ON THE VARIATIONAL PROCESS IN SINGULAR OPTIMAL CONTROL

MIHAI POPESCU

In this paper, we study the problem of the optimal control for Bolza functionals by investigating the extremals containing singular arcs utilizing the Moore-Penrose generalized inverse.

1. THE OPTIMUM PROBLEM

We consider the space:

$$D = E^1 \times E^n \times E^m . \tag{1}$$

Let:

$$D_0 \subset D \tag{2}$$

be an open domain. Suppose that

$$f(t, x(t), u(t)) : D \to E^n , \tag{3}$$

$$f_0(t, x(t), u(t)) : D \to E^1 \tag{4}$$

are C^2 functions. Assume that

$$u(t) : E^1 \to E^m \tag{5}$$

is piecewise continuous and u is piecewise continuous on $\tau = [t_0, t_f]$.

Our problem is to determine the function $u(t) \in U$ which minimizes the functional

$$J(u) = \Phi(t_f, x_f) + \int_{t_0}^{t_f} f_0(t, x, u)\,\mathrm{d}t , \tag{6}$$

such that

$$\dot{x} = f(t, x, u), \tag{7}$$

Institut de Statistique et Mathémathiques Appliquées "Gheorghe Mihoc-Caius Iacob", P.O.Box 1–24, 010145 Bucarest, Roumanie.

Rev. Roum. Sci. Tech. – Méc. Appl., Tome 52, N° 1, P. 55–64, Bucarest, 2007

$$x(t_0) = x_0, \tag{8}$$

$$\psi(t_f, x_f) = 0. \tag{9}$$

We assume that

$$\Gamma = \{x(t), u(t)\} \in D_0, \quad t \in T, \quad u \in U \tag{10}$$

are admissible arcs.

Define the Hamiltonian

$$H(t, x, u, p) = f_0(t, x, u) + p^T f(t, x, u), \tag{11}$$

where **T** designates the transposition operation. The arc Γ is an extremal arc if along it there exists a multiplier vector $\mathbf{p}(t)$ such that

$$\dot{p}^T = -H_x \tag{12}$$

$$H_u = 0. \tag{13}$$

Let

$$\delta x = \xi, \quad \delta u = \eta, \quad \delta p = \zeta \tag{14}$$

be the variations of the state **x**, control **u**, and multiplier **p** respectively.

The auxiliary minimum problem consists in the minimization of the second variation,

$$J_2(\eta) = \frac{1}{2}\int_{t_0}^{t_f} 2\omega(t, \xi, \eta)dt + \frac{1}{2}\xi^T G_{xx}\xi\Big|_{t_f}, \tag{15}$$

where

$$2\omega = \xi^T H_{xx}\xi + 2\eta^T H_{ux}\xi + \eta^T H_{uu}\eta, \tag{16}$$

$$G = \Phi + \nu^T \psi, \tag{17}$$

subject to

$$\dot{\xi} = f_x \xi + f_u \eta, \quad \xi(t_0) = 0 \tag{18}$$

$$\psi_x \xi\Big|_{t_f} = 0. \tag{19}$$

The matrices $f_x, f_u, H_{xx}, H_{ux}, H_{uu}$ are evaluated along the extremal arc Γ.

The equations for the extremals are provided by the vanishing of the first variation of the functional J, given by (6),

$$\dot{x} = f, \tag{20a}$$

$$\dot{p} = -H_x^T, \tag{20b}$$

$$H_u = 0. \tag{20c}$$

The equations of a neighboring extremal result from the variation of the system (20),

$$\dot{\xi} = f_x \xi + f_u \eta, \tag{21}$$

$$\dot{\zeta} = -H_{xx}\zeta - H_{xu}\eta - f_x^T, \tag{22}$$

$$H_{uu}\eta + H_{ux}\xi + f_u^T = 0. \tag{23}$$

Equations (21)-(23), representing the Jacobi equations, are satisfied for an admissible variation $\eta \in U$.

