ON THE VARIATIONAL PROCESS
IN SINGULAR OPTIMAL CONTROL

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In this paper, we study the problem of the optimal control for Bolza functionals by investigating the extremals containing singular arcs utilizing the Moore-Penrose generalized inverse.

1. THE OPTIMUM PROBLEM

We consider the space:

\[ D = E^1 \times E^n \times E^m. \]  

Let:

\[ D_0 \subset D \]  

be an open domain. Suppose that

\[ f(t, x(t), u(t)) : D \to E^n, \]  

\[ f_0(t, x(t), u(t)) : D \to E^1 \]  

are $C^2$ functions. Assume that

\[ u(t) : E^1 \to E^m \]  

is piecewise continuous and $u$ is piecewise continuous on $\tau = [t_0, t_f].$

Our problem is to determine the function $u(t) \in U$ which minimizes the functional

\[ J(u) = \Phi(t_f, x_f) + \int_{t_0}^{t_f} f_0(t, x, u) \, dt, \]  

such that

\[ \dot{x} = f(t, x, u), \]  

\[ \Phi(t_f, x_f) = \Phi(x(t_f), x(t_f)). \]
We assume that
\[ \Gamma = \{ x(t), u(t) \} \in D_0, \quad t \in T, \quad u \in U \] (10)
are admissible arcs.

Define the Hamiltonian
\[ H(t, x, u, p) = f_0(t, x, u) + p^T f(t, x, u), \] (11)
where $T$ designates the transposition operation. The arc $\Gamma$ is an extremal arc if along it there exists a multiplier vector $p(t)$ such that
\[ p^T = -H_x \] (12)
\[ H_u = 0. \] (13)

Let
\[ \delta x = \xi, \quad \delta u = \eta, \quad \delta p = \zeta \] (14)
be the variations of the state $x$, control $u$, and multiplier $p$ respectively.

The auxiliary minimum problem consists in the minimization of the second variation,
\[ J_2(\eta) = \frac{1}{2} \int_{t_0}^{t_f} 2\omega(t, \xi, \eta)dt + \frac{1}{2} \xi^T G \xi \bigg|_{t_f}, \] (15)
where
\[ 2\omega = \xi^T H_{xx} \xi + 2\eta^T H_{ux} \xi + \eta^T H_{uu} \eta, \] (16)
\[ G = \Phi + \nu^T \psi, \] (17)
subject to
\[ \dot{\xi} = f_x \xi + f_u \eta, \quad \xi(t_0) = 0 \] (18)
\[ \psi \xi \bigg|_{t_f} = 0. \] (19)

The matrices $f_x, f_u, H_{xx}, H_{ux}, H_{uu}$ are evaluated along the extremal arc $\Gamma$. 
The equations for the extremals are provided by the vanishing of the first variation of the functional $J$, given by (6),

\begin{align}
\dot{x} &= f, \\
\dot{p} &= -H^T_x, \\
H_u &= 0.
\end{align}

The equations of a neighboring extremal result from the variation of the system (20),

\begin{align}
\dot{e} &= f_x e + f_u \eta, \\
\dot{\xi} &= -H_x e e - H_u \eta - f_x^T, \\
H_u e + H_u \xi + f_u^T &= 0.
\end{align}

Equations (21)-(23), representing the Jacobi equations, are satisfied for an admissible variation $\eta \in U$.

2. MOORE-PENROSE INVERSE

We assume that $H_{uu}$ is singular along $\Gamma$. We denote by $H^*$ the generalized Moore-Penrose inverse of $H_{uu}$ evaluated along an extremal arc, defined by

\begin{align}
H^+ = \lim_{\varepsilon \to 0} \left( H_{uu}^T H_{uu} + \varepsilon^2 I \right)^{-1} H_{uu}^T, \tag{24}
\end{align}

where $I$ represents the identity matrix with the dimension required in (24). One can prove (Ref. [5]) that the limit in (24) exists and

\begin{align}
H^+ = \lim_{x \to 0} H_{uu}^T \left( H_{uu} H_{uu}^T + \varepsilon^2 I \right)^{-1}. \tag{25}
\end{align}

We introduce the notations

\begin{align}
\text{rank}(H_{uu}) &= \text{null}(H_{uu}), \\
N(H_{uu}) &= \{H_{uu} | H_{uu} = 0\}, \tag{26}
\end{align}

