MATHEMATICAL STUDY OF THE PLANAR OSCILLATIONS OF A HEAVY, ALMOST HOMOGENEOUS, INCOMPRESSIBLE, INVISCID LIQUID PARTIALLY FILLING A CONTAINER*

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In this paper, the authors, after showing that the problem of the small oscillations of a heavy heterogeneous liquid, which fills partially a container, is not a classical problem with discrete spectrum, study in details the two-dimensional problem in the particular case where the density of the liquid in the equilibrium position can be approximated by a linear function of the height of the particle, which differs very little from a constant, in the fluid domain. Then, the fluid is called "almost homogeneous in the fluid domain". They prove that, in this case, the spectrum is real and decomposed in two parts: an essential spectrum which fills a bounded interval and a point spectrum formed by a sequence of eigenvalues tending to infinity, by means of the methods of the functional analysis. Finally, they explicit the spectrum in the particular case of a rectangular container.

1. INTRODUCTION

The problem of the small oscillations of a homogeneous, incompressible, inviscid liquid in a container, taking into account the gravity or under zero gravity, has been dealt with in very many works, which are analyzed in the books by Moiseyev and Rumiantev [9], Myshkis et al. [10], Kopacheskii et al. [5]. The case of a container containing two immiscible, incompressible liquids has been studied in the book by Kopachevskii et al. [5], where is also treated the problem of a finite number of liquids, and by Capodanno [2] who considers also the case of viscous liquids.

But it seems that, since the old papers by Rayleigh [11] and Love [7], which are summarized in the classical book by Lamb [6, pp.378–380], the case of the heterogeneous liquid has not interested much the scientists.

In the paper (Capodanno [1]), the author has studied, in the two- dimensional case, the oscillations of a liquid whose density is an increasing function of the depth and which fills completely or partially an arbitrary container. He has proved that, because of the non-compactness of the operator of the problem, this one was not a standard vibration problem (Sanchez Hubert and Sanchez Palencia[13]) [i.e. a problem such that the eigenvalues are real and positive and form an increasing

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sequence which tends to infinity and the associated eigenfunctions form an orthogonal basis in a suitable Hilbert space] and he has studied in details the case of a rectangular container, in particular for the Rayleigh exponential law of density. But he could not study the spectrum of the problem in the general case.

Though it is theoretical, this paper has interested companies, which make software in fluids mechanics.

In a following paper (Capodanno, [3]), the same author has studied in details the case of a container closed by an elastic cover.

In this work, the authors, restricting themselves, like Rayleigh and Love, to the two-dimensional problem, consider the case where the density of the liquid in the equilibrium position is approximately a decreasing linear function of the height of the particle, which differs very little from a constant in the domain occupied by the liquid: the liquid is called “almost homogeneous in the domain”.

It is possible to use an approximated equation, analogous to the Boussinesq equation of the theory of convective fluid motion (Kopachevskii et al.. [5, pp. 268–269]. By using the method of the orthogonal projection, the authors obtain the operatorial equation of motion in a suitable Hilbert space and prove that the spectrum of the problem is real and decomposed in two parts: an essential spectrum which fills a bounded interval and a point spectrum formed by a sequence of eigenvalues, which tends to infinity. Finally, they explicit the spectrum in the particular case of a rectangular container.

2. EQUATION OF THE PLANAR MOTION OF AN HETEROGENEOUS LIQUID IN A CONTAINER

The axis Ox_2 is directed vertically upwards; the axis Ox_1 is horizontal and contains the free line in its equilibrium position (Fig.1).

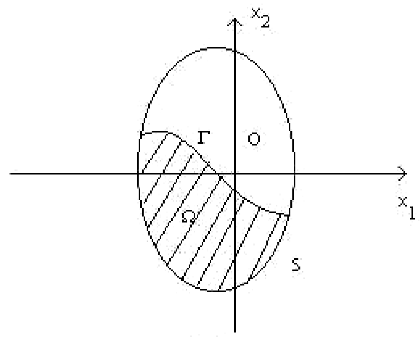

Fig. 1 – Liquid in a container.

We denote by $\vec{u}^*(x,t)$ the small displacement of the particle of the liquid, which occupies the position $x(x_1,x_2)$ at the instant *t* with respect to its position in the equilibrium configuration at the instant $t=0$, by $\rho^*(x,t)$ and $p^*(x,t)$ the density and the pressure at the point *x* at the instant *t*.

1) The equation of the motion are

$$\left.\begin{aligned}&\rho^*\ddot{\vec{u}}^* = -\nabla p^* - \rho^* g\vec{x}_2,\\ &\operatorname{div}\dot{\vec{u}}^* = 0,\end{aligned}\right\}\text{in } \Omega \quad \left(\ddot{\vec{u}}^* = \frac{\partial^2 \vec{u}^*}{\partial t^2}\right),$$

where *g* is the constant acceleration due to gravity and Ω the domain occupied by the liquid.

The equation of continuity is

$$\frac{\partial\rho^*}{\partial t} + \operatorname{div}(\rho\dot{\vec{u}}^*) = 0\,;$$

taking into account of the equation $\operatorname{div}\dot{\vec{u}}^* = 0,$ we have

$$\frac{\partial\rho^*}{\partial t} = -\dot{\vec{u}}^*\cdot\nabla\rho^*.$$

Let ε be a small parameter which characterizes the smallness of the oscillations; we set

$$\begin{aligned}\rho^*(x,t) &= \rho_0(x) + \varepsilon\,\rho^1(x,t) + \ldots,\\ p^*(x,t) &= p_0(x) + \varepsilon\,p^1(x,t) + \ldots,\\ \vec{u}^*(x,t) &= \qquad\quad \varepsilon\,\vec{u}^1(x,t) + \ldots,\end{aligned}$$

where $\rho_0(x)$ and $p_0(x)$ are the density and the pressure in the equilibrium position.

We have

$$-\nabla p_0 - \rho_0(x) g\vec{x}_2 = 0\,;$$

then, p_0 and ρ_0 are functions of x_2 only and we have

$$\frac{\mathrm{d}p_0(x_2)}{\mathrm{d}x_2} = -\rho_0(x_2)g\,.$$

We will assume that, at the equilibrium position, the density is an increasing function of the height, i.e

$$\frac{\mathrm{d}\rho_0(x_2)}{\mathrm{d}x_2} = \rho_0^{'} < 0.$$

We have, at the first order in ε

$$\mathrm{div}\dot{\vec{u}}^1 = 0 \ ;$$

integrating from $t = 0$ to t, we obtain

$$\mathrm{div}\vec{u}^1 = 0 .$$

In the same way, the linearized equation of continuity, the linearized Euler's equation and the kinematic condition on the wall take the form:

$$\rho^1(x,t) = -u_2^1(x,t)\rho_0'(x_2),$$
$$\rho_0\ddot{\vec{u}}^1 = -\nabla p^1 - \rho^1 g\, \vec{x}_2,$$
$$\vec{u}^1 \cdot \vec{n} = 0 \ \text{on } S,$$

where $\vec{n}$ is the unit normal vector on the wall.

The equation of the free line can be written in the form

$$x_2 = \varepsilon\, u_2^1(x_1,0,t) + \ldots .$$

We must have $p^* = p_a$ on the free line, $p_a = p_0(0)$ being the constant atmospheric pressure, i.e.

$$p_0[\varepsilon\, u_2^1(x_1,0,t)+\ldots] + \varepsilon\, p^1[x_1, \varepsilon\, u_2^1(x_1,0,t)+\ldots, t] + \ldots = p_0(0),$$

and therefore, at the first order in ε:

$$p^1(x_1,0,t) = \rho_0(0)\, g\, u_2^1(x_1,0,t).$$

Suppressing the upper index 1 in the preceding equations, we obtain the equations of the small oscillations of the liquid (Capodanno, 1993):

$$\rho_0(x_2)\ddot{\vec{u}} = -\nabla p - \rho(x,t)\, g\, \vec{x}_2, \tag{1}$$

$$\mathrm{div}\vec{u} = 0, \tag{2}$$

$$\rho(x,t) = -u_2(x,t)\rho_0^{'}(x_2), \tag{3}$$

in Ω

$$\vec{u}\cdot\vec{n}=0 \quad \text{on } S, \tag{4}$$

$$p(x_1,0,t)=\rho_0(0)\, g\, u_2(x_1,0,t). \tag{5}$$

Eliminating $\rho(x,t)$ between the equation (1) and (3), we have

$$\rho_0(x_2)\ddot{\vec{u}}=-\nabla p+\rho_0'(0)\, g\, u_2(x,t)\vec{x}_2 \quad \text{in } \Omega. \tag{6}$$

2) Let us look for the variational formulation of the problem. Taking the scalar product of the equation (6) and the vector $\vec{w}$ and integrating on Ω, we have

$$\int_\Omega \rho_0(x_2)\ddot{\vec{u}}\cdot\vec{w}\,\mathrm{d}\Omega=-\int_\Omega \nabla p\cdot\vec{w}\,\mathrm{d}\Omega+\int_\Omega \rho_0'(x_2)\, g\, u_2\, w_2\,\mathrm{d}\Omega.$$