2. MOORE-PENROSE INVERSE

We assume that H_{uu} is singular along Γ. We denote by H^* the generalized Moore-Penrose inverse of H_{uu} evaluated along of an extremal arc, defined by

$$H^+ = \lim_{x \to 0} \left(H_{uu}^T H_{uu} + \varepsilon^2 I \right)^{(-1)} H_{uu}^T, \tag{24}$$

where I represents the identity matrix with the dimension required in (24). One can prove (Ref. [5]) that the limit in (24) exists and

$$H^+ = \lim_{x \to 0} H_{uu}^T \left(H_{uu} H_{uu}^T + \varepsilon^2 I \right)^{-1}. \tag{25}$$

We introduce the notations

$$rank(H_{uu}) = \Re(H_{uu}), \tag{26}$$

$$N(H_{uu}) = \{ H_{uu} | H_{uu} = 0 \}, \tag{27}$$

The following properties are satisfied:

$$R(H_{uu}^T) = R(H^+), \tag{28}$$

$$H_{uu}H^{+} = I \tag{29}$$

and we have

$$H^{+} = \begin{cases} H_{uu}^{-1}, \text{ if } \det(H_{uu}) \neq 0 \\ 0, \text{ if } \det(H_{uu}) = 0. \end{cases} \tag{30}$$

If $\nu \in E^{m}$, we can write

$$\nu = \hat{\nu} + \bar{\nu}, \tag{31}$$

where

$$\nu = \begin{cases} \hat{\nu} \text{ is the projection of } \nu \text{ on } R(H_{uu}), \\ \bar{\nu} \text{ is the projection of } \nu \text{ on } \mathrm{N}(H_{uu}). \end{cases} \tag{32}$$

From (30) and (32), it follows that

$$H^{+}(H_{uu}\nu) = H_{uu}^{-1}(H_{uu}\hat{\nu}) + 0 \cdot \bar{\nu} = \hat{\nu}. \tag{33}$$

Hypothesis (H). There exists a symmetric continuously differentiable matrix $P(t)$ such that, along Γ, we have

$$N(H_{xu} + Pf_{u}) \supseteq N(H_{uu}). \tag{34}$$

Then, we have

$$\text{either } H_{uu} = 0 \rightarrow H_{xu} + Pf_{u} = 0, \tag{35}$$

$$\text{or } (H_{xu} + Pf_{u})^{T} = H_{xu} + f_{u}^{T}P = 0. \tag{36}$$

Therefore,

$$\det(H_{uu}) = 0 \rightarrow \det(H_{ux} + f_{u}^{T}P) = 0. \tag{37}$$

By negation, from (37) we have

$$R(H_{ux} + f_{u}^{T}P) \subseteq R(H_{uu}). \tag{38}$$

Consequently, we have proved the equivalence between Hypothesis (H) and the relation (38).

3. CLEBSCH TRANSFORMATION

From (23), using (38), it follows that

$$H^{+}H_{uu}\eta = -H^{+}\left(H_{ux}\xi + f_u^T\zeta\right) = \hat{\eta}\,. \tag{39}$$

Introducing (31) and (39(in (21) and (23) respectively, one obtains

$$\dot{\xi} = f_x\xi + f_u\left[-H^{+}\cdot\left(H_{xx}\xi + f_u^T\zeta\right) + \bar{\eta}\right] = \hat{\eta}\,, \tag{40}$$

$$\dot{\zeta} = -H_{xx}\xi - H_{ux}\left(\hat{\eta} + \bar{\eta}\right) - f_x^T\zeta = -C\xi - A^T\zeta - H_{xu}\bar{\eta}\,, \tag{41}$$

where

$$A = f_x - f_u H^{+} H_{ux}, \quad B = f_u H^{+} f_u^T, \quad C = H_{xx} - H_{ux} H^{+} H_{ux}\,. \tag{42}$$

