KINEMATIC STUDY OF A PLANAR REDUNDANT MANIPULATOR WITH THREE DEGREES OF FREEDOM*

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The joint angles, velocities and accelerations, for a planar redundant manipulator with three degrees of freedom, are analysed, while the characteristic point of the manipulator end-effector tracks an imposed planar curve. In the first stage, only the end-effector displacement, velocity and acceleration are considered, while, in the second stage, the velocity and acceleration transmitted by means of Jacobian null space are added. These last ones determine the redundant manipulator self-motion and aim the optimisation of different performance criteria imposed by means of an arbitrary function. The obtained comparative results for the two stages are presented graphically.

1. INTRODUCTION

Redundant manipulators represent an efficient solution for solving kinematically complex tasks in the robot workspace. A redundant manipulator provides flexibility and ability of the end-effector movement on requested trajectories as well as the possibility to optimise different various criteria. Using the Jacobian null space and modifying properly the kinematic parameters of the manipulator joints one determines new kinematic configurations that ensures the right end-effector movement on imposed trajectories and, simultaneously, satisfies the selected performance criteria. These modifications of the joint kinematic parameters do not affect the fulfilment of the main end-effector task, while the obtained increased manipulator dexterity is very important in solving the robot complex work tasks [1, 2].

The aim of this work is a kinematic study of a planar redundant manipulator with three degrees of freedom, as an intermediate stage before the dynamic analysis, because, in case of redundant robots, a specific control of the dynamic properties produces an important influence on the dynamic behaviour of the system.

The proposed kinematic analysis is performed in two stages. In the first stage, the end-effector positions, velocities and accelerations on the imposed trajectory are considered known and the joint angles, velocities and accelerations are determined [3, 4]. In the second stage, the Jacobian null space influence is

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considered [5]. An arbitrary function is introduced in the Jacobian null space and generates a self-motion of the manipulator in order to satisfy some performance criteria. The used function is the sum of the squares of the joint angles sinus and aims to avoid the singular configurations of the manipulator.

2. KINEMATICS

A redundant serial manipulator with n degrees of freedom is considered working in an m-dimensional space, with $n > m$. The degree of redundancy is ($n - m$). The manipulator configuration is described by a vector $\mathbf{q}$ with n elements that describe the n joints or degrees of freedom; the end-effector configuration (position and orientation) is described by a vector $\mathbf{x}$ with m elements.

The manipulator kinematics is described by a function f, which represents an m-dimensional vector and which relates the vectors $\mathbf{x}$ and $\mathbf{q}$:

$$\mathbf{x} = f(\mathbf{q}). \tag{1}$$

If one differentiates equation (1), the relation between end-effector velocities and joint velocities is obtained:

$$\dot{\mathbf{x}} = \frac{\partial f(q)}{\partial q}\dot{\mathbf{q}}. \tag{2}$$

Denoting $\frac{\partial f(q)}{\partial q} = \mathbf{J} \in \Re^{m\times n}$ – manipulator Jacobian, results:

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}. \tag{3}$$

Considering only the end-effector movement, the solution of the equation (3) is given by:

$$\dot{\mathbf{q}} = \mathbf{J}^{+}\dot{\mathbf{x}}, \tag{4}$$

where: $\mathbf{J}^{+} = \mathbf{J}^{\mathrm{T}}\left(\mathbf{J}\mathbf{J}^{\mathrm{T}}\right)^{-1} \in \Re^{n\times m}$ – Jacobian pseudoinverse;

$\dot{\mathbf{x}} = [\dot{x}_{1\mathrm{E}} \quad \dot{x}_{2\mathrm{E}} \quad \ldots \quad \dot{x}_{m\mathrm{E}}]^{\mathrm{T}} \in \Re^{m\times 1}$ – end-effector velocity vector;

$\dot{\mathbf{q}} = [\dot{q}_1 \quad \dot{q}_2 \quad \ldots \quad \dot{q}_n]^{\mathrm{T}} \in \Re^{n\times 1}$ – joint velocity vector [3, 4].