The following properties are satisfied:

\begin{align}
\text{R}(H_{uu}^T) = \text{R}(H^*), \tag{28}
\end{align}
\[ H_{uu}H^+ = I \]  

(29)

and we have

\[ H^+ = \begin{cases} 
H^{-1}_{uu}, & \text{if } \det(H_{uu}) \neq 0 \\ 
0, & \text{if } \det(H_{uu}) = 0. 
\end{cases} \]  

(30)

If \( \nu \in E^m \), we can write

\[ \nu = \hat{\nu} + \bar{\nu}, \]  

(31)

where

\[ \nu = \begin{cases} 
\hat{\nu} & \text{is the projection of } \nu \text{ on } R(H_{uu}), \\
\bar{\nu} & \text{is the projection of } \nu \text{ on } N(H_{uu}). 
\end{cases} \]  

(32)

From (30) and (32), it follows that

\[ H^+ (H_{uu} \nu) = H^{-1}_{uu} (H_{uu} \hat{\nu}) + 0 \cdot \bar{\nu} = \hat{\nu}. \]  

(33)

**Hypothesis (H).** There exists a symmetric continuously differentiable matrix \( P(t) \) such that, along \( \Gamma \), we have

\[ N(H_{uu} + Pf_u) \supseteq N(H_{uu}). \]  

(34)

Then, we have

\[ \text{either } H_{uu} = 0 \rightarrow H_{uu} + Pf_u = 0, \]  

(35)

or \( (H_{uu} + Pf_u)^T = H_{uu} + f_u^T P = 0. \)  

(36)

Therefore,

\[ \det(H_{uu}) = 0 \rightarrow \det(H_{uu} + f_u^T P) = 0. \]  

(37)

By negation, from (37) we have

\[ R(H_{uu} + f_u^T P) \subseteq R(H_{uu}). \]  

(38)

Consequently, we have proved the equivalence between Hypothesis (H) and the relation (38).

3. **CLEBSCH TRANSFORMATION**

From (23), using (38), it follows that
On the variational process in singular optimal control

\[ H^+ H_{uu} \eta = -H^+ \left( H_{uu} \xi + f_u^T \zeta \right) = \hat{\eta}. \] (39)

Introducing (31) and (39) in (21) and (23) respectively, one obtains

\[ \dot{\xi} = f_c \xi + f_u \left[ -H^+ \cdot \left( H_{uu} \xi + f_u^T \zeta \right) + \bar{\eta} \right] = \hat{\eta}, \] (40)

\[ \dot{\zeta} = -H_{xx} \xi - H_{uu} \left( \bar{\xi} + \bar{\eta} \right) - f_u^T \zeta = -C \xi - A^T \zeta - H_{uu} \bar{\eta}, \] (41)

where

\[ A = f_c - f_u H^+ H_{uu}, \quad B = f_u H^+ f_u^T, \quad C = H_{xx} - H_{uu} H^+ H_{uu}. \] (42)

In [7] we evaluate the second variation by adding an identically null expression, i.e.,

\[ \begin{align*}
\frac{1}{2} \xi^T \xi' & - \frac{1}{2} \xi^T \xi' \bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} \frac{1}{2} \frac{d}{dt} \left[ \xi^T(t) \xi(t) \right] dt = 0.
\end{align*} \] (43)

It follows that

\[ J_2 = \frac{1}{2} \int_{t_0}^{t_f} \left[ \xi^T W^* \xi + \sigma^T B \sigma + 2 \xi^T \left( H_{uu} + W_{uu} \right) \bar{\eta} \right] dt + \frac{1}{2} \xi^T \left( G_{xx} - W \right) \xi \bigg|_{t_f}, \] (44)

where

\[ W^* = \dot{\dot{W}} + A^T W + W A - W^T B W + C, \] (45)

\[ \sigma = \zeta - W \xi. \] (46)

In expression of \( J_2 \) one observes that

\[ \begin{align*}
\psi_x \xi(t_f) & = 0
\end{align*} \] (47)

or, by expression,

\[ \begin{align*}
\sum_{i=1}^{s} \psi_{x_i} \xi_i(t_f) + \sum_{i=s+1}^{n} \psi_{x_i} \xi_i(t_f) & = 0.
\end{align*} \] (48)