In [7] we evaluate the second variation by adding an identically null expression, i.e.,

$$\frac{1}{2}\xi^T W\xi\Big|_{t_0} - \frac{1}{2}\xi^T W\xi\Big|_{t_f} + \int_{t_0}^{t_f}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\xi^T(t)W\xi(t)\right]\mathrm{d}t = 0. \tag{43}$$

It follows that

$$J_2 = \frac{1}{2}\int_{t_0}^{t_f}\left[\xi^T W^*\xi + \sigma^T B\sigma + 2\xi^T\left(H_{xu} + Wf_u\right)\bar{\eta}\right]\mathrm{d}t + \frac{1}{2}\xi^T\left(G_{xx} - W\right)\xi\Big|_{t_f}, \tag{44}$$

where

$$W^* = \dot{W} + A^T W + WA - W^T BW + C\,, \tag{45}$$

$$\sigma = \zeta - W\xi\,. \tag{46}$$

In expression of J_2 on observes that

$$\psi_x\xi\left(t_f\right) = 0 \tag{47}$$

or, by expression,

$$\sum_{i=1}^{S}\psi_x^i\xi_i\left(t_f\right) + \sum_{i=S+1}^{n}\psi_x^i\xi_i\left(t_f\right) = 0\,. \tag{48}$$

It follows that

$$M_1\bar{\xi}_S\left(t_f\right) + M_2\bar{\xi}_{n-S}\left(t_f\right) = 0, \tag{49}$$

where

$$\begin{aligned} M_1 &= \left(\psi^i_{x_j}\right), \quad i, j = 1,\ 2, \ldots ..s, \\ M_2 &= \left(\psi^k_{xl}\right), \quad k, l = s+1, s+2, \ldots ..., n, \end{aligned} \tag{50}$$

and

$$\xi_s = \left(\xi_1, \xi_2, ..., \xi_s\right)^T, \tag{51}$$

$$\xi_{n-s} = \left(\xi_{s+1}, \xi_{s+2}, ..., \xi_{n+s}\right)^T. \tag{52}$$

From (49), one obtains

$$\bar{\xi}_S\left(t_f\right) = -M_1^{-1} M_2 \xi_{n-S}\left(t_f\right). \tag{53}$$

Then,

$$\bar{\xi}\left(t_f\right) = \begin{bmatrix} \bar{\xi}_S\left(t_f\right) \\ \cdots \\ \bar{\xi}_{n-S}\left(t_f\right) \end{bmatrix} = \begin{bmatrix} -M_1^{-1} M_2 \\ \cdots \\ I \end{bmatrix} \cdot \bar{\xi}_{n-S}\left(t_f\right) = X \bar{\xi}_{n-S}\left(t_f\right). \tag{54}$$

Consequently,

$$\begin{aligned} J_2 &= \frac{1}{2} \int_{t_0}^{t_f} \left[\xi^T W^* \xi + \sigma^T B \sigma + 2\xi^{TT}\left(H_{xu} + W f_u\right) \bar{\eta}\right] \mathrm{d}t + \\ &+ \frac{1}{2} \xi_{n-S}^T \left(F_{xx} + \nu^T \psi_{xx} - W\right) X \bar{\xi}_{n-S} \Big|_{t_f}. \end{aligned} \tag{55}$$

We have the following two theorems:

Theorem 3.1. *If* Γ *satisfies Hypothesis (H), and if it is an extremal of the variations, then*

$$J_2 = \frac{1}{2} \int_{t_0}^{t_f} \left[\xi^T W^* \xi + \sigma^T B \sigma\right] \mathrm{d}t + \frac{1}{2} \xi_{n-S}^T X^T \left(F_{xx} + \nu^T \psi_{xx} - W\right) X \bar{\xi}_{n-S} \Big|_{t_f}. \tag{56}$$

