ON A PARAMETRIC STABILITY IDENTIFICATION METHOD WITH APPLICATIONS*

Marcel MIGDALOVICI[1], Daniela BARAN[2]

The research is focused on the theoretical study of the Lyapunov stability of the evolution of a dynamical system depending on parameters. It is described an original method of separation between the stable and unstable zones, in the plane of chosen principal parameters. In the paper is presented, using this results, an original method for identification, in the plane of principal parameters of the mathematical model of the dynamical system, the stable and unstable regions of the motion. We analyze also some theorems, as the Floquet stability theorem, concerning the stability of a dynamical system described by differential equation systems with periodical coefficients. The results are applied to the study of the couple pantograph – contact wire of a electrical locomotive. The parameters of the system are the two concentrated masses, the bending stiffness, the wire tension, the viscous damping and the mass per unit length of the wire, the other damping coefficients and stiffness elements of the system and any constant speed specified in the model. We study the stable and unstable regions of the dynamical system motion using these parameters and our method of stability zones identification.

1. INTRODUCTION

Firstly we describe some results about the differential linear equations and systems. Consider a linear differential equation of order n for the unknown function y:

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1 y' + a_0 y = f\ , \tag{1}$$

where $a_{n-1},\ldots,\ a_1,\ a_0, f$ are functions defined on an interval $J \subset R$ with complex values, and initial conditions $y(x_0) = y_0,\ y'(x_0) = y_1,\ldots, y^{(n-1)}(x_0) = y_{n-1}$.

Using the notation $w_{n+1} = y^{(n)}, w_n = y^{(n-1)},\ldots,\ w_2 = y',\ w_1 = y,$ the equation (1) can be written in a matrix form as

$$W' = A\ W + g\ . \tag{2}$$

We present without proof [5], the following theorem:

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[1] Institute of Solid Mechanics, Bucharest, email: marcel_migdalovici@yahoo.com
[2] INCAS "Elie Carafoli", Bucharest, email: dbaran@aero.incas.ro

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Theorem 1. *If in matrix equation (2) the functions* $f, a_0, ..., a_{n-1}$ *are continuous on the definition interval* $J \subset R$, *then equation (2) has a unique solution* $W(x)$, *a column vector, so that* $W(x_0) = W_0$.

Are defined any n linear independent solutions of the homogenous system $W' = A\,W$ as the fundamental system of solutions. With the linear independent vectors, placed one after the other, one forms a square matrix W called the fundamental matrix of homogenous system which verifies the matrix equation $W' = AW$.

Theorem 2. *If W is any fundamental matrix of system* $W' = A\,W$, *then any solution of this system can be written as* $w = Wc$ *where* c *is a constant vector; if the initial condition is* $w(x_0) = w_0$ *then the solution is* $w(x) = W(x)W^{-1}(x_0)w_0$. *Any fundamental matrix of system can be deduced from another multiplying at right with a constant matrix.*

Theorem 3. *If* W *is any fundamental matrix of the system* $w' = A\,w$, *and* $w(x_0) = w_0$, *then any solution of the inhomogeneous system* $w' = A\,w + g,\quad A,\, g \in C^0(J)$, *is* $w(x) = W(x)W^{-1}(x_0)w_0 + W(x)\int_{x_0}^{x}(W^{-1}g)(t)\mathrm{d}t$.

2. MATRIX FUNCTIONS

We present in the following paragraphs some details about the matrix functions. For the beginning we consider the polynomial function. If $A \in M_n$ with proper values $\lambda_1, \lambda_2, ..., \lambda_n$, then $A^k \in M_n,\ k \in N$ and $p(A)$ is:

$$p(A) = A^m + b_1A^{m-1} + ... + b_{m-1}A + b_m,\ b_j \in C,\ j = 1,...,n. \tag{3}$$