Taking $\vec{w}$ such that $\mathrm{div}\vec{w}=0$ and $\vec{w}\cdot\vec{n}=0$ on S we have

$$\int_\Omega \nabla p\cdot\vec{w}\,\mathrm{d}\Omega=\int_\Omega[\mathrm{div}(p\,\vec{w})-p\,\mathrm{div}\vec{w}]\,\mathrm{d}\Omega=\int_{S+\Gamma} p\,\vec{w}\cdot\vec{n}\,\mathrm{d}S=\int_\Gamma p\, w_2\,\mathrm{d}\Gamma$$

and, therefore, using (5)

$$\int_\Omega \nabla p\cdot\vec{w}\,\mathrm{d}\Omega=\int_\Gamma \rho_0(0)\, g\, u_2\, w_2\,\mathrm{d}\Gamma.$$

Then, we obtain the variational equation

$$\int_\Omega \rho_0(x_2)\ddot{\vec{u}}\cdot\vec{w}\,\mathrm{d}\Omega-\int_\Omega \rho_0'(x_2)\, g\, u_2\, w_2\,\mathrm{d}\Omega+\int_\Gamma \rho_0(0)\, g\, u_{2|\Gamma}\, w_{2|\Gamma}\,\mathrm{d}\Gamma=0$$

for every $\vec{w}$ "admissible".

Looking for solutions in the form $\vec{u}(x,t)=\mathrm{e}^{\mathrm{i}\omega t}\vec{U}(x)$, we must find $\vec{U}(x)$ and a positive real number ω^2 such that

$$\omega^2\int_\Omega \rho_0(x_2)\vec{U}\cdot\vec{w}\,\mathrm{d}\Omega=-\int_\Omega \rho_0'(x_2)\, g\, U_2\, w_2\,\mathrm{d}\Omega+\int_\Gamma \rho_0(0)\, g\, U_{2|\Gamma}\, w_{2|\Gamma}\,\mathrm{d}\Gamma$$

for every $\vec{w}$ "admissible".

Let us consider the case of the completely filled container.

Since $\mathrm{div}\vec{U}=0$, $\mathrm{div}\vec{w}=0$, we can write

$$\vec{U}=\begin{pmatrix}\dfrac{\partial\psi}{\partial x_2}\\ -\dfrac{\partial\psi}{\partial x_1}\end{pmatrix};\quad \vec{w}=\begin{pmatrix}\dfrac{\partial\varphi}{\partial x_2}\\ -\dfrac{\partial\varphi}{\partial x_1}\end{pmatrix}.$$

Ψ is the stream-function which we can take equal to zero on the boundary $\partial\Omega$ of Ω.

Then, we introduce the space V formed by the function $\psi \in H_0^1(\Omega)$ equipped with the scalar product

$$(\psi, \varphi)_V = \int_\Omega \rho_0(x_2) \nabla \psi \cdot \nabla \varphi \, \mathrm{d}\Omega \, ;$$

the associated norm is obviously equivalent to the classical norm of $H_0^1(\Omega)$.

We have the problem: to find $\psi(x) \in V$ and a positive real number ω^2 such that

$$\omega^2 \, (\psi, \varphi)_V = -\int_\Omega \rho_0(x_2) \, g \, \frac{\partial \psi}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, \mathrm{d}\Omega \quad \forall \varphi \in V.$$

Since the bilinear form

$$a\,(\psi, \varphi) = -\int_\Omega \rho_0(x_2) \, g \, \frac{\partial \psi}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \, \mathrm{d}\Omega$$

is obviously continuous on $V \times V$, there exists a bounded linear operator A from V into V such that

$$a\,(\psi, \varphi) = (A\,\psi, \varphi)_V \quad \forall \varphi \in V,$$

so that we must study the spectral problem

$$A\,\psi = \omega^2 \psi \,, \quad \psi \in V.$$

A is bounded and symmetrical. It is positive definite; indeed, we have $(A\,\psi, \phi)_V \geq 0,$ zero for $\dfrac{\partial \psi}{\partial x_1} = 0$ a.e.

It can be shown that from this equality and the condition $\psi_{|\partial\Omega} = 0,$ we can deduce $\psi = 0$ a.e (Miklin, 1970, p.31).

Therefore, the spectrum of A is real and lies in the interval $[0, \| A \|]$. But, A is not compact and its spectrum is not discrete.

In the case of a partially filled container, appears an integral on the free line Γ and the problem is more complicated.

The problem of the small oscillations of a heavy heterogeneous liquid in a container is not a classical vibration problem with discrete spectrum. It seems difficult to study the spectrum in the general case; but (Capodanno, 1993), it is possible to explicit it in the particular case of a rectangular container.

In order to study the general case of an arbitrary container, we restrict ourselves to a “simplified” problem.

3. EQUATIONS OF MOTION OF AN ALMOST HOMOGENEOUS LIQUID

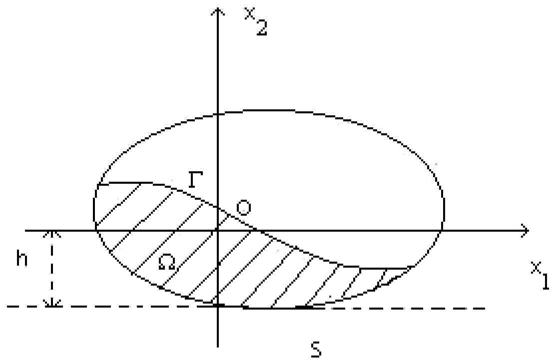

Fig. 2 – Small oscillations of liquid in the container.

We denote by $-h$ ($h>0$) the ordinate of the lowest point on the wall S of the container (Fig.2). In Ω we have $|x_2| \le h$.

Let us suppose that the density of the liquid in the equilibrium position can be written in the form

$$\rho_0(x_2) = f(\beta x_2) \ ,$$

with $f(0)>0, f'(0)<0$, β being a positive constant such as βh is sufficently small, so that $(\beta h)^2$, $(\beta h)^3$, $\cdots$, are negligible with respect to βh.

Since $|\beta x_2| \le \beta h$ in Ω, we have

$$\rho_0(x_2) = f(0) + \beta x_2 f'(0) + ...$$

Then the liquid is called *almost homogeneous* in Ω.

Changing the notation, we shall write

$$\rho_0(x_2) = \rho(1-\beta x_2) + o(\beta h).$$

In the following, we replace, in the equation (6), $\rho_0(x_2)$ by the positive constant ρ and $\rho_0^{'}(x_2)$ by the negative constant $-\rho\beta$.

Then, we obtain the approximated equation, analogous to the Boussinesq equation of the theory of the convective fluid motion (Kopachevkii et al.,1989, pp.268-269):

$$\ddot{\vec{u}} = -\frac{1}{\rho}\nabla p - \beta g u_2 \vec{x}_2 \qquad \text{in } \ \Omega. \tag{7}$$

The equation of the small oscillations of a heavy almost homogeneous incompressible inviscid liquid in the container are (7),(2),(4),(5).

By virtue of the incompressibility of the liquid, we must have $\int_\Gamma u_{2|\Gamma}\, \mathrm{d}\Gamma = 0$ and hence $\int_\Gamma p_{|\Gamma}\, \mathrm{d}\Gamma = 0$. But p is indeterminate to the constant of an additive function of time; the condition gives this function.

4. OPERATORIAL EQUATION OF THE PROBLEM

1) We are going to use the method of the orthogonal projection. We introduce the following spaces (Kopachevkii et al.,(1989), p.106):

$$J_0(\Omega) = \{\vec{u} \in L^2(\Omega) = [L^2(\Omega]^2; \mathrm{div}\vec{u} = 0, u_n = 0 \text{ in } H^{-1/2}(\partial\Omega)\},$$

$$G(\Omega) = \{\vec{u} = \nabla p\, ; p \in H^1(\Omega); \int_\Gamma p_{|\Gamma}\, \mathrm{d}\Gamma = 0\},$$

$$J_{0,s}(\Omega) = \{\vec{u} \in L^2(\Omega);\ \mathrm{div}\vec{u} = 0, u_n = 0 \text{ in } [H^{1/2}_{00}(S)]'\},$$

$$G_{h,s}(\Omega) = \{\vec{u} = \nabla p\, ;\ p \in H^1(\Omega); \Delta p = 0; u_n = 0 \text{ in } [H^{1/2}_{00}(S)]'; \int_\Gamma p_{|\Gamma}\, \mathrm{d}\Gamma = 0\},$$

$$G_{0,\Gamma}(\Omega) = \{\vec{u} = \nabla p\, ;\ p \in H^1(\Omega); p_{|\Gamma} = 0\},$$

equipped with the classical norm of $L^2(\Omega)$, the space (Dautray Lions, 1988, Vol.4, pp.1223–1224)

$$H(\Delta,\Omega) = \{v \in H^1(\Omega); \Delta v \in L^2(\Omega)\},$$

equipped with the norm

$$\| v \|_{H(\Delta,\Omega)} = \{\| v \|^2_{H^1(\Omega)} + \| \Delta v \|^2_{L^2(\Omega)}\}^{1/2},$$

(it is well-know that $\dfrac{\partial v}{\partial n}$ makes sense as element of $H^{-1/2}(\partial\Omega)$), and the space

$$H^1_{h,S} = \{p \in H^1(\Omega)\, ; \Delta p = 0\, ; \frac{\partial p}{\partial n} \text{ in } [H^{1/2}_{00}(S)]'\, ; \int_\Gamma p_{|\Gamma}\, \mathrm{d}\Gamma = 0\},$$

equipped with Dirichlet's norm $(\int_\Omega |\nabla p|^2\, \mathrm{d}\Omega)^{1/2}$, equivalent in this space to the norm of $H^1(\Omega)$ and $H(\Delta,\Omega)$. The space $[H^{1/2}_{00}(S)]'$ is defined in (Dautray et Lions, Vol.4, p.1241).