Theorem 3.2. *If Hypothesis (H) holds along an extremal arc* Γ. *If* H_{uu} *is non-negative definite along* Γ, *then for each extremal variation there exists a matrix W=O such that*

$$F_{xx} + \nu^T \psi_{xx} - W \geq 0 \tag{57}$$

for $t \in \left[t_0, t_f\right]$. *Moreover,* J_2 *is non-negative.*

Then the adjoint variations take the from

$$\zeta = P\xi. \tag{58}$$

If $P^{*}\xi \equiv 0$, then the expression of the second variation becomes

$$J_2 = \frac{1}{2}\xi_{n-S}^{T}X^{T}\left(F_{xx} + \nu^{T}\psi_{xx} - P\right)X\bar{\xi}_{n-S}\Big|_{t_f} \tag{59}$$

Therefore, we have the following Lemma.

Lemma 3.1. *Along the extremal arc, if the condition* $P^{*}\xi \equiv 0$ *holds for all extremal variations, then there exists a matrix* $P(t)$, *with*

$$P\left(t_f\right) + F_{xx} + \nu^{T}\psi_{xx}\Big|_{t_f}, \tag{60}$$

such that J_2 *takes the minimum value.*

Obviously, according to the previous results, the second variation given by (59) vanishes for the matrix *P* defined in (60), and thus J_2 becomes minimal.

Lemma 3.2. *Let* Γ *be an extremal defined on* $\left[t_0, t_p\right]$ *satisfying Hypothesis (H). On the arc* Γ, $P^{*}\xi \equiv 0$ *for all the extremal values, then*

$$N\left(f_u\right) \subseteq N\left(H_{uu}\right). \tag{61}$$

In addition, if H_{uu} is non-negative definite and if

$$\xi(t_1) = \xi(t_2) = 0 \quad \text{for} \quad t_0 \le t_1 < t_2 \le t_f$$

it follows that

$$\xi(t) \equiv 0, \quad \text{on } [t_1, t_2].$$

Proof. By Hypothesis (H)

$$P^{*}\xi = 0.$$

Then,

$$\dot{\sigma} = -\left(A^{T} - PB\right)\sigma. \tag{62}$$

Let $\Phi_{\sigma}\left(t, t_1\right)$ be the matrix of the fundamental solution for (62). Then,

$$\Phi_{\sigma}\left(t_1, t_1\right) = I. \tag{63}$$

Let

$$v(t) \in N\left(H_{uu}\right).$$

Using Hypothesis (H), from the equation of Jacobi adding the term $f_u^{T}P\xi$, it results that

$$f_u^T \sigma = 0 . \tag{64}$$

The solution of (62) can be written as

$$\sigma = \Phi_\sigma(t, t_1) v . \tag{65}$$

Consequently, (106) becomes

$$\left(\Phi_\sigma^T(t, t_1) f_u\right)^T \cdot v = 0 . \tag{66}$$

Then,

$$\Phi_\sigma^T(t, t_1) f_u v = 0, \ t \in \left[t_0, t_f\right] . \tag{67}$$

Since Φ_σ^T is a nonsingular, we have

$$f_u v = 0 \tag{68}$$

for all $t \in \left[t_0, t_f\right]$. Consequently, from (68), we have

$$v \in N(f_u) \tag{69}$$

and then,

$$N(f_u) \supseteq N(H_{uu}) . \tag{70}$$

From $\overline{\eta} \in N(H_{uu})$, it results that, $\overline{\eta} \in N(f_u)$. Then, $f_u \overline{\eta} = 0$, and it results

$$\dot{\xi} = (A - BP)\xi - B\sigma . \tag{71}$$

The homogeneous part of (71) represents the adjoint of (62). From (64), we can write

$$\Phi_\sigma^T(t, t_1)\xi(t) = \xi(t_1) - \int_{t_1}^{t} \Phi_\sigma^T(\Gamma, t_1) B\sigma \mathrm{d}\Gamma \cdot \tag{72}$$

Since

$$\xi(t_1) = \xi(t_2) = 0$$

we have

$$\int_{t_1}^{t} \Phi_\sigma^T(\Gamma, t_1) B\sigma \mathrm{d}\Gamma = 0 \cdot \tag{73}$$

