CONCENTRATION OF STRESSES NEAR A DEBONDED FLEXIBLE INCLUSION IN PLANE ELASTICITY

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We study the concentration of elastic stresses around a debonded inclusion. The integro-differerential equations are obtained by using discontinuous solutions. The latter may be obtained by integrating the solutions that are obtained due to the concentrated jumps of the displacements and stresses. The obtained system is solved by using the method of orthogonal polynomials. We also determine the displacement jumps and the influence of the inclusion flexibility on the stress intensity factor.

1. INTRODUCTION

Lately a number of problems in the theory of elasticity for bodies that have different defects such as crack, inclusions, etc. have been solved. There are several approaches for reducing such problems to integral equations. One of them consists in the application of the Fourier transfom in the normal direction to the surface of the defect. While keeping the jumps on the defect and using the inverse Fourier transform integral equations for jumps on the defects may be obtained. This approach is developed systematically by G.Ia. Popov [6]. Another approach is proposed by S.L. Crouch [2] which expresses the displacements and stresses due to the constant displacement discontinuity. The concept of the stress discontinuity is not used. Therefore this approach may not be used for solving the problem with debonded inclusions. The approach which is developed here consists of the following: first we construct the solutions due to the concentrated jumps of the stresses and displacements. We then use their as Green functions may be obtained by superposition the discontinuous solutions, *i.e.* the solutions which have a discontinuity of the first order on the defect. In this paper we show how the discontinuous solutions can be used to obtain the integro-differential equations for debonded thin inclusions with negligible bending rigidity.

2. SOLUTIONS FROM CONCENTRATED JUMPS

We assume that in the infinite plate on crossing the line $x_1 = 0$ the stresses and displacements have jumps. For the jumps we introduce the notation ($i = 1, 2$):

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$$\begin{aligned}&\sigma_{1i}(-0,x_2)-\sigma_{1i}(+0,x_2)=\langle\sigma_{1i}(x_2)\rangle;\\&u_i(-0,x_2)-u_i(+0,x_2)=\langle u_i(x_2)\rangle.\end{aligned}\tag{1}$$

First we assume that the jumps are concentrated at the point $x_2=0$. The solutions for an arbitrary point (x_1,x_2) due to the concentrated jumps of the stresses have the form [4]

$$\begin{aligned}&u_i^*(x_1,x_2)=U_{ij}^*(x_1,x_2)\langle\sigma_{1j}(0)\rangle\ \ (i=1,2);\\&\sigma_{ij}^*(x_1,x_2)=\sigma_{ijk}^*(x_1,x_2)\langle\sigma_{1k}(0)\rangle\ \ (i,j=1,2).\end{aligned}\tag{2}$$

The elements $U_{ij}^*(x_1,x_2)$ and $\sigma_{ijk}^*(x_1,x_2)$ have the form

$$U_{ij}^*(x_1,x_2)=\frac{1}{4\pi\mu}\left[(1+\chi)\ln\left(\frac{1}{r}\right)\delta_{ij}+(1-\chi)r_{,i}r_{,j}\right];$$

$$\sigma_{ijk}^*(x_1,x_2)=-\frac{1}{2\pi r}\left[\chi\left(r_{,j}\,\delta_{ik}+r_{,i}\,\delta_{jk}-r_{,k}\,\delta_{ij}\right)+2(1-\chi)r_{,i}\,r_{,j}\,r_{,k}\right].\tag{3}$$

Note that the displacements and stresses due the concentrated jumps of the stresses coincide with those due to the concentrated forces $F_1=\langle\sigma_{11}\rangle$ and $F_2=\langle\sigma_{12}\rangle$ which are applied at the origin.