It follows that

\[ M_1 \bar{\xi}_S(t_f) + M_2 \bar{\xi}_{n-S}(t_f) = 0, \] (49)

where

\[ M_1 = \left( \psi_{x_i}^T \right)_{i, j = 1, 2, \ldots, s}, \]
\[ M_2 = \left( \psi_{x_i}^T \right)_{k, l = s + 1, s + 2, \ldots, n}, \] (50)
and
\[ \xi_x = (\xi_1, \xi_2, \ldots, \xi_x)^T, \quad (51) \]
\[ \xi_{n-x} = (\xi_{x+1}, \xi_{x+2}, \ldots, \xi_{n+x})^T. \quad (52) \]

From (49), one obtains
\[ \bar{\xi}_S(t_f) = -M_1^{-1}M_2\xi_{n-S}(t_f). \quad (53) \]

Then,
\[
\begin{bmatrix}
\bar{\xi}(t_f) \\
\bar{\xi}_S(t_f) \\
\bar{\xi}_{n-S}(t_f)
\end{bmatrix} = 
\begin{bmatrix}
-1 & -M_1 & -M_2 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{\xi}_S(t_f) \\
\bar{\xi}_{n-S}(t_f)
\end{bmatrix} = X\bar{\xi}_{n-S}(t_f). \quad (54)
\]

Consequently,
\[ J_2 = \frac{1}{2} \int_{t_0}^{t_f} \left[ \xi^T W^* \xi + \sigma^T B \sigma + 2 \xi^T T (H_{xx} + Wf_u) \eta \right] dt + \]
\[ + \frac{1}{2} \xi_{n-S}^T (F_{xx} + v^T \psi_{xx} - W) X\bar{\xi}_{n-S} \bigg|_{t_f}. \quad (55) \]

We have the following two theorems:

**Theorem 3.1.** If \( \Gamma \) satisfies Hypothesis \((H)\), and if it is an extremal of the variations, then
\[ J_2 = \frac{1}{2} \int_{t_0}^{t_f} \left[ \xi^T W^* \xi + \sigma^T B \sigma \right] dt + \frac{1}{2} \xi_{n-S}^T X^T (F_{xx} + v^T \psi_{xx} - W) X\bar{\xi}_{n-S} \bigg|_{t_f}. \quad (56) \]

**Theorem 3.2.** If Hypothesis \((H)\) holds along an extremal arc \( \Gamma \). If \( H_{uu} \) is non-negative definite along \( \Gamma \), then for each extremal variation there exists a matrix \( W = 0 \) such that
\[ F_{xx} + v^T \psi_{xx} - W \geq 0 \quad (57) \]
for \( t \in [t_0, t_f] \). Moreover, \( J_2 \) is non-negative.

Then the adjoint variations take the from
\[ \zeta = P \xi. \quad (58) \]
If $P^*\xi \equiv 0$, then the expression of the second variation becomes

$$J_2 = \frac{1}{2} \xi_{\bar{m}-3}^T X \{ F_{\alpha} + v^{*} \psi_{\alpha} - P \} \bar{X}_{\bar{m}-3} \mid f,$$

(59)

Therefore, we have the following Lemma.

**Lemma 3.1.** Along the extremal arc, if the condition $P^*\xi \equiv 0$ holds for all extremal variations, then there exists a matrix $P(t)$, with

$$P(t) + F_{\alpha} + v^{*} \psi_{\alpha} \mid f,$$

(60)

such that $J_2$ takes the minimum value.

Obviously, according to the previous results, the second variation given by (59) vanishes for the matrix $P$ defined in (60), and thus $J_2$ becomes minimal.