Because **B** is nonnegative, due to the non-negativity of H_{uu}, it follows that

$$\Phi_{\sigma}^{T}(\Gamma, t_1) B\sigma \equiv 0, \qquad (74)$$

on $\left[t_0, t_f\right]$. As Φ_{σ}^{T} is nonsingular, it follows that

$$\xi(t) \equiv 0, \quad \text{on } [t_1, t_2].$$

Theorem 3.3. *Let the arc* Γ*, and let* (ξ, η, ζ) *be a set of variations on the extremal arc* Γ. *If* $\zeta = P\xi$*, it follows* $P^{*}\xi \equiv 0$ *and* $\overline{\eta} \in N(H_{uu} + Pf_u)$.

Proof. Let

$$\zeta = P\xi. \qquad (75)$$

Then, $\sigma = 0$. This implies that

$$\dot{\sigma} = -P^{*}\xi - \left(A^{T} - PB\right)\sigma = -P^{*}\xi = 0. \qquad (76)$$

By differentiation in (75), it follows that

$$\dot{\zeta} = \dot{P}\xi + P\dot{\xi}, \qquad (77)$$

where $\dot{P}$ is given by

$$\dot{P} = P^{*} - AP - PA + PBC - C. \qquad (78)$$

Then, for ξ given by (75) and $\sigma = 0$, the Jacobi equations become

$$\dot{\zeta} = (A - BP)\xi + f_u\overline{\eta}, \qquad (79)$$

$$\zeta = -C\xi - A^{T}\zeta - H_{xu_u}\overline{\eta}. \qquad (80)$$

Substituting (78)-(79) in (77), one obtains that

Either $\quad (H_{xu} + Pf_u)\overline{\eta} = 0 \qquad (81)$

or $\quad \overline{\eta} \in N(H_{xu} + Pf_u), \qquad (82)$

This ends the proof of Theorem 3.3.

4. CONCLUSIONS

One was studied the problem of the optimal control for Bolza functionals by investigating the extremals containing singular arcs utilizing the Moore-Penrose generalized inverse.

Aknowledgemnts. This research is partially supported by the research contract CEEX05-D11-44/2005 and Grant CNCSIS 911/2006.

Received 4 October 2006

REFERENCES

1. BERCOVITZ, I.D., *Variational methods in problems of control and programming*, J. Math. Anal., **3**, pp. 145-169 (1961).
2. ALBERT, A., *Regression and the Moore-Penrose Inverse*, Academic Press, New York, 1972.
3. MOLINARI, B. P., *Non-negativity of Quadratic Functional*, SIAM J. Control, **13**, pp. 792-806 (1975).
4. HULL, D. G., *Sufficient Conditions for a Minimum of the Free-Final-Time Optimal Control Problem*, Journal of Optimization Theory and Applications, **68**, *2*, pp. 2275-2287 (1991).
5. POPESCU, M., *Applications of Canonical Transformations in Optimizing Orbital Transfer*, Journal of Guidance, Control and Dynamics, **20**, pp. 774-779 (1995).
6. POPESCU, M., PELLETIER, F., *Optimal Control for a Lagrangeian Functionals*, 00-036, 2000 CNRS France.
7. PELLETIER, F., POPESCU, M., *Critère de normalité et critères d'optimalité pour des contrôles singulière*, Rev. Roum. Math. Pures et Appl., **46**, *2-3*, pp. 311-333 (2001).
8. POPESCU, M., *Singular Optimal Control for dynamics system*, Editura Academiei Române, Bucharest, 2002.
9. POPESCU, M., *Singular Normal Extremals and Conjugate Point for Bolza Functionals*, Journal of Optimization Theory and Applications, **114**, *2*, pp. 267–282 (2002).
10. POPESCU, M., *Control of nonlinear systems in singular problems*, Nonlinear Analysis, **63**, *5-7*, pp. 1985-1997 (2005).