The solutions due to the concentrated displacement jumps have the form [4]

$$u_i^{**}(x_1,x_2)=U_{ij}^{**}(x_1,x_2)\langle u_j(0)\rangle,$$

where

$$\begin{aligned}&U_{11}^{**}(x_1,x_2)=-\frac{1}{2\pi r}r_{,1}\left[\chi+2(1-\chi)\ \ r_{,1}^2\right]\ \ ;\\&U_{12}^{**}(x_1,x_2)=-\frac{1}{2\pi r}r_{,2}\left[\chi+2(1-\chi)\ \ r_{,1}^2\right]\ \ ;\\&U_{21}^{**}(x_1,x_2)=-\frac{1}{2\pi r}r_{,2}\left[-\chi+2(1-\chi)\ \ r_{,1}^2\right]\ \ ;\\&U_{22}^{**}(x_1,x_2)=-\frac{1}{2\pi r}r_{,1}\left[\chi+2(1-\chi)\ \ r_{,2}^2\right]\ \ .\end{aligned}\tag{4}$$

The stresses due to the concentrated displacement jumps may be written as

$$\sigma_{ij}^{**}(x_1,x_2)=\sigma_{ijk}^{**}(x_1,x_2)\langle u_k(0)\rangle\ \ (i,j=1,2),$$

where

$$\sigma_{111}^{**}(x_1,x_2)=\frac{\mu(1-\chi)}{\pi r^2}\left(3r_{,1}^4-6r_{,1}^2\,r_{,2}^2-r_{,2}^4\right);$$

$$\sigma_{112}^{**}(x_1,x_2)=\frac{2\mu(1-\chi)}{\pi r^2}r_{,1}r_{,2}(3r_{,1}^2-r_{,2}^2);$$

$$\sigma_{221}^{**}(x_1,x_2)=-\frac{\mu(1-\chi)}{\pi r^2}\left(r_{,1}^4-6r_{,1}^2r_{,2}^2+r_{,2}^4\right); \tag{5}$$

$$\sigma_{222}^{**}(x_1,x_2)=\frac{2\mu(1-\chi)}{\pi r^2}r_{,1}r_{,2}(3r_{,2}^2-r_{,1}^2);$$

$$\sigma_{121}^{**}=\sigma_{112}^{**};\ \sigma_{122}^{**}=\sigma_{221}^{**}.$$

In the formulas (3)–(5) μ is the shear modulus, $\chi=(1-\nu)/2$ for the plane stress state; $\chi=(1-2\nu)/[2(1-\nu)]$ for the plane strain; ν is the Poisson ratio, δ_{ij} is the Kronecker delta; $r_{,i}=x_i/r$; $r^2=x_ix_i$ $(i=1,2)$.

We note

$$U_{11}^{**}=\sigma_{111}^{*};\ U_{12}^{**}=\sigma_{121}^{*};\ U_{21}^{**}=\sigma_{112}^{*};\ U_{22}^{**}=\sigma_{122}^{*}.$$

If the jumps are located at the point $x_1=0;\, x_2=\xi_2$, then in the relations (3)–(5) x_2 must be replaced by $x_2-\xi_2$.

3. CONSTRUCTION OF THE DISCONTINUOUS SOLUTIONS

On using the solutions for the concentrated jumps as Green functions we can write the discontinuous solutions

$$\begin{aligned}\tilde{u}_i(x_1,x_2)&=\int_L U_{ij}^{*}\left(x_1,x_2-\xi_2\right)\left\langle\sigma_{1j}\left(\xi_2\right)\right\rangle\mathrm{d}\xi_2+\int_L U_{ij}^{**}(x_1,x_2-\xi_2)\left\langle u_j\left(\xi_2\right)\right\rangle\mathrm{d}\xi_2;\\ \tilde{\sigma}_{ij}(x_1,x_2)&=\int_L \sigma_{ijk}^{*}\left(x_1,x_2-\xi_2\right)\left\langle\sigma_{1k}\left(\xi\right)\right\rangle\mathrm{d}\xi_2+\int_L \sigma_{ijk}^{**}\left(x_1,x_2-\xi_2\right)\left\langle u_k\left(\xi_2\right)\right\rangle\mathrm{d}\xi_2.\end{aligned} \tag{6}$$

The formulas (3)–(6) are sufficient to solve various problems for the body with defects located on the co-ordinates line $x_1=0$. For that the stress-deformation state will be presented as the sum of two states: main (denoted by zero), due the outside load and perturbed, due to the jumps of the stresses and displacements

$$u_i=u_i^0+\tilde{u}_i;\ \sigma_{ij}=\sigma_{ij}^0+\tilde{\sigma}_{ij}.$$

In the general case while crossing the defect the four jumps – two for stresses and two for the displacements – are unknown. Using the boundary conditions on the sides of the defect we obtain a system of integral equations for the unknown jumps. In the some particular cases some jumps may be equal to zero.