For a matrix which admits a diagonal form, that means that there is a matrix D with non zero values only on its diagonal, and an invertible matrix S with $A = SDS^{-1}$ then $A^k = S\,D^kS^{-1}$; $p(A) =,\ = Sp(D)S^{-1}$, where:

$$p(D) = \begin{bmatrix} p(\lambda_1) & 0 & ... & 0 \\ \multicolumn{4}{c}{.............................} \\ 0 & 0 & ... & p(\lambda_n) \end{bmatrix}. \tag{4}$$

We extend the matrix function definition for differentiable functions in a domain in C which contains the proper values of A. For a closed rectifiable curve γ which includes inside a point ς, where g is differentiable, but which does not include a singularity of g, is known that $g(\varsigma)=\frac{1}{2\pi i}\int_{\gamma}(\varsigma-z)^{-1}g(z)\mathrm{d}z$. We define $g(A)$ as $g(A)=\frac{1}{2\pi i}\int_{\gamma}(A-zI)^{-1}g(z)\mathrm{d}z$, where γ is a closed rectifiable curve which includes the spectrum of A, but does not include any singularity of g. For the exponential function $g(z)=e^{xz}$, $x, z\in C$, one defines $g(A)=e^{xA}$ as $g(A)=\frac{1}{2\pi i}\int_{\gamma}(A-zI)^{-1}e^{xz}\mathrm{d}z$, where γ is a closed rectifiable curve which includes the proper values of A. We differentiate:

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{xA})=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{\gamma}(A-zI)^{-1}e^{xz}\mathrm{d}z\right)=\frac{1}{2\pi i}\int_{\gamma}(A-zI)^{-1}ze^{xz}\mathrm{d}z=Ae^{xA} \tag{5}$$

because the last integral is the matrix function for $g(z)=ze^{xz}$. We obtain that e^{xA} verifies the matrix equation $W'=AW$ and for $x=0$ we have $e^{0A}=\frac{1}{2\pi i}\int_{\gamma}(A-zI)^{-1}1\mathrm{d}z=I$, where I is the unit matrix.

The matrix e^{xA} is a fundamental matrix for the differential system: $W'=A\,W+g,\;\; A, g\in C^{0}(J)$ and the general solution, with the initial conditions $w(x_0)=w_0$, is:

$$w(x)=e^{(x-x_0)A}w_0+e^{xA}\int_{x_0}^{x}e^{-tA}g\mathrm{d}t\,.$$

3. DIFFERENTIAL EQUATIONS SYSTEMS WITH PERIODICAL COEFFICIENTS

Consider the linear homogenous differential system:

$$W'=A\,W,\;\; A\in M_n,\, A\in C^{0}(J),\, J\subset R\,.$$

We suppose that there is $p\in R_+$ so that $A(x+p)=A(x)$ for any $x\in J$. The system is periodic with the period p. We mention the following theorem (Floquet):

Theorem 4. *If the system $W'=A\,W$ is periodic, with the period $p>0$, then any fundamental matrix* W *of the system, can be expressed as $W(x)=W_1(x)e^{x\mathrm{R}}$, where $W_1(x)\in M_n$ is a periodical matrix with the period p, and $R\in M_n$ is a* *constant matrix* $R = \frac{1}{p}\ln C$, *with constant matrix* C *defined by* $W(x+p) = W(x)C,\ C \in M_n$.

4. STABILITY THEORY ASPECTS

Consider the differential system $y' = A\,y$, $A \in M_n$ with components defined and continuous on $I \subset R$. Consider also $t_0 \in I$ and $\tilde{y}_0 \in R^n$. From theorem 1, the solution $\tilde{y} : I \to R^n$, exists, it is unique, so that $\tilde{y}(t_0) = \tilde{y}_0$. Another solution $y : I \to R^n$ of the system with the initial condition $y(t_0) = y_0$, and $y_0 \neq \tilde{y}_0$, is called a perturbed solution of system, reported to $\tilde{y}$. The solution $y : I \to R^n$ is called Lyapunov stable if for any $\varepsilon > 0$ exists δ so that, for $|y_0 - \tilde{y}_0| < \delta$ then $|y - \tilde{y}| < \varepsilon$ for any $t > t_0$, where $|y| = \max\{\ |y_1(t)|,\ \ |y_2(t)|, \ldots,\ |y_n(t)|\ ;\ t \geq t_0\ \}$. If, supplementary, $|y_j(t) - \tilde{y}_j(t)| \to 0$, for any $j = 1,2,\ldots,n$, and $t \to \infty$, then the solution is called asymptotic stable.