Considering also the manipulator self-motion, the solution of equation (3) results as follows:

$$\dot{\mathbf{q}} = \mathbf{J}^{+}\dot{\mathbf{x}} + \mathbf{N}\dot{\mathbf{q}} = \dot{\mathbf{q}}_x + \dot{\mathbf{q}}_{\mathrm{N}}, \tag{5}$$

where: $\mathbf{N} = (\mathbf{I} - \mathbf{J}^{+}\mathbf{J}) \in \Re^{n\times n}$ – *projector* matrix. The vector $\dot{\mathbf{q}}_x$ is the part of joint velocities that contributes to the main task accomplishment, *i.e.* end-effector movement, while the vector $\dot{\mathbf{q}}_{\mathrm{N}}$, representing a vector of arbitrary joint velocities that satisfies the secondary objectives, is the part of joint velocities that produces the manipulator self-motion, [2].

Differentiating the equation (5), one obtains:

$$\ddot{\mathbf{q}} = \mathbf{J}^{+}\ddot{\mathbf{x}} + \dot{\mathbf{J}}^{+}\dot{\mathbf{x}} + \mathbf{N}\ddot{\mathbf{q}} + \dot{\mathbf{N}}\dot{\mathbf{q}} = \ddot{\mathbf{q}}_{x} + \ddot{\mathbf{q}}_{\mathrm{N}} \quad (6)$$

where: $\ddot{\mathbf{q}} = [\ddot{q}_1 \quad \ddot{q}_2 \quad \dots \quad \ddot{q}_n]^{\mathrm{T}} \in \Re^{n\times 1}$ – joint accelerations vector; $\ddot{\mathbf{q}}_{x} = \mathbf{J}^{+}\ddot{\mathbf{x}} + \dot{\mathbf{J}}^{+}\dot{\mathbf{x}}$; $\ddot{\mathbf{q}}_{\mathrm{N}} = \mathbf{N}\ddot{\mathbf{q}} + \dot{\mathbf{N}}\dot{\mathbf{q}}$. The vector $\ddot{\mathbf{q}}_{x}$ is the part of joint accelerations related to the end-effector movement, while the vector $\ddot{\mathbf{q}}_{\mathrm{N}}$ is the part of joint accelerations, transmitted by means of the null space, aiming to satisfy an arbitrary optimisation function [5].

3. APLICATION

For the kinematic study, a planar redundant manipulator with three degrees of freedom ($n = 3$, $m = 2$) is proposed. The initial position of the manipulator is shown in the figure 1.

The characteristic point M_E of the end-effector must move along the curve (C), which represents a circle segment. The initial position and the manipulator links dimensions are:

$$\theta_0 = [120^{\circ}\ {-90^{\circ}}\ 75^{\circ}]^{\mathrm{T}};$$
$$l_0 = 16.8\ \mathrm{u};\quad l_1 = l_2 = l_3 = 30\ \mathrm{u};\quad R = 20\ \mathrm{u};\quad x_C = 70\ \mathrm{u};\quad y_C = 0;\quad \alpha_0 = \pi/2, \quad (7)$$

where: θ_0 – vector of initial joint coordinates; l_0, l_1, l_2, l_3 – lengths of the links; R – radius of the circle segment; (x_C, y_C) – circle centre coordinates; α_0 – initial angular position (at $t = 0$); u – length unit.

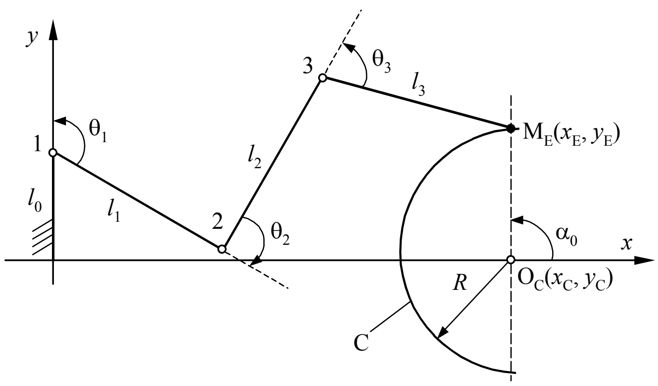

Fig. 1 – The initial position of the studied planar redundant manipulator.