4. REDUCTION OF THE PROBLEM FOR DEBONDED INCLUSION TO INTEGRO-DIFFERENTIAL EQUATIONS

We assume that the side $x_1 = +0$ of the flexible inclusion with negligible bending rigidity which was in contact with the plate is debonded (Fig. 1). The plate is loaded at infinity by the stresses $\sigma_{11} = p = \text{const.}$ On passing through the defect the displacements u_1, u_2 and stresses σ_{12} have jumps. Using the discontinuous solutions we may write

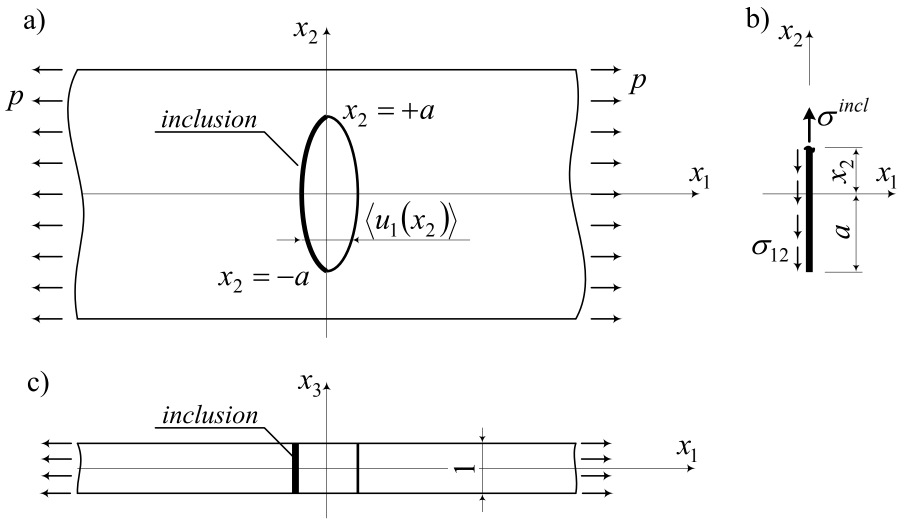

Fig. 1 – The plate under extension with debonded inclusion.

$$\sigma_{1j}(x_1,x_2) = \int_{-a}^{a} \sigma^{*}_{1j2}(x_1, x_2-\xi_2)\langle\sigma_{12}(\xi_2)\rangle \mathrm{d}\xi_2 + \int_{-a}^{a} \sigma^{**}_{1j1}(x_1, x_2-\xi_2)\langle u_1(\xi_2)\rangle \mathrm{d}\xi_2 + \\ + \int_{-a}^{a} \sigma^{**}_{1j2}(x_1, x_2-\xi_2)\langle u_2(\xi_2)\rangle \mathrm{d}\xi_2 \ (j=1,2); \tag{7}$$

$$u_2(x_1,x_2) = \int_{-a}^{a} U^{*}_{22}(x_1, x_2-\xi_2)\langle\sigma_{12}(\xi_2)\rangle \mathrm{d}\xi_2 + \int_{-a}^{a} U^{**}_{21}(x_1, x_2-\xi_2)\langle u_1(\xi_2)\rangle \mathrm{d}\xi_2 + \\ + \int_{-a}^{a} U^{**}_{22}(x_1, x_2-\xi_2)\langle u_2(\xi_2)\rangle \mathrm{d}\xi_2.$$

If the plate is not loading on the sides of the defect we have the following boundary conditions:

$$x_1 = +0 : \sigma_{11} = \sigma_{12} = 0 \quad ; \ x_1 = -0 : u_2 = u^{incl}. \tag{8}$$

