathscr{K}} \cdot \mathbf{L}^p + (\frac{1}{\tilde{\rho}} \boldsymbol{\Upsilon}^h_{\mathscr{K}} - \partial_{\mathbf{H}} \varphi) \cdot \frac{d}{dt} \mathbf{H} + \\ &+ \partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}} \varphi \cdot \{ \mathbf{L}^p \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}} - \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}} [\mathbf{I}, \mathbf{L}^p] - \overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}} \mathbf{L}^p \} + \partial_{\nabla_{\mathscr{K}} \mathbf{H}} \varphi \cdot ((\nabla_{\mathscr{K}} \mathbf{H}) \mathbf{L}^p) \geq 0. \end{aligned} \tag{43}$$

We remark that the terms contained in the dissipation inequality (43) are linear with respect to the rate of damage variable and rate of plastic distorsion.

Remark. By introducing the operators ${}_r\odot$ and $\odot$, which associates to the third order tensors $\mathscr{A}, \mathscr{B}$ the second order tensor, denoted $\mathscr{A}_r \odot \mathscr{B}$, $\mathscr{A} \odot \mathscr{B}$, and $\mathscr{A} \odot {}_l \mathscr{B}$, respectively, the linear dependence on $\mathbf{L}^p$ can be explicitly put into evidence in the inequality (43).

The following operators are defined for all second order tensors $\mathbf{L}$

$$\begin{aligned} (\mathscr{A} \,{}_r\!\odot \mathscr{B}) \cdot \mathbf{L} &= \mathscr{A} \cdot \mathbf{L} \mathscr{B} = \mathscr{A}_{ijk} L_{in} \mathscr{B}_{njk}, \\ (\mathscr{A} \odot \mathscr{B}) \cdot \mathbf{L} &= \mathscr{A} [\mathbf{I}, \mathbf{L}] \cdot \mathscr{B} = \mathscr{A}_{isk} L_{sn} \mathscr{B}_{ink}, \\ (\mathscr{A} \odot_l \mathscr{B}) \cdot \mathbf{L} &= \mathscr{A} \mathbf{L} \cdot \mathscr{B} = \mathscr{A}_{ijk} L_{kn} \mathscr{B}_{ijn}, \end{aligned} \tag{44}$$

being attached to any pairs of third order tensors $(\mathscr{A}, \mathscr{B}.)$ **Proposition 3.** Under the hypothesis that the specific micro momenta are defined in terms of the free energy function, the reduced dissipation inequality can be finally written as

$$
\begin{aligned}
&\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p_{\mathscr{K}}\cdot\mathbf{L}^p+(\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^h-\partial_{\mathbf{H}}\varphi)\cdot\frac{d}{dt}\mathbf{H}+\\
&+\{\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi\;{}_r\odot\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}-\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}\odot\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi-\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}\odot_l\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi\}\cdot\mathbf{L}^p+\\
&+((\nabla_{\mathscr{K}}\mathbf{H})\odot_l\partial_{\nabla_{\mathscr{K}}\mathbf{H}}\varphi)\cdot\mathbf{L}^p\geq 0.
\end{aligned}
\tag{45}
$$

Axiom 5. The time derivative of the plastic distorsion and damage tensorial variable, $\mathbf{F}^p$ and $\mathbf{H}$, are postulated to be given by the following viscoplastic type evolution equations

$$
\begin{aligned}
\xi_1\,\mathbf{L}^p &=\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p_{\mathscr{K}}+\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi\;{}_r\odot\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}-\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}\odot\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi-\\
&-\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}\odot_l\partial_{\overset{(\mathrm{p})}{\mathscr{A}}_{\mathscr{K}}}\varphi+(\nabla_{\mathscr{K}}\mathbf{H})\odot_l\partial_{\nabla_{\mathscr{K}}\mathbf{H}}\varphi, \qquad (46)\\
\xi_2\,\dot{\mathbf{H}} &=\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^h_{\mathscr{K}}-\partial_{\mathbf{H}}\varphi
\end{aligned}
$$

where $\mathbf{L}^p=\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}$, with the constitutive functions ξ_k, for k=1,2, restricted by the condition

$$
\xi_1\,\mathbf{L}^p\cdot\mathbf{L}^p+\xi_2\,\dot{\mathbf{H}}\cdot\dot{\mathbf{H}}\geq 0 \tag{47}
$$