The angular domain of the end-effector movement on the circle (C) is $\left[\frac{\pi}{2},\ \frac{3\pi}{2}\right]$. The movement is accomplished in three phases:

- uniformly accelerated motion for $t \in [0,\ t_1)$,
- uniform motion for $t \in (t_1,\ t_2)$,
- uniformly decelerated motion for $t \in (t_2,\ t_3]$,

where: $t_1 = 0.5$ s ; $t_2 = 2$ s ; $t_3 = 2.5$ s .

During the interval $t \in [0,\ t_1)$ of the uniformly accelerated motion phase with the angular acceleration $\varepsilon = \pi$ rad/s^2, the displacement, velocity and acceleration of the end-effector tip are given by following equations:

$$\begin{aligned}
x_{E1} &= x_C + R\cos\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right),\\
y_{E1} &= y_C + R\sin\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right),\\
\dot{x}_{E1} &= -R\varepsilon t\sin\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right),\\
\dot{y}_{E1} &= R\varepsilon t\cos\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right),\\
\ddot{x}_{E1} &= -R\varepsilon\sin\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right) - R\varepsilon^2 t^2\cos\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right),\\
\ddot{y}_{E1} &= R\varepsilon\cos\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right) - R\varepsilon^2 t^2\sin\left(\alpha_0 + \frac{\varepsilon t^2}{2}\right).
\end{aligned} \tag{8}$$

During the interval $t \in (t_1,\ t_2)$ of the uniform motion phase with angular velocity $\omega = \frac{\pi}{2}$ rad/s, the variation of the kinematic parameters of the end-effector tip is described by:

$$\begin{aligned}
x_{E2} &= x_C + R\cos\left(\alpha_0^{'} + \omega t\right),\\
y_{E2} &= y_C + R\sin\left(\alpha_0^{'} + \omega t\right),\\
\dot{x}_{E2} &= -R\omega\sin\left(\alpha_0^{'} + \omega t\right),
\end{aligned}$$

$$\begin{aligned}
\dot{y}_{E2} &= R\omega\cos\left(\alpha_0^{'} + \omega t\right) \\
\ddot{x}_{E2} &= -R\omega^2\cos\left(\alpha_0^{'} + \omega t\right) \\
\ddot{y}_{E2} &= -R\omega^2\sin\left(\alpha_0^{'} + \omega t\right)
\end{aligned} \tag{9}$$

where $\alpha_0^{'} = \alpha_0 + \frac{\varepsilon t_1^2}{2}$.

The interval $t \in (t_2,\ t_3]$ of the uniformly decelerated motion phase with the angular acceleration $\varepsilon = -\pi$ rad/s^2 is described by:

$$\begin{aligned}
x_{E3} &= x_C + R\cos\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right), \\
y_{E3} &= y_C + R\sin\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right), \\
\dot{x}_{E3} &= -R\left(\omega + \varepsilon t\right)\sin\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right), \\
\dot{y}_{E3} &= R\left(\omega + \varepsilon t\right)\cos\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right), \\
\ddot{x}_{E3} &= -R\varepsilon\sin\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right) - R\left(\omega + \varepsilon t\right)^2\cos\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right), \\
\ddot{y}_{E3} &= R\varepsilon\cos\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right) - R\left(\omega + \varepsilon t\right)^2\sin\left(\alpha_0^{''} + \omega t + \frac{\varepsilon t^2}{2}\right),
\end{aligned} \tag{10}$$

where $\alpha_0^{''} = \alpha_0^{'} + \omega t_2$.

In Fig. 2, the imposed variation laws for displacement, velocity and acceleration (s, ds/dt and $\mathrm{d}_2 s/\mathrm{d}t^2$, respectively) of the end-effector tip along the given circle segment, for the three mentioned motion phases, are illustrated.