Theorem 5 (Floquet). *If the system* $W' = A\,W$ *is periodic, with the period* $p > 0$, *and* W, *any fundamental matrix of the system, expressed as:* $W(x) = W_1(x)e^{x\mathrm{R}}$, *where* $W_1(x) \in M_n$ *is a periodical matrix with the period* p, *and* $R \in M_n$ *is a constant matrix, then, if the proper values of* R *have negative real part, the solution of the periodical system is asymptotically stable, and if at least a proper value of the matrix* R *is strictly positive, the solution of the periodical system is unstable. If the proper values of the matrix* R *have zero real part, then the solution of the periodical system is undecided (stable, unstable or periodical).*

We search the stable and unstable zones of the solution $y : I \to R^n$ for the system $y' = A\,y$ in the following our hypothesis about the possibility of separation of stable and unstable zones of the solution y:

If $y : I \to R^n$ *is a stable solution of the system* $y' = A\,y$, *with periodical matrix and continuous components, defined by parameters, for fixed parameters, we suppose that there is a neighbourhood of fixed parameters where the solution* y *is also stable. For an unstable solution of the system we suppose that we can formulate analogue property.*

This hypothesis is used by us for separation of the stable and unstable zones in the plane of two chosen principal parameters by curves of periodical solutions of the system.

Determining the domain of periodical solutions for the two chosen parameters plane, one determines the image of stability zone in the plane.

We use the following procedure to identify the boundary of the points with periodical solution from the chosen parameters plane. The fixed domain for analysis, of the parameters plane, is covered with a sufficient fine mesh and we study the evolution of the specified displacement solution in the mesh points. In the neighborhood of the periodic points of the parameters plane one can use a refined mesh.

5. APPLICATION

The dimensionless system of equations [9] that specifies the stated problem for the dynamical system described by pantograph and contact wire, compound from two sprung superposed masses in contact with a wire, is as follows:

$$
\begin{aligned}
&(1-\mu)\ddot{\tilde{y}}_3 + 2\,\varsigma_{\mathrm{s}}\tilde{\omega}_{\mathrm{ns}}\,(\dot{\tilde{y}}_3-\dot{\tilde{y}}_2) + \tilde{\omega}_{\mathrm{ns}}{}^2(\tilde{y}_3-\tilde{y}_2) + \tilde{\omega}_{\mathrm{nL}}{}^2\tilde{y}_3 + 2\varsigma_{\mathrm{L}}\tilde{\omega}_{\mathrm{nL}}\dot{\tilde{y}}_3 = 0\\
&\mu\,\ddot{\tilde{y}}_2 + (1-\mu)\ddot{\tilde{y}}_3 + \tilde{\Omega}_{\mathrm{n}}{}^2(\tilde{y}_2 - \sum_{j=1}^{\infty}[Tj(\tau)+wj]\,\frac{\sin j\Delta}{j\Delta}\sin j\tau) +\\
&\qquad + \tilde{\omega}_{\mathrm{nL}}{}^2\tilde{y}_3 + 2\varsigma_{\mathrm{L}}\tilde{\omega}_{\mathrm{nL}}\dot{\tilde{y}}_3 = 0 \qquad\qquad (6)\\
&\frac{\mathrm{d}^2T_j}{\mathrm{d}\tau^2} + \frac{1.}{\tilde{v}_\beta}\frac{\mathrm{d}T_j}{\mathrm{d}\tau} + (\frac{j^4}{\tilde{v}_{\mathrm{EI}}{}^2} + \frac{j^2}{\tilde{v}_{\mathrm{T}}{}^2})T_j =\\
&= -2\,\tilde{M}\,(\mu\,\ddot{\tilde{y}}_2 + (1-\mu)\ddot{\tilde{y}}_3 + \tilde{\omega}_{\mathrm{nL}}{}^2\tilde{y}_3 + 2\varsigma_{\mathrm{L}}\tilde{\omega}_{\mathrm{nL}}\dot{\tilde{y}}_3)\,\sin j\tau;\ j = 1,\ldots,5
\end{aligned}
$$