We mention here that the micro forces, namely micro stresses $\boldsymbol{\Upsilon}^p_{\mathscr{K}}$ and $\boldsymbol{\Upsilon}^d_{\mathscr{K}}$ together with the corresponding micro momenta $\boldsymbol{\mu}^p_{\mathscr{K}}$ and $\boldsymbol{\mu}^d_{\mathscr{K}}$, satisfy their own micro balance equations, see for instance [5] and [6]. If the micro momenta are replaced by the energetic type constitutive equations (42) the micro balance equations allow us to determine the constitutive equations for micro stresses. Consequently the well defined evolution equations can be derived, see for instance [12].

5. CONCLUSIONS

Within the framework of continuum mechanics the complete set of constitutive and evolution equations have been formulated in order to describe the behaviour of elasto-plastic materials with damaged structure.

The constitutive and evolution equations have been derived to be compatible with certain dissipation principle, formulated here as the free energy imbalance.

Two type of finite deformation models have been formulated for elasto-plastic damaged body. In order to formulate the boundary value problems say for equilibrium, the balance equation has to be added. In the considered here models, the balance equations for macro forces are reduced to $Div\mathbf{T} + \mathbf{b} = 0$, were $\mathbf{b}$ denotes the body force and $\mathbf{T}$ is Cauchy stress tensor, as the macro momentum is vanishing in the second model, see formula (38).

Finally we pointed out some remarks concerning the formulated problems. In order to solve appropriate boundary problem the damage variable has to be chosen, say $\mathbf{F}^d$, and the free energy function has to be given, say by (20). For shake of simplicity the presence of internal variables $\boldsymbol{\alpha}$ is neglected.

We emphasized that the plastic distortion and damage variable, say $\mathbf{F}^p$ and $\mathbf{F}^d$, are defined by differential type evolution equations. The initial value have to be given in order to have formally a well defined evolution system.

The Cauchy stress $\mathbf{T}$, the deformation gradient $\mathbf{F}$, the plastic distortion $\mathbf{F}^p$ and damage variable $\mathbf{F}^d$ are the unknowns of the problem. All these fields are considered to be functions of the material point of the body, i.e. $\mathbf{X} \in \mathscr{B}$, at any time t.

By solving the boundary value problem, see for instance the papers by Cleja-Ţigoiu and Paşcan [11], and Cleja-Ţigoiu et al. [12], we make distinctions between the nature of the unknowns. The Cauchy stress $\mathbf{T}$, the deformation gradient $\mathbf{F}$, are the basic unknowns of the problem to be solved at any time t, while the plastic distortion $\mathbf{F}^p$ and damage variable $\mathbf{F}^d$ are supposed to be given, as parameter functions.

We proceed as follows:

- The Cauchy stress $\mathbf{T}$ has to satisfy the equilibrium equation $Div\mathbf{T} = 0$, written here in the absence of the body forces;
- The Cauchy stress is considered to be function derived from the free energy function through the formula $(26)_1$ at any time t, as function of the current values of $\mathbf{F}$;
- In order to put into evidence the presence of $\mathbf{F}$ and $\mathbf{F}^p$ and $\mathbf{F}^d$, as like-parameter functions, the tensor field $\hat{\mathbf{F}}$ has to be eliminated for instance, via the formula $\hat{\mathbf{F}} = \mathbf{F}(\mathbf{F}^p)^{-1}$, at any time t.
- The elastic type non-linear problem has to be solved at time $t + \delta t$, when the values of the all unknowns are given at the previous moment of time t. Here is the argument for solving incremental equilibrium equations, at which the evolution equations have to be added.
- By using the update algorithms associate with the evolution equations for $\mathbf{F}^d$ and $\mathbf{F}^p$ the parameters functions were determined.

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