We recall the orthogonal decomposition (Kopachevskii et al. 1989), p.106 :

$$L^2(\Omega) = J_0(\Omega) \oplus G(\Omega) \quad ; \quad L^2(\Omega) = J_{0,S}(\Omega) \oplus G_{0,\Gamma}(\Omega),$$
$$J_{0,S}(\Omega) = J_0(\Omega) \oplus G_{h,S}(\Omega) \quad ; \quad G(\Omega) = G_{h,S}(\Omega) \oplus G_{0,\Gamma}(\Omega).$$

From first and the fourth decompositions, we have

$$L^2(\Omega) = J_0(\Omega) \oplus G_{h,S}(\Omega) \oplus G_{0,\Gamma}(\Omega).$$

We can suppose $\vec{u} \in L^2(\Omega)$ and $p \in H^1(\Omega)$. Since $\operatorname{div} \vec{u} = 0$ and $u_n = 0$ on S, we take $\vec{u} \in J_{0,S}(\Omega)$; on the other hand, we have $\nabla p \in G(\Omega)$.

By virtue of the preceding decompositions, we are looking for $\vec{u}$ and $\nabla\Phi$ in the form (Kopachevskii et al. 1989, pp.200-204) :

$$\vec{u} = \vec{v} + \nabla\Phi, \text{ with } \vec{v} \in J_0(\Omega), \nabla\Phi \in G_{h,S}(\Omega),$$
$$\nabla p = \nabla\varphi + \nabla K, \text{ with } \nabla\varphi \in G_{h,S}(\Omega), \nabla K \in G_{0,\Gamma}(\Omega).$$

Writing the equation of the motion in the form

$$\frac{\partial^2 \vec{v}}{\partial t^2} + \frac{\partial^2}{\partial t^2}(\nabla\Phi) = -\frac{1}{\rho}\nabla\varphi - \frac{1}{\rho}\nabla K - \beta\, g\left(v_2 + \frac{\partial\Phi}{\partial x_2}\right)\vec{x}_2$$

and calling P_0, P_S, P_Γ the orthogonal projections of $L^2(\Omega)$ onto $J_0(\Omega), G_{h,S}(\Omega), G_{0,\Gamma}(\Omega)$, we obtain

$$\frac{\partial^2 \vec{v}}{\partial t^2} = -\beta\, g\, P_0\left[\left(v_2 + \frac{\partial\Phi}{\partial x_2}\right)\vec{x}_2\right], \tag{8}$$

$$\frac{\partial^2}{\partial t^2}(\nabla\Phi) = -\frac{1}{\rho}\nabla\varphi - \beta\, g P_S\left[\left(v_2 + \frac{\partial\Phi}{\partial x_2}\right)\vec{x}_2\right], \tag{9}$$

$$0 = -\frac{1}{\rho}\nabla K - \beta\, g P_\Gamma\left[\left(v_2 + \frac{\partial\Phi}{\partial x_2}\right)\vec{x}_2\right]. \tag{10}$$

In the following, we will see that $\varphi_{|\Gamma}$ depends on $\frac{\partial\Phi}{\partial n}_{|\Gamma}$, so that, if Φ is known, it is possible to calculate $\varphi_{|\Gamma}$. But, since $\varphi \in H^1_{h,S}(\Omega)$, we have: $\nabla\Phi = 0$ in Ω, $\frac{\partial\varphi}{\partial n} = 0$ on S, $\int_\Gamma \varphi_{|\Gamma} \mathrm{d}\Gamma = 0$; if $\varphi_{|\Gamma}$ is known, φ is solution of Zaremba's problem and hence, is determined. Therefore, if Φ is known, it is possible to calculate φ.

Then, we need only the first two equations, the third gives ∇K.

Since, $P_S(v_2\vec{x}_2)$ and $P_S(\frac{\partial\Phi}{\partial x_2}\vec{x}_2)$ belong to $G_{h,S}(\Omega)$, we can set

$$P_S(v_2\vec{x}_2) = \nabla\psi \quad , \quad P_S(\frac{\partial\Phi}{\partial x_2}\vec{x}_2) = \nabla\Psi,$$

where ψ and Ψ, like φ and Φ, belong to $H^1_{h,S}(\Omega)$.

Then, from the equation (9), we can deduce the first integral

$$\frac{\partial^2\Phi}{\partial t^2} = -\beta\, g\,(\psi + \Psi) + C(t) \quad \text{in}\,\Omega\,.$$

Integrating on Γ, we have $C(t) = 0$ and finally

$$\frac{\partial^2\Phi}{\partial t^2} = -\beta\, g\,(\psi + \Psi) \quad \text{in}\ \Omega. \tag{11}$$

Let us transform the dynamic condition on the free line

$$p_{|\Gamma} = \rho\, g\, u_{2|\Gamma} = \rho\, g\, u_{n|\Gamma} \quad .$$

From $\nabla p = \nabla\varphi + \nabla K$, we deduce $p = \varphi + K + f(t)$ and, after integration on Γ, $f(t) = 0$, i.e.

$$p = \varphi + K \ \ \text{in}\ \Omega$$

and therefore

$$p_{|\Gamma} = \varphi_{|\Gamma}\ .$$

On the other hand, we have $u_n = v_n + \frac{\partial\Phi}{\partial n}$ and since $\vec{v} \in J_0(\Omega)$:

$$u_{n|\Gamma} = \frac{\partial\Phi}{\partial n}|_\Gamma$$

so that the dynamic condition on the free line can be written

$$\varphi_{|\Gamma} = \rho\, g\,\frac{\partial\Phi}{\partial n}|_\Gamma\ ;$$

therefore, $\varphi_{|\Gamma}$ depends on $\frac{\partial\Phi}{\partial n}|_\Gamma$.

Let us write the first integral (11) on Γ

$$\frac{\partial^2 \Phi_{|\Gamma}}{\partial t^2} = -\, g \, \frac{\partial \Phi}{\partial n} |_{\Gamma} - \beta \, g \, (\psi + \Psi)_{|\Gamma} \, . \qquad (12)$$

Since $\Phi \in H^1_{h,S}(\Omega)$, if $\Phi_{|\Gamma}$ is knowm, Φ is determined as solution of Zaremba's problem and we can calculate φ.

Consequently, we may take $\vec{v}$ and $\Phi_{|\Gamma}$ as unknown functions and the equations of the problem are (8) and (12).

2) We are going to transform these equations into an operatorial equation in a suitable Hilbert space.

Let us introduce the spaces

$$H^1_{\Gamma}(\Omega) = \{u \in H^{(}\Omega) \, ; \int_{\Gamma} u_{|\Gamma} \mathrm{d}\Gamma = 0\} \, ,$$

equipped with Dirichlet's norm, and

$$\widetilde{L}^2(\Gamma) = \{f \in L^2(\Gamma) \, ; \int_{\Gamma} f \, \mathrm{d}\Gamma = 0\} \, .$$

We denote by γ_{Γ} the restriction to Γ of the trace operator on $\delta\Omega$; we write

$$\gamma_{\Gamma} \, \Phi = \Phi_{|\Gamma} \quad , \quad \Phi \in H^1_{\Gamma}(\Omega) \ .$$

It is well-known that γ_{Γ} is bounded and compact from $H^1_{\Gamma}(\Omega)$ into $\widetilde{L}^2(\Gamma)$, that its range is

$$\widetilde{H}^{1/2}(\Gamma) = \{f \in H^{1/2}(\Gamma) \, ; \int_{\Gamma} f \, \mathrm{d}\Gamma = 0\} \, .$$

It can be proved (Kopachevskii et al., 1989, p.45) that the orthogonal complement of the kernel $H^1_{0,\Gamma}(\Omega)$ of γ_{Γ} in $H^1_{\Gamma}(\Omega)$ is the space $H^1_{0,S}(\Omega)$ and that γ_{Γ} is an isometry from $H^1_{h,S}(\Gamma)$ onto $\widetilde{H}^{1/2}(\Omega)$.