The direct geometric model of the studied manipulator is given by:

$$\begin{aligned}
x_1 = x &= l_1 S_1 + l_2 S_{12} + l_3 S_{123}, \\
x_2 = y &= l_0 + l_1 C_1 + l_2 C_{12} + l_3 C_{123},
\end{aligned} \tag{11}$$

where: $\sin\theta_1 = S_1$; $\cos\theta_1 = C_1$; $\sin(\theta_1 + \theta_2) = S_{12}$, etc.

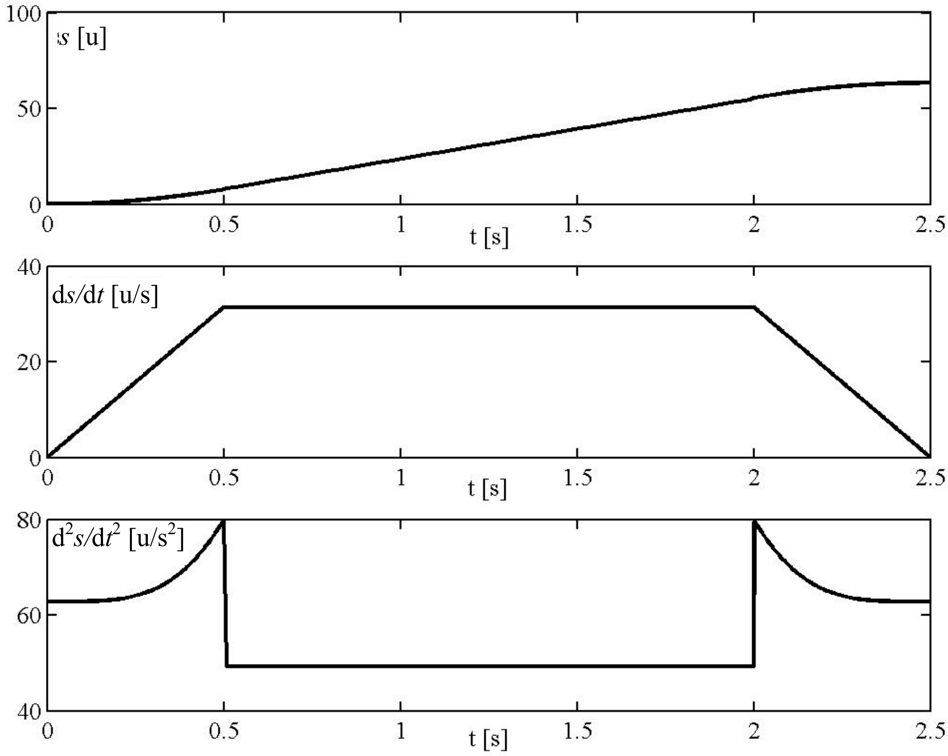

Fig. 2 – The variation laws for displacement, velocity and acceleration of the end-effector tip.

In the first stage, the manipulator self-motion transmitted by means of Jacobian null space is not considered. To determine the joint angular velocity, the equation (4) is applied:

$$\begin{gathered}
\mathbf{J}=\begin{bmatrix}\dfrac{\partial x_1}{\partial\theta_1} & \dfrac{\partial x_1}{\partial\theta_2} & \dfrac{\partial x_1}{\partial\theta_3}\\ \dfrac{\partial x_2}{\partial\theta_1} & \dfrac{\partial x_2}{\partial\theta_2} & \dfrac{\partial x_2}{\partial\theta_3}\end{bmatrix}\in\Re^{2\times3},\\
\mathbf{J}^{+}=\mathbf{J}^{\mathrm{T}}\left(\mathbf{J}\mathbf{J}^{\mathrm{T}}\right)^{-1}\in\Re^{3\times2},\\
\dot{\mathbf{x}}=\left[\dot{\mathbf{x}}_{\mathrm{E}}\quad\dot{\mathbf{y}}_{\mathrm{E}}\right]^{\mathrm{T}}\in\Re^{2\times1},\\
\dot{\mathbf{q}}=\left[\omega_1\quad\omega_2\quad\omega_3\right]^{\mathrm{T}}\in\Re^{3\times1}.
\end{gathered}\tag{12}$$