with:

$$
\tilde{v}_{\mathrm{EI}}^2 = \frac{mL^2}{EI\pi^2}v^2,\quad \tilde{v}_{\mathrm{T}}^2 = \frac{m}{T}v^2,\quad \tilde{v}_\beta = \frac{m\pi}{\beta L}v\,,
$$

where T is the tension in the wire, EI is the bending rigidity of the wire, m is the mass per unit length of the wire, β is the viscous damping of the wire, v is the uniform speed of the dynamical system and where we consider known the initial conditions for the problem:

$$
\begin{aligned}
&\tilde{y}_3(0) = \tilde{y}_{\mathrm{o}3},\ \dot{\tilde{y}}_3(0) = \dot{\tilde{y}}_{03},\ \tilde{y}_2(0) = \tilde{y}_{\mathrm{o}2},\\
&\dot{\tilde{y}}_2(0) = \dot{\tilde{y}}_{02},\ T_i(0) = T_{\mathrm{o}i},\ \dot{T}_i(0) = \dot{T}_{\mathrm{o}i}.
\end{aligned}
$$

Now we consider the participation of the external forces by additional values in the coefficients of the series development of the contact force between pantograph and contact wire, in the right hand of the third equation from the system.

Are denoted by $\tilde{A}_j,\ j \in N$ the additional term of the coefficient for $\sin j\tau$ that intervene in the third equation of the system (6). In the case of analysis we consider the following fixed values of parameters:

$$\tilde{\Omega}_n = 4.77, \quad \varsigma_s = 0.3, \quad \tilde{M} = 0.58, \quad \tilde{v}_\beta = 6.4,$$
$$\mu = 0.1, \quad \tilde{\omega}_{nL} = 0.72, \quad \varsigma_L = 0.45, \quad \tilde{v}_{EI} = 85.6.$$

The free dimensionless parameters in the plane of parameters are chosen, in this case, $\tilde{\lambda}$ and $\tilde{v}_T$, where $\tilde{\lambda} = \tilde{\omega}_{nL}/\tilde{\omega}_{ns}$. We analyse the stability of motion for the concentrated mass M_u with the displacement $\tilde{y}_2$.

In Fig. 1 is plotted with continuous line the domain of periodic solutions of $\tilde{y}_2$ in the two chosen parameters plane in the case $\tilde{A}_j = 0$ for $j \in N$ and with discontinuous line the domain of periodic solutions of $\tilde{y}_2$ in the case $\tilde{A}_1 = 0.03$ and $\tilde{A}_j = 0,\ j \neq 1$.

In Fig. 2, left and right, are plotted evolutions of the displacement $\tilde{y}_2$ for the following values of the parameters, respectively:

$$\tilde{M} = 0.071, \ \mu = 0.1, \ \tilde{\omega}_{ns} = 1.98, \ \tilde{\omega}_{nL} = 2.13, \ \tilde{\Omega}_n = 12.128,$$
$$\tilde{v}_\beta = 22.777, \ \zeta_s = 0.358, \ \zeta_L = 0.953, \ \tilde{v}_{EI} = 34.497, \ \tilde{v}_T = 0.719, \ \tilde{A}_1 = 0.$$

and

$$\tilde{M} = 0.071, \ \mu = 0.1, \ \tilde{\omega}_{ns} = 0.99, \ \tilde{\omega}_{nL} = 1.065, \ \tilde{\Omega}_n = 6.064,$$
$$\tilde{v}_\beta = 45.553, \ \zeta_s = 0.358, \ \zeta_L = 0.953, \ \tilde{v}_{EI} = 68.993, \ \tilde{v}_T = 1.616, \ \tilde{A}_1 = 0.$$