On the other hand, the embedding from $\widetilde{H}^{1/2}(\Gamma)$ into $\widetilde{L}^2(\Gamma)$ is, classically, dense and compact.

The adjoint T of γ_{Γ} is defined by

$$(T \, \psi, v)_{H^1_{\Gamma}(\Omega)} = \langle \psi, \gamma_{\Gamma} \, v \rangle \quad \forall \, \psi \in (\widetilde{H}^{1/2}(\Gamma))' \, , \quad \forall v \in H^1_{h,S}(\Omega) \, ,$$

where $(\widetilde{H}^{1/2}(\Gamma))'$ is the antiduality of $\widetilde{H}^{1/2}(\Gamma)$ and $\langle \cdot, \cdot \rangle$ is the antidualicty between $(\widetilde{H}^{1/2}(\Gamma))'$ and $\widetilde{H}^{1/2}(\Gamma)$. T is, classically, an isometry from $(\widetilde{H}^{1/2}(\Gamma))'$ onto $H^1_{h,S}(\Omega)$.

We can interpret $T^{-1}\Phi \in (\widetilde{H}^{1/2}(\Gamma))'$, $\Phi \in H^1_{h,S}(\Omega)$, as the normal derivative $\dfrac{\partial \Phi}{\partial n}|_\Gamma$.

Indeed, setting $\psi = T^{-1}u, u \in H^1_{h,S}(\Omega)$ in the preceding equation, we have

$$\int_\Omega \nabla u \cdot \nabla \bar{v}\, \mathrm{d}\Omega = \langle T^{-1}u, \gamma_\Gamma \rangle \quad \forall\, u, v \in H^1_{h,S}(\Omega).$$

Since $H^1_{h,S}(\Omega)$ is a subspace of $H(\Delta,\Omega)$, we can applied the generalized Green formula (Dautray, Lions, 1988, Vol.4, p.1224)

$$\langle \frac{\partial u}{\partial n}, v \rangle_{H^{-1/2}(\partial\Omega),\, H^{1/2}(\partial\Omega)} = \int_\Omega \Delta u \cdot \bar{v}\, \mathrm{d}\Omega + \int_\Omega \nabla u \cdot \nabla \bar{v}\, \mathrm{d}\Omega$$

$$\forall u \in H(\Delta,\Omega), \forall v \in H^1(\Omega).$$

For $u \in H^1_{h,S}$, we have then

$$\int_\Omega \nabla u \cdot \nabla \bar{v}\, \mathrm{d}\Omega = \langle \frac{\partial u}{\partial n}, v \rangle_{H^{-1/2}(\partial\Omega),\, H^{1/2}(\partial\Omega)},$$

so that

$$\langle T^{-1}u, \gamma_\Gamma \rangle = \langle \frac{\partial u}{\partial n}, v_{|\partial\Omega} \rangle_{H^{-1/2}(\partial\Omega),\, H^{1/2}(\partial\Omega)}.$$

Taking into account of $\dfrac{\partial u}{\partial n} = 0$ on S, we may consider $T^{-1}u$ as the normal derivative $\dfrac{\partial u}{\partial n_{|\Gamma}}$.

We are using also the following well-known result (Sanchez Hubert and Sanchez Palencia, 1989).

The operator $C = \gamma_\Gamma T$ is an isometry from $(\widetilde{H}^{1/2}(\Gamma))'$ onto $\widetilde{H}^{1/2}(\Gamma)$. Its restriction to $\widetilde{L}^2(\Gamma)$ is bounded, self- adjoint, definite positive and compact. $C^{1/2}$ is an isometry from $(\widetilde{H}^{1/2}(\Gamma))'$ onto $\widetilde{L}^2(\Gamma)$ and $C^{-1/2}$ is an isometry from $\widetilde{H}^{1/2}(\Gamma)$ onto $\widetilde{L}^2(\Gamma)$.

Because of the interpretation of T^{-1}, we have

$$\Phi_{|\Gamma} = \left(C \frac{\partial \Phi}{\partial n} \right)_{|\Gamma}.$$

Let us introduce in the equations (8) and (12) the function

$$\eta = C^{-1/2} \left(\frac{\partial \Phi}{\partial n} \right)_{|\Gamma} = C^{1/2}\, \Phi_{|\Gamma} \in \tilde{L}^2(\Gamma)$$

instead of Φ.

The equation (12), after applying the operator $C^{-1/2}$, takes the form

$$\frac{\mathrm{d}\eta^2}{\mathrm{d}t^2} = -g\, C^{-1}\eta - \beta\, g (C^{-1/2}\psi_{|\Gamma} + C^{-1/2}\Psi_{|\Gamma}). \tag{13}$$

Let us consider the equation (8), i.e.

$$\frac{\mathrm{d}^2\vec{v}}{\mathrm{d}t^2} = -\beta\, g\, P_0(v_2\vec{x}_2) - \beta\, g\, P_0\left(\frac{\partial \Phi}{\partial x_2} \vec{x}_2 \right).$$

We may put

$$\beta\, g\, P_0(v_2\vec{x}_2) = A_{11}\, \vec{v},$$

where A_{11} is a linear operator from $J_0(\Omega)$ into $J_0(\Omega)$.

Since Φ defined linearly on η, we may put

$$\beta\, g\, P_0\left(\frac{\partial \Phi}{\partial x_2} \vec{x}_2 \right) = A_{12}\, \eta,$$

where A_{12} is a linear operator from $\tilde{L}^2(\Gamma)$ into $J_0(\Omega)$.

Then the equation (8) takes the form

$$\frac{\mathrm{d}^2\vec{v}}{\mathrm{d}t^2} = A_{11}\, \vec{v} - A_{12}\, \eta. \tag{14}$$

In the same way, since $\nabla\psi = P_S(v_2\vec{x}_2)$, we can put

$$\beta\, g\, C^{-1/2}\, \psi_{|\Gamma} = A_{21}\, \vec{v},$$

where A_{21} is a linear operator from $J_0(\Omega)$ into $\tilde{L}^2(\Gamma)$.

Since $\nabla\psi = P_S\left(\frac{\partial\Phi}{\partial x_2}\vec{x}_2\right)$ depends linearly on η, we may put

$$\beta\, g\, C^{-1/2}\,\psi_{|\Gamma} = A_{22}\,\vec{v}\,,$$

where A_{22} is a linear operator from $\tilde{L}^2(\Gamma)$ into $\tilde{L}^2(\Gamma)$.

Then the equation (13) takes the form

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} = -g\,C^{-1}\eta - A_{21}\,\vec{v} - A_{22}\,\eta\,. \tag{15}$$

Let us put

$$y = \begin{pmatrix}\vec{v}\\ \eta\end{pmatrix} \in H = J_0(\Omega)\oplus\tilde{L}^2(\Gamma)\,,$$

H being equipped with the scalar product

$$(y,\tilde{y})_H = (\vec{v},\tilde{\vec{v}})_{L^2(\Omega)} + (\vec{\eta},\tilde{\vec{\eta}})_{\tilde{L}^2(\Gamma)}\,,$$

and

$$A = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\quad B = \begin{pmatrix}0 & 0\\ 0 & g\,C^{-1}\end{pmatrix},$$

we obtain finally the operatorial equation of the problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (A+B)y = 0\,,\qquad y\in H. \tag{16}$$

5. PROPERTIES OF THE OPERATORS OF THE PROBLEM

1) *Study of the operator* A_{11}. In this study , we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the norm of $L^2(\Omega)$ or $J_0(\Omega)$.

a) A_{11} *is symmetrical.* Indeed, for every pair $\vec{u},\vec{v}\in J_0(\Omega)$, we have

$$(A_{11}\vec{u},\vec{v}) = \beta\,g\int_\Omega P_0(u_2\vec{x}_2)\cdot\bar{\vec{v}}\,\mathrm{d}\Omega = \beta\,g\int_\Omega(u_2\vec{x}_2)\cdot\bar{\vec{v}}\,\mathrm{d}\Omega = \beta\,g\int_\Omega u_2\cdot\overline{v_2}\,\mathrm{d}\Omega$$

and

$$(\vec{u},A_{11}\vec{v}) = \overline{(A_{11}\vec{v},\vec{u})} = \beta\,g\overline{\int_\Omega u_2\cdot\overline{v_2}\,\mathrm{d}\Omega}\,.$$

Therefore, we obtain

$$(A_{11}\vec{u},\vec{v}) = (\vec{u}, A_{11}\vec{v}\ ).$$

b) A_{11} *is bounded.* We have

$$\| A_{11}\vec{u} \| = \beta\, g \,\| P_0(u_2\vec{x}_2) \| \le \beta\, g \,\| u_2\vec{x}_2 \|,$$

and consequently

$$\| A_{11}\vec{u} \| \le \beta\, g \,\| \vec{u} \|.$$

c) *The spectrum* $\sigma(A_{11})$ *of* A_{11} *is the closed intervall* $[0,\beta\, g]$. Indeed, we have at first:

$$(A_{11}\vec{u},\vec{u}) = \beta\, g \int_{\Omega} | u_2 |^2 \,\mathrm{d}\Omega \ge 0.$$

Let us consider the ratio

$$\frac{(A_{11}\vec{u},\vec{u})}{\| \vec{u} \|^2} = \beta\, g \frac{\int_{\Omega} | u_2 |^2 \,\mathrm{d}\Omega}{\int_{\Omega} | \vec{u} |^2 \,\mathrm{d}\Omega}.$$

Its inf is obviously zero and its sup, i.e. $\| A_{11} \|$ is $\le \beta\, g$.