Using the equilibrium equation for the element of the inclusion $(-a, x_2)$ (Fig. 1b) we obtain

$$\int_{-a}^{x_2} \langle \sigma_{12}(\xi_2) \rangle \mathrm{d}\xi_2 = \frac{1}{2} \int_{-a}^{a} \operatorname{sign}(x_2 - \xi_2) \langle \sigma_{12}(\xi_2) \rangle \mathrm{d}\xi_2 = \sigma^{incl} A^{incl}, \tag{9}$$

where σ^{incl} are the normal stresses on the cross section of the inclusion; A^{incl} is the area of the cross section and we suppose that the inclusion is not loaded.

Differentiating the last boundary condition (8) and using Hooke's law we obtain

$$\frac{\partial u_2}{\partial x_2} = \frac{\mathrm{d}u^{incl}}{\mathrm{d}x_2} = \varepsilon^{incl} = \frac{\sigma^{incl}}{E^{incl}}. \tag{10}$$

On substituting the relations (7) into first two boundary conditions (8) and (10) and using the formulas

$$\lim_{x_2 \to \pm 0} \frac{x_2}{x_2^2 + \xi_2^2} = \pm \pi \delta(\xi_2); \quad \lim_{x_2 \to \pm 0} \frac{x_2 \xi_2^2}{\left(x_2^2 + \xi_2^2\right)^2} = \pm \frac{\pi}{2} \delta(\xi_2);$$

$$\lim_{x_2 \to \pm 0} \frac{x_2^3}{\left(x_2^2 + \xi_2^2\right)^2} = \pm \frac{\pi}{2} \delta(\xi_2); \quad \frac{\partial}{\partial x_2} \operatorname{sign}(x_2 - \xi_2) = 2\delta(x_2 - \xi_2), \tag{11}$$

where $\delta(x)$ is Dirac delta function.

In the obtained system we derive the last equation with respect to x_2. After normalising the interval to $(-1, +1)$ and introducing the notation

$$\varphi_{12}(x_2) = \langle \sigma_{12}(a x_2) \rangle / \mu; \; \varphi_1(x_2) = \langle u_1(a x_2) \rangle / a; \; \varphi_2(x_2) = \langle u_2(a x_2) \rangle / a, \tag{12}$$

we obtain the following system of the integro-differential equations

$$\frac{\chi}{2\pi} \int_{-1}^{1} \frac{\varphi_{12}(\xi_2) \mathrm{d}\xi_2}{x_2 - \xi_2} - \frac{(1-\chi)}{\pi} \int_{-1}^{+1} \frac{\varphi_1(\xi_2)}{(x_2 - \xi_2)^2} \mathrm{d}\xi_2 = -\frac{p}{\mu};$$

$$-\frac{1}{2} \varphi_{12}(x_2) - \frac{(1-\chi)}{\pi} \int_{-1}^{1} \frac{\varphi_2(\xi_2)}{(x_2 - \xi_2)^2} \mathrm{d}\xi_2 = 0; \tag{13}$$

$$-\frac{1}{4\pi(1-\chi)} \int_{-1}^{1} \frac{\varphi_{12}(\xi_2)}{x_2 - \xi_2} \mathrm{d}\xi_2 - \frac{1}{2\lambda} \int_{-1}^{1} \operatorname{sign}(x_2 - \xi_2) \varphi_{12}(\xi_2) \mathrm{d}\xi_2 + \frac{1}{2} \frac{\partial}{\partial x_2} \varphi_2(x_2) = \frac{1}{4(1-\chi)} \frac{p}{\mu},$$

where $\lambda = E^{incl} A^{incl} / \mu a$.

