ON THE MODELLING OF EULER-BERNOULLI BEAMS WITH AUXETIC PATCHS*

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In this paper, the Euler-Bernoulli beams with auxetic patchs are described in the light of the nonlocal theory of damping. Unlike ordinary local damping models, the damping force in a nonlocal model is obtained as a weighted average of the velocity field over the spatial domain, determined by a kernel function based on distance measures.

1. BEAMS WITH AUXETIC PATCHS

One way to manipulate the eigenfrequencies of a structure is to vary its damping properties. The structures that are optimized in this paper are beams, which are modelled by coupling the Euler-Bernoulli theory with a nonlocal damping theory. The nonlocal theory describes long-range interactions among the particles, the stress at a location being determined by interatomic interactions in the neighbours around that location. The deformations at one position produce forces and moments at other points in the structure (Eringen and Edelen [1], Polizzotto [2]). The interest in the subject has resulted in a large number of papers which describe nonlocal damping models based on viscoelasticity (Ahmadi [3]), on the harmonic waves motion in Voigt-Kevin and Maxwell media (Nowinski [4]), or on composites with the internal damping torque (Russell [5], Ghoneim [6]), and so on. Lei, Friswell and Adhikari [7] developed a nonlocal damping model including time and spatial hysteresis effects for Euler-Bernoulli beams and Kirchoff plates.

The ability of tailoring the best behavior of materials and structures consists in a qualitative and quantitative understanding of the damping properties. Currently, it is impossible to obtain an optimal solution by maximizing eigenfrequencies or gaps between them, or by minimizing the possibility of internal resonance (Abrete [8], Pedersen [9–11]) only on the base of ordinary local damping models. In the nonlocal theory, the damping force is obtained as a weighted average of the velocity field over the spatial domain, determined by a kernel function based on distance measures.

In this paper we consider a nonlocal damping model for Euler-Bernoulli

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beams with auxetic patchs. The beam has the length L, and the damping is introduced by auxetic patchs between some locations $(x_1, x_1 + \Delta x_1)$, $(x_2, x_2 + \Delta x_2)$, $(x_k, x_k + \Delta x_k)$, $x_2 \geq x_1 + \Delta x_1$, $x_i \geq x_{i-1} + \Delta x_{i-1}$, $i = 2,...,k$, as shown in Fig. 1.1.

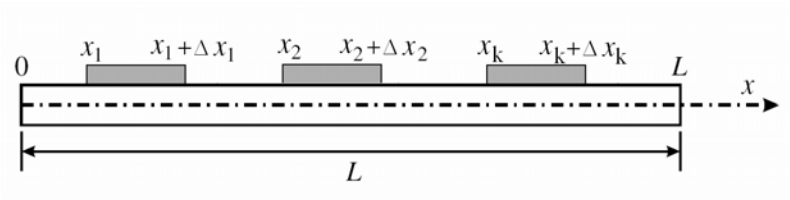

Fig. 1.1 – The beam with auxetic patchs.

Materials with a negative Poisson ratio ν are auxetic materials. Auxetic behaviour is found in materials from the molecular level, the microscopic, and to macroscopic structures. All the major classes of materials ((honeycombs and fibre-reinforced composites, polymers, composites, metals and ceramics) can exist in auxetic form and the natural and synthetic auxetic materials are known over several orders of magnitude of stiffness, or Young's modulus (Lakes [12]). The behaviour of auxetic materials under deformation can be enhanced as a result of having a negative Poisson's ratio. The beams with damping patchs have distinct advantages over monolithic beams of equal mass in applications to structures designed to dissipate in patch a significant fraction of the kinetic energy initially acquired. Consequently, the beam retains its integrity with only limited loading, but its patchs has enhanced the energy-absorbing capacity.