Therefore, the spectrum $\sigma(A_{11})$ is contained in the interval $[0, \| A_{11} \|] \subseteq [0,\beta\, g]$.

In order to prove that it is $[0,\beta\, g]$, we can apply the following Weyl's criterion (Reed and Simon (1970), p.273):

Let A be a bounded self-adjoint operator of a Hilbert space. λ belongs to the spectrum $\sigma(A)$ of A, if and only if, there exists a sequence $\{\theta_l\}$ such that $\| \theta_l \| = 1$ and $\lim_{l\to\infty} \| (A - \lambda\, I)\theta_l \| = 0$.

In order to prove that every λ such that $0 < \lambda < \beta\, g$ belongs to the spectrum $\sigma(A_{11})$ of A_{11}, we prove that, for every $\mu = \dfrac{\lambda}{\beta\, g}$ such that $0 < \mu < 1$, there exists a sequence $\{\vec{u}_l\} \in J_0(\Omega)$ verifying

$$\frac{\left\| \dfrac{1}{\beta g} K\vec{u}_l - \mu\vec{u}_l \right\|}{\| \vec{u}_l \|} \to 0 \ \text{ when } \ l \to \infty.$$

Then, since $\sigma(A_{11})$ is closed, it is the closed interval $[0, \beta\, g]$ and, consequently, $\| A_{11} \| = \beta\, g$.

We notice that, since there is not a discrete spectrum, $\sigma(A_{11})$ conicides with the essential spectrum $\sigma_{ess}(A_{11})$ of A_{11}.

We must construct a suitable sequence $\{\vec{u}_l\}$. With this aim in view, we adjoint a process used in (Kopachevskii, 1989, pp.193-196).

$q(x)$ being an element of $D(\Omega)$, let us put

$$\vec{u} = \begin{pmatrix} u_1 = \dfrac{\partial \Delta q}{\delta x_2} \\ u_2 = -\dfrac{\partial \Delta q}{\delta x_1} \end{pmatrix}.$$

Obviously, $\vec{u} \in J_0(\Omega)$.

We need to calculate $A_{11}\, \vec{u}_l$, and hence $P_0(u_2\, \vec{x}_2)$. Then, we introduce the auxiliary Neumann problem:

$$\Delta\varphi = \operatorname{div}(u_2\, \vec{x}_2) \quad \text{in } \Omega \qquad \frac{\partial \varphi}{\partial n} = (u_2\, \vec{x}_2) \cdot \vec{n} \quad \text{on} \quad \partial\Omega .$$

It is easy to see that

$$\vec{w} = u_2\, \vec{x}_2 - \nabla \varphi$$

belongs to $J_0(\Omega)$. Since $\nabla \varphi \in G(\Omega)$, we have $\vec{w} = P_0(u_2\, \vec{x}_2)$. Then, we can write

$$P_0(u_2\, \vec{x}_2) = u_2\, \vec{x}_2 - \nabla \varphi .$$

Consequently, we can calculate simply $P_0(u_2\, \vec{x}_2)$ and then $A_{11}\, \vec{u}$, if we know one solution of the auxiliary problem.

Indeed, the auxiliary problem is

$$\Delta\varphi = -\frac{\partial^2 \Delta\, q}{\partial x_1 \partial x_2} \ \text{ in } \ \Omega \ ; \ \ \frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega ,$$

and we can take $\varphi = -\dfrac{\partial^2 q}{\partial x_1 \partial x_2}$.

Then, we have

$$A_{11}\,\vec{u} = \beta\, g(u_2\,\vec{x}_2 - \nabla\varphi),$$

i.e.

$$\frac{1}{\beta g} A_{11}\vec{u} = \begin{pmatrix} \dfrac{\partial^3 q}{\partial^2 x_1 \partial x_2} \\ \dfrac{\partial^3 q}{\partial x_1 \partial^2 x_2} - \dfrac{\partial \Delta q}{\partial x_1} \end{pmatrix}.$$

In order to construct the sequence $\{\vec{u}_l\}$, we take

$$q(x) = q_{nm}(x) = \mathrm{e}^{\mathrm{i}(nx_1 + mx_2)}\theta(x),$$

where $\theta(x) \in D(\Omega)$ is independent on n and m and is equal 1 in the circle $|x - x_0| \le r$ ($x_0 \in \Omega$), whose the radius r is so small that this circle C lies entirely in Ω.

We consider the sequence

$$\vec{u}_{mn} = \begin{pmatrix} \dfrac{\partial \Delta q_{mn}}{\partial x_2} \\ -\dfrac{\partial \Delta q_{mn}}{\partial x_1} \end{pmatrix}$$

and we assume that n and m tend to infinity, the ratio $\dfrac{m}{n}$ remaining constant.

We obtain easily

$$\frac{\partial \Delta q_{mn}}{\partial x_1} = -in(n^2 + m^2)\theta(x)\mathrm{e}^{\mathrm{i}(nx_1 + mx2)} + O(n^2 + m^2),$$

where $\dfrac{O(n^2 + m^2)}{n^2 + m^2}$ contains only the derivatives of $\theta(x)$ and therefore, is uniformly bounded in Ω and equal to zero in the circle $|x - x_0| \le r$.

We calculate also $\dfrac{\partial \Delta q_{mn}}{\partial x_2}$ and we obtain finally

$$\vec{u}_{mn} = \begin{pmatrix} -im \\ in \end{pmatrix}(n^2 + m^2)\mathrm{e}^{\mathrm{i}(nx_1 + mx_2)}\theta(x) + O(n^2 + m^2).$$

In the same way, we have

$$\frac{1}{\beta\, g} A_{11}\vec{u}_{mn} = \begin{pmatrix} -im \\ in \end{pmatrix} (n^2+m^2)\mathrm{e}^{\mathrm{i}(nx_1+mx_2)}\theta(x) + O(n^2+m^2)\,.$$

From the preceding results , we deduce

$$\frac{1}{\beta\, g} A_{11}\vec{u}_{mn} - \frac{n^2}{n^2+m^2}\vec{u}_{mn} = O(n^2+m^2),$$

and therefore

$$\|\frac{1}{\beta\, g} A_{11}\vec{u}_{mn} - \frac{n^2}{n^2+m^2}\vec{u}_{mn}\| \le c(n^2+m^2),$$

where c is a positive constant.

Now, let us study $\|\vec{u}_{mn}\|$.

We have,

$$|\vec{u}_{mn}|^2 = (n^2+m^2)^3\,|\theta|^2 + O[(n^2+m^2)^{5/2}]\,,$$

so that

$$|\vec{u}_{mn}|^2 \le c_1(n^2+m^2)^3 \qquad (c_1 \text{ positive constant})$$

and

$$\|\vec{u}_{mn}\|^2 \le c_2(n^2+m^2)^3 \qquad (c_2 = c_1 \cdot \text{ meas.of } \Omega)\,.$$

On the other hand, in the circle $|x-x_0| \le r$, we have $|\vec{u}_{mn}|^2 = (n^2+m^2)^3$, since $\theta = 1$ and $O(n^2+m^2) = 0$. Then, we have

$$\|\vec{u}_{mn}\|^2 = \int_\Omega |\vec{u}_{mn}|^2\, \mathrm{d}\Omega \ge \int_{|x-x_0|\le r} \mathrm{d}\Omega = c_0(n^2+m^2)^3 \qquad (c_0 = \pi r^2).$$

Finally, we can write,

$$c_0(n^2+m^2)^3 \le \|\vec{u}_{mn}\|^2 \le c_0(n^2+m^2)^3.$$

Let μ such that $0 < \mu < 1$. For every $\varepsilon > 0$, no matter how small, a rational number $\dfrac{\overline{m}}{\overline{n}}$ can be found such that

$$\mu < \frac{\overline{n}^2}{\overline{n}^2+\overline{m}^2} = \frac{1}{1+\left(\dfrac{\overline{m}}{\overline{n}}\right)^2} < \mu + \varepsilon\,.$$

Let us choose, $m = l\overline{m}, n = l\overline{n}$, where l is an integer number which tends to infinity; we have $\frac{n^2}{n^2+m^2} = \frac{\overline{n}^2}{\overline{n}^2+\overline{m}^2}$ and then,

$$\mu < \frac{n^2}{n^2+m^2} < \mu + \varepsilon .$$

From the inequality

$$\| \frac{1}{\beta\, g} A_{11}\, \vec{u}_{mn} - \mu\, \vec{u}_{mn} \| \leq \| \frac{1}{\beta\, g} A_{11}\, \vec{u}_{mn} - \frac{n^2}{n^2+m^2} \vec{u}_{mn} \| + | \frac{n^2}{n^2+m^2} - \mu | \| \vec{u}_{mn} \|$$

and the preceding result, we deduce

$$\frac{\| \frac{1}{\beta\, g} A_{11}\vec{u}_{mn} - \mu\, \vec{u}_{mn} \|}{\vec{u}_{mn} \|} \leq \frac{\frac{c}{\sqrt{c_0}}}{\sqrt{\overline{n}^2+\overline{m}^2}} \frac{1}{l} + \varepsilon \sqrt{\frac{c_2}{c_0}}$$

and therefore

$$\frac{\| \frac{1}{\beta\, g} A_{11}\vec{u}_{mn} - \mu\, \vec{u}_{mn} \|}{\| \vec{u}_{mn} \|} \leq 2\varepsilon \sqrt{\frac{c_2}{c_0}},$$

for $m = l\overline{m}$, $n = l\overline{n}$ and l sufficiently great.