The solutions of the system (13) which take into account the behaviour of the functions of the end interval $(-1, +1)$ may be represented in the form of the series of the Chebyshev orthogonal polynomials

$$\varphi_1(x_2)=\sqrt{1-x_2^2}\sum_{m=1}^{\infty}X_mU_{m-1}(x_2);$$

$$\varphi_2(x_2)=\sqrt{1-x_2^2}\sum_{m=1}^{\infty}Y_mU_{m-1}(x_2); \tag{14}$$

$$\varphi_{12}(x_2)=\frac{1}{\sqrt{1-x_2^2}}\sum_{m=1}^{\infty}Z_mT_{m-1}(x_2),$$

where X_m, Y_m, Z_m are the unknown coefficients; $T_m(x_2), U_m(x_2)$ the Chebyshev polynomials of the first and second kinds respectively.

We introduce the series (14) in the system (13) and use the relations [1, 3, 4, 6]

$$\int_{-1}^{1}\frac{T_{m-1}(\xi_2)\mathrm{d}\xi_2}{(x_2-\xi_2)\sqrt{1-\xi_2^2}}=\begin{cases}0 & (m=1),\\ -\pi U_{m-2}(x_2) & (m=2,3,\ldots,);\end{cases}$$

$$\int_{-1}^{1}\frac{\sqrt{1-\xi_2^2}U_{m-1}(\xi_2)\mathrm{d}\xi_2}{(x_2-\xi_2)^2}=-\pi mU_{m-1}(x_2)\ \ (m=1,2,\ldots,); \tag{15}$$

$$\int_{-1}^{1}\frac{T_{m-1}(\xi_2)}{\sqrt{1-\xi_2^2}}\operatorname{sign}(x_2-\xi_2)\mathrm{d}\xi_2=\begin{cases}2\arcsin x_2 & (m=1),\\ -\dfrac{2}{m}\sqrt{1-x_2^2}U_{m-2}(x_2) & (m=2,3,\ldots,)\end{cases}.$$

Then each obtained equation is multiplied by $\sqrt{1-x_2^2}U_{k-1}(x_2)$ $(k=1,2,\ldots,)$ and then integrated on the interval $(-1,+1)$ using the orthogonality property of the polynomials $U_k(x_2)$. We then obtain the following system of the linear algebraic equations for the coefficients X_k, Y_k, Z_k $(k=1,2,\ldots,)$

$$(1-\chi)kX_k-\frac{\chi}{2}Z_{k+1}=-\frac{p}{\mu}\delta_{1k};$$

$$\pi(1-\chi)kY_k-\sum_{m=1}^{\infty}\gamma_{k-1,m-1}^{(1)}Z_m=0; \tag{16}$$

$$-\sum_{m=1}^{\infty}m\gamma_{k-1,m}^{(1)}Y_m+\frac{\pi}{4(1-\chi)}Z_{k+1}+\frac{2}{\lambda}\sum_{m=2}^{\infty}\frac{1}{m}\gamma_{k-1,m-2}^{(2)}Z_m=\frac{\pi}{4(1-\chi)}\frac{p}{\mu}\delta_{1,k-1},$$

where

$$\gamma_k^{(1)}=\int_{-1}^{1}\arcsin x_2\sqrt{1-x_2^2}U_k(x_2)\mathrm{d}x_2=\frac{2(k+1)\left[1-(-1)^k\right]}{k^2(k+2)^2}\ \ (k=0,1,\ldots,);$$

$$\gamma_{km}^{(2)} = \int_{-1}^{1} \left(1 - x_2^2\right) U_k\left(x_2\right) U_m\left(x_2\right) \mathrm{d}x_2 = -\frac{2(k+1)(m+1)\left[1 + (-1)^{k+m}\right]}{\left[(m+1)^2 - k^2\right]\left[(m+1)^2 - (k+1)^2\right]}; \quad (17)$$

$$\gamma_{km}^{(3)} = \int_{-1}^{1} U_k\left(x_2\right) T_m\left(x_2\right) \mathrm{d}x_2 = \frac{(k+1)\left[1 + (-1)^{k+m}\right]}{(k+1)^2 - m^2} \quad (k, m = 0, 1, ...,).$$

The coefficients $\gamma_k^{(1)}, \gamma_{km}^{(2)}, \gamma_{km}^{(3)}$ are equal to zero for k even and $k+m$-odd, respectivelly. For other values they were evaluated using the new variable $\theta = \arccos(x_2)$ and using the formulas [3]

$$T_m(\cos\theta) = \cos m\theta; \; U_m(\cos\theta) = \sin(m+1)\ \ \theta / \sin\theta.$$

The system (16) may be solved by reduction method.