**Lemma 3.2.** Let $\Gamma$ be an extremal defined on $[t_0, t_f]$ satisfying Hypothesis (H). On the arc $\Gamma$, $P^*\xi \equiv 0$ for all the extremal values, then

$$N(f_\alpha) \subseteq N(H_{\mu}).$$

(61)

In addition, if $H_{\mu}$ is non-negative definite and if

$$\xi(t) = 0, \quad \text{for} \quad t_0 \leq t_1 < t_2 \leq t_f$$

it follows that

$$\xi(t) = 0, \quad \text{on} \quad [t_1, t_2].$$

**Proof.** By Hypothesis (H)

$$P^*\xi = 0.$$

Then,

$$\bar{\sigma} = -(A^* - PB)\bar{\sigma}.$$  

(62)

Let $\Phi_{\alpha}(t, t_1)$ be the matrix of the fundamental solution for (62). Then,

$$\Phi_{\alpha}(t, t_1) = I.$$  

(63)

Let

$$v(t) \in N(H_{\mu}).$$

Using Hypothesis (H), from the equation of Jacobi adding the term $\int_{t_0}^{t_f} P \xi$, it results that
The solution of (62) can be written as
\[ \sigma = \Phi_\alpha (t, t_f) v. \] (65)

Consequently, (106) becomes
\[ \Phi_\alpha^T(t, t_f) f_u^T \cdot v = 0. \] (66)

Then,
\[ \Phi_\alpha^T(t, t_f) f_u^T v = 0, \quad t \in [t_0, t_f]. \] (67)

Since \( \Phi_\alpha^T \) is a nonsingular, we have
\[ f_u^T v = 0 \] (68)
for all \( t \in [t_0, t_f] \). Consequently, from (68), we have
\[ v \in N(f_u) \] (69)
and then,
\[ N(f_u) \supseteq N(H_{uu}). \] (70)

From \( \tilde{\eta} \in N(H_{uu}) \), it results that, \( \tilde{\eta} \in N(f_u) \). Then, \( f_u \tilde{\eta} = 0 \), and it results
\[ \xi = (A - BP) \tilde{\xi} - B_\sigma \varepsilon. \] (71)

The homogeneous part of (71) represents the adjoint of (62). From (64), we can write
\[ \Phi_\alpha^T(t, t_f) \xi(t) = \xi(t_f) - \int_t^{t_f} \Phi_\alpha^T(\Gamma, t_f) B \sigma d\Gamma. \] (72)

Since
\[ \xi(t_f) = \xi'(t_f) = 0 \]
we have
\[ \int_t^{t_f} \Phi_\alpha^T(\Gamma, t_f) B \sigma d\Gamma = 0. \] (73)

Because \( B \) is nonnegative, due to the non-negativity of \( H_{uu} \), it follows that
\[ \Phi^*_\sigma(\Gamma, t_1)B\sigma \equiv 0, \]  
(74)
on \left[ t_0, t_f \right]. As \( \Phi^*_\sigma \) is nonsingular, it follows that
\[ \xi(t) \equiv 0, \text{ on } [t_1, t_2]. \]

**Theorem 3.3.** Let the arc \( \Gamma \), and let \( (\xi, \eta, \zeta) \) be a set of variations on the extremal arc \( \Gamma \). If \( \zeta = P\xi \), it follows \( P^*\xi \equiv 0 \) and \( \overline{\eta} \in N(H_{fu} + Pf_u) \).

**Proof.** Let
\[ \zeta = P\xi. \]  
(75)
Then, \( \sigma = 0 \). This implies that
\[ \dot{\sigma} = -P^*\xi - (A^r - PB)\sigma = -P^*\xi = 0. \]  
(76)
By differentiation in (75), it follows that
\[ \dot{\xi} = \dot{P}\xi + P\dot{\xi}, \]  
(77)
where \( \dot{P} \) is given by
\[ \dot{P} = P^* - AP - PA + PBC - C. \]  
(78)
Then, for \( \xi \) given by (75) and \( \sigma = 0 \), the Jacobi equations become
\[ \dot{\zeta} = (A - BP)\xi + f_u \overline{\eta}, \]  
(79)
\[ \zeta = -C\xi - A^r\zeta - H_{fu} \overline{\eta}. \]  
(80)
Substituting (78)-(79) in (77), one obtains that
Either \( (H_{fu} + Pf_u) \overline{\eta} = 0 \)
(81)or \( \overline{\eta} \in N(H_{fu} + Pf_u), \)
(82)
This ends the proof of Theorem 3.3.

**4. CONCLUSIONS**

One was studied the problem of the optimal control for Bolza functionals by investigating the extremals containing singular arcs utilizing the Moore-Penrose generalized inverse.

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