The sequence $\{\vec{u}_{l\overline{m},\, l\overline{n}}\}$ satisfies the Weyl criterion, so that we have $\sigma(A_{11}) = [0, \beta\, g] = \sigma_{ess}(A_{11})$.

2) *Study of the operator* A

2a) A *is symmetrical*. We have

$$(Ay_1, y_2)_H = (A_{11}\vec{v}_1 + A_{12}\eta, \vec{v}_2)_{L^2(\Omega)} + (A_{21}\vec{v}_1 + A_{22}\eta, \eta_2)_{\tilde{L}^2(\Gamma)}$$

and, therefore

$$\frac{1}{\beta\, g}(Ay_1, y_2)_H = (P_0[(v_{12} + \frac{\partial \Phi_1}{\partial x_2})\vec{x}_2], \vec{v}_2))_{L^2(\Omega)} + (C^{-1/2}(\psi_{1|\Gamma} + \Psi_{1|\Gamma}))_{\tilde{L}^2(\Gamma)} .$$

The first scalar product can be written

$$((v_{12} + \frac{\partial \Phi_1}{\partial x_2}), \vec{v}_2)_{L^2(\Omega)} .$$

Let us transform the second product scalar. We have

$$(\eta_2, C^{-1/2}(\psi_{1|\Gamma} + \Psi_{1|\Gamma}))_{\tilde{L}^2(\Gamma)} = (C^{1/2}\frac{\partial \Phi}{\partial n}|_\Gamma, C^{-1/2}(\psi_{1|\Gamma} + \Psi_{1|\Gamma}))_{\tilde{L}^2(\Gamma)} =$$

$$= \langle \frac{\partial \Phi}{\partial n}|_\Gamma, \psi_{1|\Gamma} + \Psi_{1|\Gamma} \rangle_{(\tilde{H}^{1/2}(\Gamma))', \tilde{H}^{1/2}(\Gamma)}$$

since $C_{1/2}$ is self-adjoint. Using the formula

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \langle T^{-1}u, v_{|\Gamma} \rangle_{(\tilde{H}^{1/2}(\Gamma))', \tilde{H}^{1/2}(\Gamma)} \qquad \forall\, u, v \in H^1_{h,S}(\Omega),$$

we obtain

$$(\eta_2, C^{-1/2}(\psi_{1|\Gamma} + \Psi_{1|\Gamma}))_{\tilde{L}^2(\Gamma)} = (\nabla\Phi_2, \nabla(\psi_1 + \Psi_1))_{\tilde{L}^2(\Omega)} =$$

$$= (\nabla\Phi_2, P_S[(v_{12} + \frac{\partial \Phi_1}{\partial x_2})\vec{x}_2])_{L^2(\Omega)} = (\nabla\Phi_2, (v_{12} + \frac{\partial \Phi_1}{\partial x_2})\vec{x}_2)_{L^2(\Omega)}.$$

Finally, we find

$$\frac{1}{\beta\, g}(Ay_1, y_2)_H = ((v_{12} + \frac{\partial \Phi_1}{\partial x_2})\vec{x}_2, \vec{v}_2 + \nabla\Phi_2)_{L^2(\Omega)} =$$

$$= \int_\Omega (v_{12} + \frac{\partial \Phi_1}{\partial x_2})\overline{(v_{22} + \frac{\partial \Phi_2}{\partial x_2})}\, \mathrm{d}\Omega,$$

from we deduce easily the symmetry of the operator A.

2b) *A is bounded and* $\| A \| = \beta\, g$. Taking $y_1 = y_2 = y$, we have

$$\frac{1}{\beta\, g}(Ay, y)_H = \int_\Omega \left| v_2 + \frac{\partial \Phi}{\partial x_2} \right|^2 d\Omega \le \| \vec{v} + \nabla\Phi \|^2_{L^2(\Omega)}.$$

Since, $\nabla\Phi \in H^1_{h,S}$, we have

$$\| \nabla\Phi \|^2_{L^2(\Omega)} = \int_\Omega \nabla\Phi \overline{\nabla\Phi}\, d\Omega = \langle \frac{\partial \Phi}{\partial n}|_\Gamma, \Phi_{|\Gamma} \rangle_{\left(\tilde{H}^{1/2}(\Gamma)\right)', \tilde{H}^{1/2}(\Gamma)} =$$

$$= \left\langle C^{-1/2}\eta, C^{1/2}\eta \right\rangle_{\left(\tilde{H}^{1/2}(\Gamma)\right)', \tilde{H}^{1/2}(\Gamma)}.$$

Since $C^{-1/2}$ is self-adjoint, we can write

$$\| \nabla \Phi \|^2_{L^2(\Omega)} = (\eta, \eta)_{2(\Gamma)} = \| \eta \|^2_{2(\Omega)},$$

so that

$$\| \vec{v} + \nabla \Phi \|^2_{L^2(\Omega)} = \| y \|^2_H .$$

Consequently, we have

$$\| A \| \leq \beta \, g.$$

But, from

$$\| A \| = \sup_{y \in H} \frac{\| Ay \|_H}{\| y \|_H} = \sup_{y \in H} \frac{(\| A_{11}\vec{v} + A_{12}\eta \|^2_{L^2(\Omega)} + \| A_{21}\vec{v} + A_{22}\eta \|^2_{2(\Gamma)})^{1/2}}{(\| v \|^2_{L^2(\Omega)} + \| \eta \|^2_{2(\Gamma)})^{1/2}},$$

we deduce

$$\| A \| \geq \sup_{\vec{v} \in_0(\Omega)} \frac{\| A_{11}\vec{v} \|_{2(\Omega)}}{\| \vec{v} \|_{2(\Omega)}} = \| A_{11} \| = \beta \, g$$

and finally

$$\| A \| = \beta \, g.$$

2c) *Study of the norms of the operators* A_{ij}. Writing the inequality

$$\frac{1}{\beta \, g} (Ay, y) \;\; \leq \| y \|^2 \qquad \forall \; y \in,$$

for $\vec{v} = 0$, we obtain

$$\frac{1}{\beta \, g} (A_{22}\eta, \eta)_{2(\Gamma)} \leq \| \eta \|^2_{2(\Gamma)} \qquad \forall \; \eta \in^2 (\Gamma),$$

and then

$$\| A_{22} \| \leq \beta \, g.$$

By definition of $\| A \|$, we obtain

$$\| A_{21} \| \leq \beta \, g \quad , \quad \| A_{12} \| \leq \beta \, g.$$

Taking $y_1 = \begin{pmatrix} 0 \\ \eta_1 \end{pmatrix}$ and $y_2 = \begin{pmatrix} \vec{v}_2 \\ 0 \end{pmatrix}$, we obtain, by virtue of the symmetry of A:

$$(A_{12}\eta_1, \vec{v}_2)_{L^2(\Omega)} = (\eta_1, A_{21}\vec{v}_2)_{L^2(\Gamma)} \qquad \forall\, \eta_1 \in \tilde{L}^2(\Gamma), \vec{v}_2 \in I_0(\Omega),$$

so that

$$A_{12}^* = A_{12}.$$

3) *Study of the operator* B. By virtue of the properties of C, C^{-1} is an unbounded operator of $\tilde{L}^2(\Gamma)$, self-adjoint, and strongly positive, so that B is an unbounded operator of H, self-adjoint and not-negative.

4) *First remark on the operator* $\mathcal{A} = A + B$. Looking for solutions of the equation (16) in the form

$$y(x,t) = y(x)\, \mathrm{e}^{\mathrm{i}\omega t}$$

and putting $\mathcal{A} = A + B$, we obtain

$$\mathcal{A}y = \omega^2 y,$$

$\mathcal{A}$ being self-adjoint like A and B, the spectrum of the problem is real. On the other hand, since

$$(\mathcal{A}y, y) = (Ay, y) + (gC^{-1}\eta, \eta)_{2(\Gamma)},$$

$\mathcal{A}$ is not negative, so that its spectrum lies on the real positive halfaxis and $\omega^2 = 0$ is an eigenvalue.