To determine the joint accelerations, the inverse differential kinematic model from the equation (6) is used, without considering the Jacobian null influence:

$$\ddot{\mathbf{q}}=\mathbf{J}^{+}\ddot{\mathbf{x}}+\dot{\mathbf{J}}^{+}\dot{\mathbf{x}},\tag{13}$$

where $\ddot{\mathbf{x}} = [\ddot{\mathbf{x}}_E \quad \ddot{\mathbf{y}}_E]^T \in \Re^{2\times1}$; $\ddot{\mathbf{q}} = [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3]^T \in \Re^{3\times1}$.

In Fig. 3a the variations of $\theta_i, d\theta_i/dt, d^2\theta_i/dt^2$ $(i = 1, 2, 3)$ are represented, for the particular case when the kinematic restrictions (the manipulator self-motion, induced by an arbitrary optimisation function), are not considered.

To consider the manipulator self-motion, one proposes to use an arbitrary function S aiming to maximise the joint angles values in order to avoid singular configurations:

$$S = \sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 . \tag{14}$$

The modified equation (5) is used for joint angular velocities determination:

$$\dot{\mathbf{q}} = \mathbf{J}^+\dot{\mathbf{x}} + \psi_1 \mathbf{N}\mathbf{W}_v \nabla S , \tag{15}$$

where: ψ_1 – scale factor; $\mathbf{W}_v$ – diagonal 3×3 matrix having the elements v_{11}, v_{22}, v_{33}. The scale factor ψ_1 is used to “balance” dimensionally the two terms in the equation (15), while the matrix $\mathbf{W}_v$ weighs the vector ∇S (the gradient of the performance function S):

$$\nabla S = \begin{bmatrix} \dfrac{\partial S}{\partial\theta_1} & \dfrac{\partial S}{\partial\theta_2} & \dfrac{\partial S}{\partial\theta_3} \end{bmatrix}^T \in \Re^{3\times1} , \tag{16}$$

$$\mathbf{W}_v = \begin{bmatrix} v_{11} & 0 & 0 \\ 0 & v_{22} & 0 \\ 0 & 0 & v_{33} \end{bmatrix} \in \Re^{3\times3} . \tag{17}$$

Finally, one obtains the joint velocities vector $\dot{\mathbf{q}} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T$.

To calculate the angular joint accelerations ε_i (i = 1, 2, 3) the modified equation (6) is used:

$$\ddot{\mathbf{q}} = \mathbf{J}^+\ddot{\mathbf{x}} + \dot{\mathbf{J}}^+\dot{\mathbf{x}} + \psi_2 \left(\mathbf{N}\mathbf{W}_a \frac{d(\nabla S)}{dt} + \dot{\mathbf{N}}\mathbf{W}_v \nabla S \right), \tag{18}$$

where: ψ_2 – scaling factor; $\mathbf{W}_a$ – diagonal (3×3) weighting matrix for the vector $\dfrac{d(\nabla S)}{dt}$:

$$\mathbf{W}_a = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \in \Re^{3\times3} . \tag{19}$$

The terms $\dot{\mathbf{J}}^+\dot{\mathbf{x}}$ and $\dot{\mathbf{N}}\mathbf{W}_v \nabla S$ from the equation (18) are used to obtain requested velocities for the end-effector and the manipulator self-motion, respectively. The Jacobian null space provides the possibility to choose an optimisation function, which satisfies some performance criteria. The diagonal weighting matrices $\mathbf{W}_v$ and $\mathbf{W}_a$ are used to influence the joint angular velocities and acceleration, respectively.