2. THE NONLOCAL DAMPING MODEL

Starting point of the nonlocal damping model is to assume that the damping force at a given point depends on the past history of a velocity field over a certain domain, through a kernel function (Lei, Friswell and Adhikari [7]). The governing equation of motion for a 1D linear damped continuous dynamic system may be expressed as

$$Lu(x,t) = f(x,t), \quad x \in \Omega, \quad t \in [0, T], \tag{2.1}$$

where $u(x,t)$ is the displacement vector, x is the spatial variable, t is time, $f(x,t)$ is the distributed external load, and L is the nonlocal operator defined by

$$Lu(x,t) = \rho(x)\frac{\partial^2}{\partial t^2}u(x,t) + M\frac{\partial}{\partial t}u(x,t), \tag{2.2}$$

where $\rho(x)$ is the distributed mass density, and the operator M is defined as

$$M\frac{\partial}{\partial t}u(x,t)=\int_{\Omega}\int_{0}^{t}C_e(x,\xi,t-\tau)\frac{\partial}{\partial t}u(\xi,\tau)\mathrm{d}\tau\mathrm{d}\xi+ \\ +\int_{\Omega}\int_{0}^{t}C_i(x,\xi,t-\tau)\frac{\partial^2}{\partial x\partial t}u(\xi,\tau)\mathrm{d}\tau\mathrm{d}\xi, \tag{2.3}$$

with $C_e(x,\xi,t-\tau)$ and $C_i(x,\xi,t-\tau)$ the kernel functions for the external and internal damping, respectively. The external damping is only dependent on the displacement, whereas the internal damping is dependent on the internal strains, which are given by spatial derivatives of the displacement. Equation (2.1) is subject to the initial and boundary conditions

$$u(x,0)=u_0(x),\ \ \frac{\partial}{\partial t}u(x,t)\big|_{t=0}=v_0(x), \\ u(x,t)=\overline{g}_1(x,t),\ \ x\in\Gamma_1, \\ \frac{\partial}{\partial x}u(x,t)=\overline{g}_2(x,t),\ \ x\in\Gamma_2, \tag{2.4}$$

where $u_0(x)$ and $v_0(x)$ are the initial displacement and velocity, Γ_1 and Γ_2 are the boundary domains, and $\overline{g}_1(x,t)$ and $\overline{g}_2(x,t)$ are known functions at the boundary. If the internal and external damping kernel functions are assumed to be separable in space and time, we can write in a general form

$$C(x,\xi,t-\tau)=H(x)c(x-\xi)g(t-\tau), \tag{2.5}$$

where the subscripts for the internal and external damping kernels are removed.

The function $H(x)$ denotes the presence of nonlocal damping. A particular case of (2.5) is the nonlocal viscous damping (or spatial hysteresis), where the kernel function is given by a delta function in time. In this case, the force depends only on the instantaneous value of the velocity or strain rate

$$g(t-\tau)=\delta(t-\tau), \tag{2.6}$$

but depends on the spatial distribution of the velocities

$$C(x,\xi,t-\tau)=H(x)c(x-\xi)\delta(t-\tau). \tag{2.7}$$

In the model (2.7), velocities at different locations within a certain domain can affect the damping force at a given point. This spatial hysteresis is similar to the damping model proposed by Banks and Inman [13] and Banks *et al.* [14] to describe the damping mechanism for a quasi-isotropic composite beam.

The spatial kernel function, $c(x-\xi)$ is normalized to satisfy the condition

$$\int_{-\infty}^{\infty} c(x)\mathrm{d}x = 1, \tag{2.8}$$

and can be choose as an exponential decay or an error function

$$c(x-\xi) = \frac{\alpha}{2}\exp(-\alpha\,|\,x-\xi\,|),$$
$$c(x-\xi) = \frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\alpha^2(x-\xi)^2\right), \tag{2.9a}$$

where α is a characteristic parameter of the damping material, or as the hat or triangular shapes

$$c(x-\xi) = \frac{1}{l_0} \quad \text{for} \quad |\,x-\xi\,| \le \frac{l_0}{2} \text{ and 0 otherwise}$$
$$c(x-\xi) = \frac{1}{l_0}\left(1-\frac{|\,x-\xi\,|}{l_0}\right) \text{ for } |\,x-\xi\,| \le l_0 \text{ and 0 otherwise.} \tag{2.9b}$$