The stress intensity factor for the opening mode may be calculated by the formula

$$K_I = \lim_{x_2 \to a} \sqrt{2\pi\left(x_2 - a\right)} \ \ \sigma_{11}\left(0, x_2\right). \quad (18)$$

The stresses σ_{11} are calculating by the relation

$$\sigma_{11} = \sigma_{11}^o + \frac{\chi}{2\pi} \int_{-a}^{+a} \frac{\left\langle \sigma_{12}\left(\xi_2\right)\right\rangle}{x_2 - \xi_2} \mathrm{d}\xi_2 - \frac{\mu(1-\chi)}{\pi} \int_{-a}^{a} \frac{\left\langle u_1\left(\xi_2\right)\right\rangle}{\left(x_2 - \xi_2\right)^2} \mathrm{d}\xi_2. \quad (19)$$

On normalising the interval of the integration, using the notation (12) and introducing the series (14) we obtain

$$\sigma_{11}\left(ax_2\right) = \sigma_{11}^o + \frac{\mu\chi}{2\pi} \sum_{m=1}^{\infty} Z_m \int_{-1}^{1} \frac{T_{m-1}\left(\xi_2\right) \mathrm{d}\xi_2}{\left(x_2 - \xi_2\right)\sqrt{1 - \xi_2^2}} - \frac{\mu(1-\chi)p}{\pi} \sum X_m \int_{-1}^{1} \frac{\sqrt{1 - \xi_2^2}\, U_{m-1}\left(\xi_2\right)}{\left(x_2 - \xi_2\right)^2}. \quad (20)$$

Using the integrals [4, 6] for $x_2 > 1$

$$\int_{-1}^{1} \frac{T_m\left(\xi_2\right) \mathrm{d}\xi_2}{\left(x_2 - \xi_2\right)\sqrt{1 - \xi_2^2}} = \frac{\pi\left(x_2 - \sqrt{x_2^2 - 1}\right)^m}{\sqrt{x_2^2 - 1}} \quad (m = 0, 1, 2, ..,); \quad (21)$$

$$\int_{-1}^{1} \frac{\sqrt{1 - \xi_2^2}\, U_{m-1}\left(\xi_2\right)}{\left(x_2 - \xi_2\right)^2} \mathrm{d}\xi_2 = \pi m \frac{\left(x_2 - \sqrt{x_2^2 - 1}\right)^m}{\sqrt{x_2^2 - 1}} \quad (m = 1, 2, ...,),$$

and first equation (16) for the stress intensity factor K_I we obtain

$$K_I = \mu\sqrt{\pi a}\left[\frac{p}{\mu} + \frac{\chi}{2} \sum_{m=1}^{\infty} \left(Z_m - Z_{m+1}\right)\right] \quad (22)$$

In Table 1 we present the jump $\langle u_1(ax_2\rangle$ for $x_2 = 0$ and the stress intensity factor K_I for $\nu = 0.3$.

Table 1

The values of the jump $\langle u_1(0)\rangle$ and the stress intensity factor K_I

$\lambda = E^{incl} A^{incl} / \mu a$	0.05	0.1	0.5	1	5	10	100
$\langle u_1(0)\rangle \mu / pa$	–1.53	–1.53	–1.50	–1.48	–1.39	–1.37	–1.34
$K_I / p\sqrt{\pi a}$	0.998	0.997	0.990	0.985	0.972	0.968	0.951

For the crack the jump $\langle u_1(0)\rangle = 1.54 pa/\mu$ and stress intensity factor $K_I = p\sqrt{\pi a}$. From the table we see that the influence of the inclusion rigidity λ over $\langle u_1(0)\rangle$ and K_I is minor.

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