The differentials $\dot{\mathbf{J}}^+$, $\dot{\mathbf{N}}$ and $\frac{\mathrm{d}(\nabla S)}{\mathrm{d}t}$ are calculated using the finite differences method:

$$\dot{\mathbf{J}}^+ = \frac{\mathbf{J}_k^+ - \mathbf{J}_{k-1}^+}{\Delta t}, \quad \dot{\mathbf{N}} = \frac{\mathbf{N}_k - \mathbf{N}_{k-1}}{\Delta t}, \quad \frac{\mathrm{d}(\nabla S)}{\mathrm{d}t} = \frac{\nabla S_k - \nabla S_{k-1}}{\Delta t}, \quad k = 1,\ 2,\ \dots,\ p, \tag{20}$$

where: Δt – sampling period; p – number of the sampling steps.

Finally, one obtains the joint accelerations vector $\ddot{\mathbf{q}} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix}^{\mathrm{T}}$.

In Fig. 3b the variations of $\theta_i, \mathrm{d}\theta_i/\mathrm{dt}, \mathrm{d}^2\theta_i/\mathrm{d}t^2$ $(i = 1,\ 2,\ 3)$ are represented for the particular case when the kinematic restrictions (the manipulator self-motion) are considered.

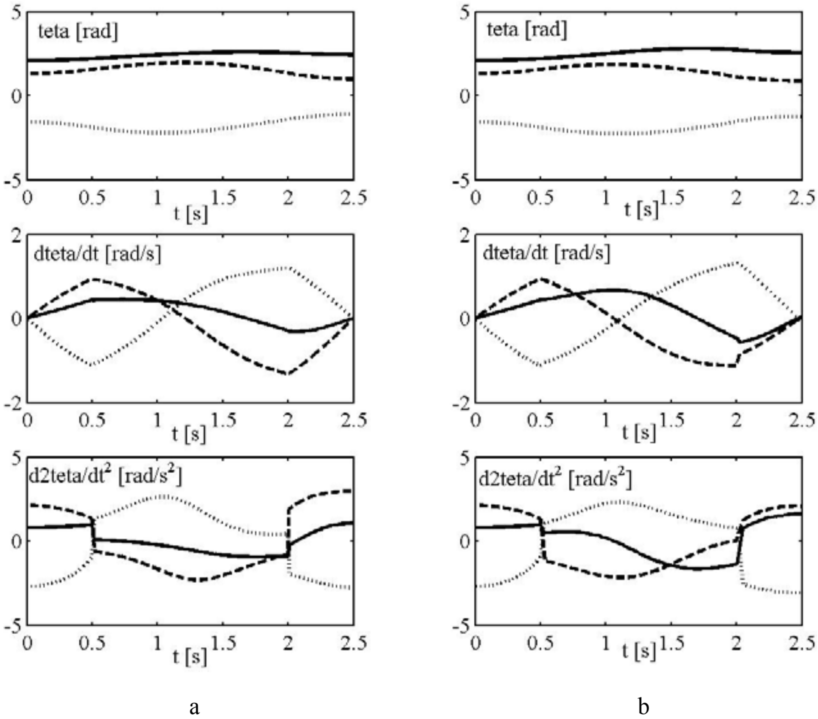

Fig. 3 – The variations of the kinematic parameters in manipulator joints (—— joint 1, ······ joint 2, - - - - joint 3) for: $\psi_2 = 0.002$; $v_{11} = 0.85$; $v_{22} = 1.1$; $v_{33} = 0.9$; $a_{11} = 1.1$; $a_{22} = 0.9$; $a_{33} = 0.95$.

4. CONCLUSIONS

The kinematic study of planar redundant manipulators represents a necessary stage for a subsequent dynamic analysis and this work proposes a method for such study. A simple planar redundant manipulator with three degrees of freedom, whose end-effector characteristic point is moved along a circle segment, is used as application, in order to ensure of easy and right evaluation of the obtained results presented as illustrative graphic representations.

The proposed algorithm for the kinematic modelling, tested on a simple example, can be applied to complex situations, which involve the end-effector movement with variable velocities on different trajectories. Also, with minimal modifications, the proposed method, tested in planar case, can be adapted for spatial applications.

The obtained results illustrate that the proposed modelling method is sensitive to the imposed restrictions introduced by means of the Jacobian null space. Thus, there is the possibility of intervention in order to improve different kinematic and dynamic parameters.

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