Here, l_0 is the influence distance parameter. Another form for $c(x-\xi)$ may be the Dirac delta function $\delta(x-\xi)$, which reflects the reacting character of the damping force

$$c(x-\xi) = \delta(x-\xi). \tag{2.10}$$

In the case of a reacting damping force, there are two cases of (2.5):

(i) viscoelastic damping (or time hysteresis) with the kernel depending on the past time histories

$$C(x,\xi,t-\tau) = H(x)\delta(x-\xi)g(t-\tau). \tag{2.11}$$

(ii) viscous damping with the force depending only on the instantaneous value of the velocity or strain rate ($g(t-\tau)$ is given by (2.6))

$$C(x,\xi,t-\tau) = H(x)\delta(x-\xi)\delta(t-\tau). \tag{2.12}$$

This model represents the well-known viscous damping model (Sorrentino *et al*. [15]).

3. THE NONLOCAL DAMPED BEAM

Consider the beam represented in Fig. 1.1. For this example, the viscoelastic damping (or time hysteresis) with the kernel depending on the past time histories given by (2.11) is considered to model the patchs. The equation of motion for this beam is

$$\frac{\partial}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} + \sum_{i=1}^{2} I_i = f(x,t), \tag{3.1}$$

where $EI(x)$ is the bending stiffness, $\rho A(x)$ is the mass per unit length, $w(x,t)$ is the transverse displacement, and $f(x,t)$ is the distributed external force.

The third and fourth terms are the nonlocal external and internal damping, defined over the spatial subdomains $(x_i, x_i+\Delta x_i)$, $i=1,2,...,k$, as

$$I_1=\sum_{i=1}^{k}\int_{x_i}^{x_i+\Delta x_i}\int_{-\infty}^{t}C_e(x,\xi,t-\tau)\frac{\partial w(\xi,\tau)}{\partial t}\mathrm{d}\tau\mathrm{d}\xi, \tag{3.2}$$

and

$$I_2=\sum_{i=1}^{k}\int_{x_i}^{x_i+\Delta x_i}\int_{-\infty}^{t}C_i(x,\xi,t-\tau)\frac{\partial^2}{\partial\xi^2}\left(\gamma(\xi)\frac{\partial^3 w(\xi,\tau(}{\partial\xi^2\partial\tau}\right)\mathrm{d}\tau\mathrm{d}\xi. \tag{3.3}$$

The internal and external damping kernels are defined (2.5). The initial conditions $(2.4)_1$ are written as

$$w(x,0)=w_0(x),\ \frac{\partial}{\partial t}w(x,t)|_{t=0}=v_0(x). \tag{3.4a}$$

The boundary conditions $(2.4)_2$ are written for different cases: for a clamped beam

$$w(x,t)=0,\ \frac{\partial w(x,t)}{\partial x}=0\ \text{ for }\ x=0,\ x=L, \tag{3.4b}$$

for a simple supported beam

$$w(x,t)=0,\ \frac{\partial^2 w(x,t)}{\partial x^2}=0\ \text{ for }\ x=0,\ x=L, \tag{3.4c}$$

and for a free end beam

$$\frac{\partial^2 w(x,t)}{\partial x^2}=0,\ \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]=0\ \text{ for }\ x=0,\ x=L, \tag{3.4d}$$