6. THE POINT SPECTRUM

Let us seek the eigenvalue ω^2 such that $\omega^2 > \beta\, g$. Setting $\mu = \omega^{-2} < \dfrac{1}{\beta\, g}$, we write the equations (14), (15) in the form

$$(I - \mu A_{11})\vec{v} = \mu A_{12}\eta, \tag{17}$$

$$\mu A_{21}\vec{v} + \mu(A_{22} + gC^{-1})\eta = \eta. \tag{18}$$

Since $\| A_{11} \| = \beta\, g$, $\| I - \mu A_{11} \|$ has a bounded inverse, which is holomorphic for $|\mu| < \dfrac{1}{\beta\, g}$ and we have for this inverse the Neumann series:

$$\widetilde{R}(\mu) = (I - \mu A_{11})^{-1} = \sum_{k=0}^{\infty} \mu^k A_{11}^k .$$

The equation (17) takes the form

$$\vec{v} = \mu \,\tilde{}\,(\mu)\, A_{12}\, \eta .$$

Substituting into the equation (18), we obtain the equation for η:

$$\mu^2\, A_{21}\, \widetilde{R}(\mu)\, A_{12}\, \eta + \mu(A_{22} + gC^{-1})\eta = \eta . \tag{19}$$

The bundle of operator $\widetilde{R}(\mu)$ is obviously self-adjoint, so that, since A_{12} and A_{21} are mutually adjoint, the bundle $A_{21}\, \widetilde{R}(\mu)\, A_{12}$ is self-adjoint.

The operator $D = A_{22} + gC^{-1}$ has an inverse which is bounded, self- adjoint, positive definite and compact from $\tilde{L}^2(\Gamma)$ into itself. Applying $D^{-1/2}$ to both sides of (16) and setting $\eta^{'} = D^{1/2}\, \eta$, we obtain the equation

$$E(\mu) = [\mu\, I - D^{-1} - \mu^2 \Phi(\mu)]\, \eta^{'} = 0, \quad \forall\, \eta' \in \tilde{L}^2(\Gamma) \tag{20}$$

where $\Phi(\mu) = D^{-1/2}\, A_{21}\, \widetilde{R}(\mu)\, A_{12}{}^{-1/2}$ is a bounded , self-adjoint bundle of operators.

The bundle $E(\mu)$ is classical (Kopacevskii et al., 1989, p.81):

– D^{-1} is self-adjoint, compact from $\tilde{L}^2(\Gamma)$ into itself;

– $\Phi(\mu)$ is an operatorial function which is self-adjoint and holomorphic for $|\mu| < \dfrac{1}{\beta\, g}$;

– $E(0) = -D^{-1}$ compact; $E'(0) = I$ strongly positive.

Therefore, there is a countable infinity of eigenvalues μ_k in the interval $[0, \beta\, g]$, which tend to zero as $k \to +\infty$, so that the corresponding eigenvalues ω_k^2 of the problem tend to infinity as $k \to +\infty$.

The corresponding eigenelements $\eta_k^{'}$ have not associated elements and form a Riesz basis* in a subspace of $\tilde{L}^2(\Gamma)$ which has a finite defect (i.e. whose the orthogonal complement in $\tilde{L}^2(\Gamma)$ has a finite dimension).

The corresponding eigenfunctions $y_k = \begin{pmatrix} \vec{v}_k \\ \eta_k \end{pmatrix}$ of the problem $Ay = \omega^2 y$ are determined by

$$\eta_k = D^{-1/2}\eta_k^{'} \ \ ; \ \vec{v}_k = \mu_k \widetilde{R}(\mu)\, A_{12}\, \eta_k .$$

If we assume that these eigenfunctions are normalized, we find easily

$$\| \vec{v}_k \|_{L^2(\Omega)} \leq \sqrt{2}\, \beta\, g \mu_k ,$$

so that $\vec{v}_k \to 0$ as $k \to \infty$ and therefore $\| \eta_k \|_{L^2(\Gamma)} \to 1$.

7. THE ESSENTIAL SPECTRUM

Now, we suppose that $\omega^2 \leq \beta\, g$ and then $\mu \geq \dfrac{1}{\beta\, g}$.

We are writing the equation (18) in the form

$$(\mu \ - I)\eta = -\mu A_{21}\vec{v} .$$

D is an unbounded, self-adjoint, strongly positive operator, with a compact inverse; let $\lambda(D)$ be its smallest eigenvalue. We have

$$((\mu \ - I)\eta,\eta)_{\tilde{L}^2(\Gamma)} \geq [\mu\lambda() - 1] \| \eta \|^2_{\tilde{L}^2(\Gamma)} .$$

We distinguish between two cases:

1) $\lambda(D) > \dfrac{1}{\beta\, g}$. Then, since $\mu \geq \dfrac{1}{\beta\, g}$, $\mu\lambda(D) - 1$ is strongly positive and compact and has a compact inverse. We can write

$$\eta = -\mu(\mu \ - I)^{-1}\, A_{21}\vec{v}.$$

---

* Let $\{e_k\}$ an orthonormal basis of $\tilde{L}^2(\Gamma)$ and Q a bounded operator with bounded inverse. Then, $\{Qe_k\}$ is a basis of $\tilde{L}^2(\Gamma)$, called Riezs basis of $\tilde{L}^2(\Gamma)$.

Substituing in the equation (17) and replacing μ by $-\omega^2$, we obtain

$$(M(\omega^2)-\omega^2 I)\vec{v}=0 \ , \ \vec{v}\in_0(\Omega)$$

with $M(\omega^2)=A_{11}-A_{12}(-\omega^2 I)^{-1}A_{21}$.

Let us study the bundle of operators $M(\omega^2)$.

If ω^2 is fixed real $\leq \beta\, g$, the operator $V(\omega^2)=A_{12}(D-\omega^2 I)^{-1}A_{21}$ is self-adjoint, and compact, because A_{12} and A_{21} are bounded and $(-\omega^2 I)^{-1}$ compact.

Let ω^2 fixed, arbitrary in $[0,\beta\, g]$; we consider the operator

$$M(\omega_1^2)=A_{11}-V(\omega_1^2).$$

Since $V(\omega_1^2)$ is compact, we have, using a classic Weyl criterion (Kopachevskii et al., 1989, p.21):

$$\sigma_{ess}[M(\omega_1^2)]=\sigma_{ess}(A_{11})=\sigma(A_{11})=[0,\beta\, g].$$

Let us use another Weyl criterion: let ω_2^2 be a point of $\sigma_{ess}[M(\omega_1^2)]$; there exists a "Weyl sequence" $\{\vec{v}_n\}$, which depends on ω_1^2 and ω_2^2 such that

$\vec{v}_n\to 0$ weakly; $\inf\|\vec{v}_n\|>0$; $(M(\omega_1^2)-\omega_2^2\, I)\vec{v}_n\to 0$ in $I_0(\Omega)$.

Let us choose $\omega_1^2=\omega_2^2$; the associated Weyl sequence $\{\overrightarrow{v_n}\}$ depends obviously on ω_1^2 only.

Then, ω_1^2 belongs to the spectrum of the problem

$$(M(\omega^2)-\omega^2\, I)\vec{v}\to 0 \quad , \quad \vec{v}\in_0(\Omega).$$

Indeed, ω_1^2 does not belong to the resolvent set of the operator $M(\omega_1^2)$, so that $M(\omega_1^2)-\omega_2^2\, I$ has not a bounded inverse. Therefore, ω_1^2 does not belong to the resolvent set of the bundle $M(\omega^2)-\omega^2\, I$ and, therefore, belongs to the spectrum of this bundle.

Since ω_1^2 is arbitrary in $[0,\beta\, g]$, every point of this interval belongs to the spectrum of the bundle $M(\omega^2)-\omega^2\, I$. Then, this spectrum is $[0,\beta\, g]$ and coincides with the essential spectrum of the problem $y=\omega^2 y$.

2) $\lambda(D)\leq\dfrac{1}{\beta\, g}$. By virtue of the properties of (D), the spectral problem

$$(D-\omega^2 I)\eta = 0 \quad , \quad \eta \in \tilde{L}^2(\Gamma)$$

is an eigenvalue classical problem: D has a countable infinity of positive eigenvalues ω_k^2, which tend to infinity as $k \to \infty$. Then, there is at most finite number of ω_k^2 in the interval $[0, \beta\, g]$.

At the points $\omega^2 \in [0, \beta\, g]$ which are different from these ω_k^2, $D-\omega^2 I$, and consequently $\mu D - I$, has a compact inverse. So, for these ω^2 the results of the first case are valid. Such ω^2 belongs to the essential spectrum of A. Since the essential spectrum is closed, every point of $[0, \beta\, g]$ belongs to the essential spectrum of the problem.

Finally, *the spectrum of the problem is formed by the essential spectrum* $[0, \beta\, g]$ *and by a point spectrum which lies outside of this interval.*

It seems difficult to explicit the spectrum of the problem, in particular the essential spectrum, except for special forms of the container.