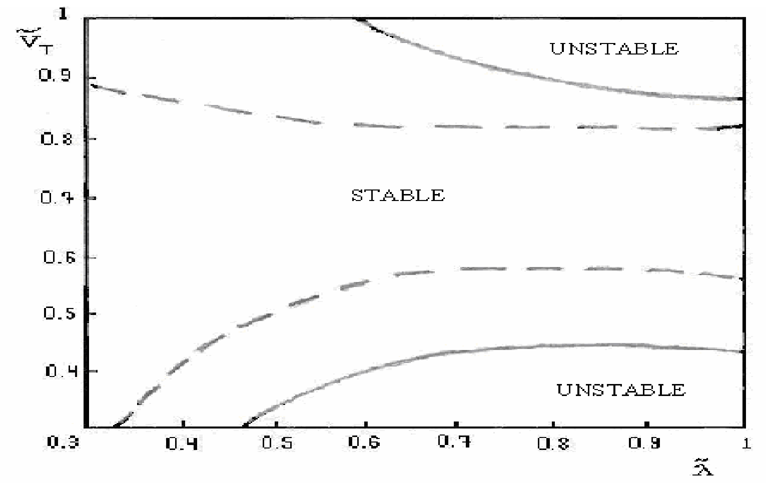

Fig. 1 – Stable and unstable zones in the plane of parameters.

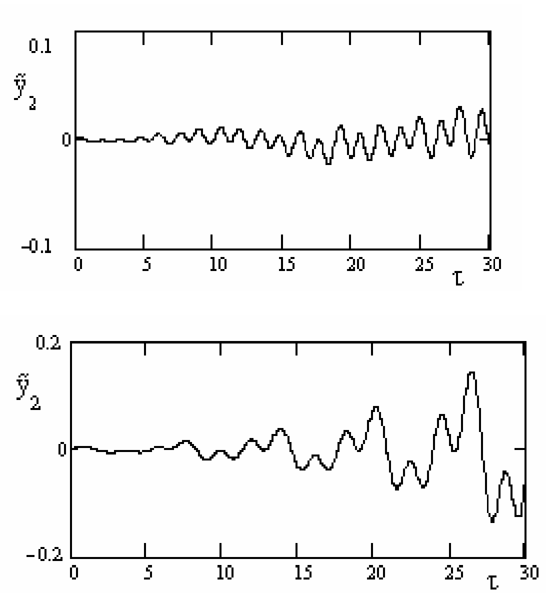

Fig. 2 – Evolutions of the displacement $\tilde{y}_2$.

The two cases of evolution of the displacement $\tilde{y}_2$ analyzed (Fig. 2) underlines the dependence of the motion stability of the dynamical system by the parameters values of the system and underlines also the possibility to choose these parameters for assure a stable evolution of the dynamical system pantograph – contact wire. The mathematical model described here assures the possibilities to optimize the values of the parameters for respect chosen criteria.

4. CONCLUSIONS

The theoretical study regarding some theorems about the solutions of differential linear equations and systems that define dynamical systems, suggest us to define an original hypothesis for searching the stable and unstable zones in the plane of principal parameters. We suppose that the dynamical system is such that the stable and unstable zones in the plane of principal parameters for the solution $y: I \to R^n$ of the system $y' = A\,y$ are separated by curves of periodical solutions. *We suppose, in consequences, in hypothesis of stability of the solution* $y: I \to R^n$ of the system $y' = A\,y$, *for fixed values of the parameters and of initial conditions (that assure the unique solution), that there is a neighbourhood of fixed parameters where the solution* y *is also stable. For an unstable solution of the system we suppose that we can formulate analogue property.*

The method of stability analysis, using our hypothesis about the possibility of separation of stable and unstable zones of the solution y of the dynamical system, described by numerical method specified in this paper, have permitted to analyze the influence of external forces on the stable or unstable motion of the pantograph – contact wire dynamical system, modeled as two sprung superposed masses in contact with a wire.

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