8. THE CASE OF RECTANGULAR CONTAINER

Let us assume that, at the equilibrium position, the liquid occupies the domain Ω: $0 < x_1 < \pi$, $-h < x_2 < 0$, the free line being $x_2 = 0$, $0 < x_1 < \pi$ (Fig.3).

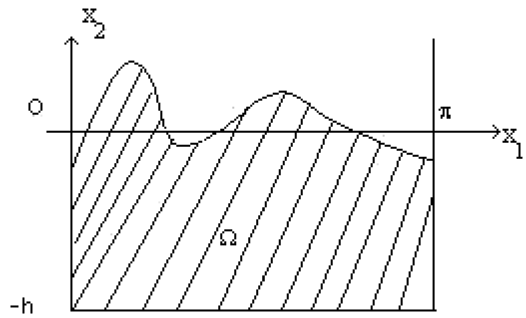

Fig. 3 – Rectangular container.

From the equation of motion (7), eliminating p and introducing the stream function $\psi(x_1, x_2, t)$ defined by $u_1 = \dfrac{\partial \psi}{\partial x_2}, u_2 = -\dfrac{\partial \psi}{\partial x_1}$, we obtain

$$\frac{\partial^2}{\partial t^2}(\Delta \psi) + \beta\, g \frac{\partial^2 \psi}{\partial x^2} = 0 \quad \text{in } \Omega. \tag{21}$$

The kinematical conditions are

$$\psi = 0 \quad for\ x_1 = 0\,,\, x_1 = \pi\,,\, x_2 = -h. \tag{22}$$

The dynamical condition on the free line is

$$p(x_1,0,t) = \rho\, g u_2(x_1,0,t).$$

Differentiating with respect to x_1 and eliminating $\frac{\partial p}{\partial x_1}$ we find

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial \psi}{\partial x_2}\right) = g\,\frac{\partial^2 \psi}{\partial x_1^2} \quad \text{for } x_2 = 0. \tag{23}$$

Looking for solutions of the systems (21),(22),(23) in the form

$$\psi(x_1,x_2,t) = X_1(x_1)\cdot X_2(x_2)\cdot \mathrm{e}^{\mathrm{i}\omega t},$$

we obtain

$$\frac{X_1''}{X_1} = \frac{\omega^2}{\beta\, g - \omega^2}\cdot\frac{X_2''}{X_2},$$

$$X_1(0) = X(\pi) = X_2(-h) = 0 \quad ; \quad \frac{X_1''}{X_1} = -\frac{\omega^2}{g}\cdot\frac{X_2'(0)}{X_2(0)}.$$

We find easily

$$X_1(x_1) = \sin n x_1 \quad (n = 1,2,\ldots)$$

and the problems

$$\begin{cases} X_2'' + n^2\,\dfrac{\beta\, g - \omega^2}{\omega^2} X_2 = 0 \\ \\ X_2(-h) = 0 \quad ; \quad X_2'(0) = \dfrac{n^2 g}{\omega^2} X_2(0). \end{cases} \qquad (n = 1,2,\ldots) \tag{24}$$

We can distinguish two cases:

1) $\omega^2 > \beta\, g$. We have

$$X_2(x_2) = \mathrm{sh} n\,\lambda\,(x_2 + h) \quad , \quad \lambda = \sqrt{\frac{\omega^2 - \beta\, g}{\omega^2}},$$

λ being root of the equation

$$n\,\mathrm{th} nh\lambda = \beta \frac{\lambda}{1-\lambda^2}, \tag{25}$$

which we can solve graphically, using the curves $y = n\,\mathrm{th} nh\lambda$ and $y = \beta \frac{\lambda}{1-\lambda^2}$.

The roots of (25) form a sequence $\{\lambda_n\}$, which is increasing and tends to 1. Therefore, the corresponding eigenvalues of the problem are

$$\omega_n^2 = \frac{\beta\, g}{1-\lambda_n^2}$$

and form a sequence which is increasing and tends to infinity. Writing (25) in the form

$$n - 2n\mathrm{e}^{-2nh\lambda} + \ldots = \beta \frac{\lambda}{1-\lambda^2}$$

and seeking λ_n in the form $\lambda_n = 1-\alpha_n$, $\alpha_n \to 0$ as $n \to \infty$, we find easily the asymptotic formulae:

$$\alpha_n \sim \frac{\beta}{2n} \;\; ; \;\; \lambda_n \sim 1 - \frac{\beta}{2n} \;,\; \omega_n^2 \sim gn.$$

2) $\omega^2 < \beta\, g$. We have

$$X_2(x_2) = \sin n\lambda(x_2 + h) \;\;,\; \lambda = \sqrt{\frac{\beta\, g - \omega^2}{\omega^2}},$$

λ being root of the equation

$$n\,\mathrm{th} nh\lambda = \beta \frac{\lambda}{1+\lambda^2}, \tag{26}$$

which we can again solve graphically.

For every $n = 1,2,3,\ldots$ we have roots λ_{nm} $(m = 1,2,\ldots)$ which verify the inequalities

$$\frac{m\pi}{nh} < \lambda_{nm} < \frac{(2m+1)\pi}{nh}.$$

So, we obtain a countable infinity of eigenvalues

$$\omega_{nm}^2 = \frac{\beta\, g}{1+\lambda_{nm}^2},$$

which lies in the interval $[0, \beta\, g]$.

From the inequalities, we deduce: if n is fixed, $\lim_{m\to\infty}\omega_{nm}^2 = 0$; if m is fixed, $\lim_{n\to\infty}\omega_{nm}^2 = \beta\, g$.

We are going to prove that every point of $\left[0, \beta\, g\right]$ is a limit-point of eigenvalues.

Denoting such point by $\alpha\beta g$, $0<\alpha<1$ and setting $\alpha = \dfrac{1}{1+\mu^2}$, $\mu>0$, we prove that there is a sequence of λ_{nm} which tends to μ .

For every integer N , no matter how great, there is a rational number $\dfrac{p_N}{q_N}$ such that

$$0<\mu-\frac{p_N}{q_N}\frac{\pi}{h}<\frac{1}{N}.$$

Let k_N be the smallest integer number such that $k_N > \dfrac{\pi N}{2q_N h}$; we set $n = k_N q_N$, $m = k_N p_N$.

It is easy to see graphically that

$$\lambda_{k_N q_N,\, k_N p_N} - \frac{k_N p_N \pi}{k_N q_N \pi} < \frac{1}{N},$$

so that

$$|\,\lambda_{k_N q_N,\, k_N p_N} - \mu\,| < \frac{2}{N}.$$

The sequence $\{\lambda_{k_N q_N,\, k_N p_N}\}$ tends to μ as $N\to\infty$.

Finally, every point of $[0, \beta\, g]$ is a limit-point of eigenvalues and another part of the spectrum is the closed interval $[0, \beta\, g]$.

9. CONCLUSION

We have proved that the spectrum of the small oscillations of an almost homogeneous heavy liquid in an open container is formed by an essential spectrum, which fills a closed interval of the real axis and by a point spectrum, which lies outside this interval.

We have explicited the spectrum in the case of a rectangular container: the point spectrum is formed by a countably infinity of eigenvalues ω_n^2, strictly greater than $\beta\, g$, which tend to infinity and an essential spectrum, which fills the closed interval $[0,\beta\, g]$, formed by a countable infinity of eigenvalues ω_{nm}^2 and by their limit-points.

Finally, let us make an interesting remark from the mechanical viewpoint.

For the rectangular container, in a neighbrhood, no matter how small, of every point $\lambda\in[0,\beta\, g]$, there are eigenvalues. Therefore, if the liquid is submitted to a sinusoidal perturbation with frequence $\sqrt{\lambda}$, there is some kind of resonance.

This remark is valid in the general case.

Let us the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = f\mathrm{e}^{i\omega t},$$

where $f\mathrm{e}^{\mathrm{i}\omega t}$ is a sinusoidale perturbation with frequence ω. Looking for solutions in the form $y = w\mathrm{e}^{\mathrm{i}\omega t}$, we obtain

$$(-\omega^2 I)w = f.$$

If A is a self-adjoint, positive definite, compact operator, it has a discrete spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots, \lambda_n \to 0$ as $n\to\infty$. Let $e_i\ (i=1,2,\ldots)$ the corresponding orthonormal eigenfunctions. Putting $f=\sum_i f_i e_i$ and seeking w in the form $w=\sum_i w_i e_i$ we obtain the classical equation

$$(\lambda_i - \omega^2)w_i = f_i \quad (i=1,2,\ldots).$$

If ω^2 is equal to one λ_i, we have the classical resonance phenomenon.

The corresponding equation is verifies only if $f_i = 0$ and, w_i is arbitrary, different from zero.

In our problem, if $\omega^2 \in \sigma_{ess}() = [0,\beta\, g]$, there exists, by virtue of the Weyl criterion, a sequence $\{y_i\}\in$ such that

$$\inf \| y_i \| > 0 \ , \ (-\omega^2 I)y_i = 0 \quad \text{in} .$$

We can say again that there is some kind of resonance.

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