The easiest way to solve (3.1) is to apply the Laplace transform to (3.1) to give the eigenvalue problem for the free vibration of the beam

$$\frac{\partial}{\partial x^2}\left(EI(x)\frac{\partial^2 W(x,s)}{\partial x^2}\right)+s^2\rho A(x)W(x,s)+\sum_{i=1}^{2}J_i=0, \tag{3.5}$$

where $W(x,s)$ is the Laplace transform of $w(x,t)$, and

$$J_1=\sum_{i=1}^{k}sG_e(s)\int_{x_i}^{x_i+\Delta x_i}c_e(x-\xi)W(\xi,s)\mathrm{d}\xi,$$

$$J_2=\sum_{i=1}^{k}sG_i(s)\int_{x_i}^{x_i+\Delta x_i}c_i(x-\xi)\frac{\partial^2}{\partial\xi^2}\left(\gamma(\xi)\frac{\partial^2 W(\xi,s)}{\partial\xi^2}\right)\mathrm{d}\xi,$$

where $G_e(s)$ and $G_i(s)$ are the Laplace transforms of the external and internal kernel functions $g_e(t)$ and $g_i(t)$. Equation (3.5) is an integro-differential equation, and obtaining the analytical solutions is possible by using the cnoidal method (Munteanu and Donescu [15], Chiroiu and Chiroiu [16]).

4. THE CNOIDAL METHOD

The inverse scattering theory generally solves certain nonlinear differential equations. The mathematical and physical structure of the inverse scattering transform solutions has been extensively studied in both one and two dimensions. The theta-function representation of the solutions is describable as a linear superposition of Jacobi elliptic functions (cnoidal functions) and additional terms, which include nonlinear interactions among them. The cnoidal method is reducible to a generalization of the Fourier series with the cnoidal functions as the fundamental basis function. This is because the cnoidal functions are much richer than the trigonometric or hyperbolic functions, that is, the modulus m of the cnoidal function, $0 \le m \le 1$, can be varied to obtain a sine or cosine function $(m \cong 0)$, a Stokes function $(m \cong 0.5)$ or a solitonic function, sech or tanh $(m \cong 1)$. The general solution of (3.5) may be written in the terms of the *theta function* representation

$$\theta(x,t) = \frac{2}{\lambda}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\Theta_n(\eta_1,\eta_2,\ldots,\eta_n), \tag{4.1}$$

where $\lambda = \alpha/6\beta$, and Θ is the *theta function* defined as

$$\Theta_n(\eta_1,\eta_2,\ldots,\eta_n) = \sum_{M\in(-\infty,\infty)} \exp(\mathrm{i}\sum_{i=1}^{n} M_i\eta_i + \frac{1}{2}\sum_{i,j=1}^{n} M_i B_{ij} M_j), \tag{4.2}$$

with n the number of degrees of freedom for a particular solution of (3.5), and

$$\eta_j = k_j x - \omega_j t + \phi_j,\ \ 1 \le j \le N. \tag{4.3}$$

In (4.3), k_j are the wave numbers, the ω_j are the frequencies and the ϕ_j are the phases. Let us introduce the vectors of wave numbers, frequencies and constant phases

$$k = [k_1,k_2,\ldots,k_n],\ \ \omega = [\omega_1,\omega_2,\ldots,\omega_n],\ \phi = [\phi_1,\phi_2\ldots,\phi_n],$$
$$\eta = [\eta_1,\eta_2,\ldots,\eta_n].$$

The vector η can be written as

$$\eta = kx - \omega t + \phi.$$

Also, we can write

$$M\eta = Kx - \Omega t + \Phi,\quad M = [M_1, M_2, ..., M_n],\ K = Mk,$$
$$\Omega = M\omega,\ \Phi = M\phi.$$

The integer components in M are the integer indices in (4.3). The matrix B can be decomposed in a diagonal matrix D and an off-diagonal matrix O, that is

$$B = D + O.$$

We present without proof the following theorem:

THEOREM. *The solution* $\theta(x,t)$ *of equation* (3.5) *can be written as*

$$\theta(x,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\log\Theta_n(\eta) = \theta_{lin}(\eta) + \theta_{int}(\eta),$$

where θ_{lin} *represents a linear superposition of cnoidal functions*

$$\theta_{lin}(\eta) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\log G(\eta),$$

$$G(\eta) = \sum_M \exp(\mathrm{i}M\eta + \frac{1}{2}M^T DM),$$

and θ_{int} *represents a nonlinear interaction among the cnoidal functions*

$$\theta_{\mathrm{int}}(\eta) = 2\frac{\partial^2}{\partial t^2}\log(1 + \frac{F\ (\eta, C)}{G\ (\eta)}),$$

$$F\ (\eta, C) = \sum_{M^\alpha} C\exp(\mathrm{i}M\eta + \frac{1}{2}M^T DM),$$

$$C = \exp(\frac{1}{2}M^T OM) - 1.$$

As a result, the cnoidal method yields to solutions consisting of a linear superposition and a nonlinear superposition of cnoidal functions. Some results are reported finally. Fig. 4.1 shows the evolution of the overall strain for an ordinary simply supported Euler-Bernoulli beam and for two simply supported Euler-Bernoulli beam with two and respectively four nonlocal viscous external damping auxetic patchs. The calculus was carried up for an aluminum beam of length $L = 2\mathrm{m}$, width $b = 0.005\mathrm{m}$, thickness $h = 0.005\mathrm{m}$, Young's modulus $E = 109$ GPa, and mass density $\rho = 2700\ \mathrm{kg/m^3}$ and Poisson's ratio $\nu = 0.34$. The positions of the left ends of the nonlocal damping patchs are $x_1 = L/5$, $x_2 = 3L/5$ and $\Delta x = 0.1\mathrm{m}$, for the first case study and $x_1 = L/5$, $x_2 = 2L/5$, $x_3 = 3L/5$, $x_4 = 4L/5$ and $\Delta x = 0.1\mathrm{m}$ for the second case study. For the auxetic material we have considered Young's modulus $E = 1.55$ GPa, mass density ρ = $= 837\ \text{kg/m}^3$, and Poisson's ratio $\nu = -0.25$.

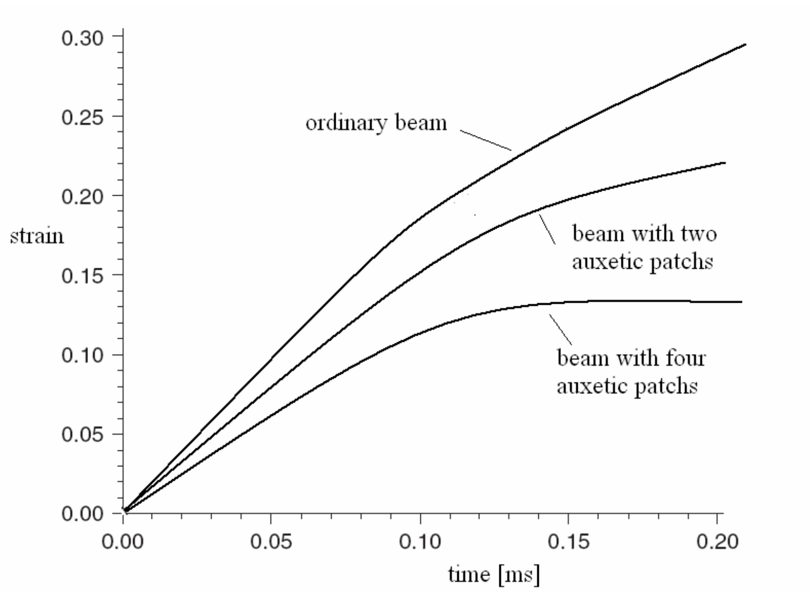

Fig. 4.1 – The evolution of of the overall strain for an ordinary beam and for two beams with two and respectively four auxetic patchs.

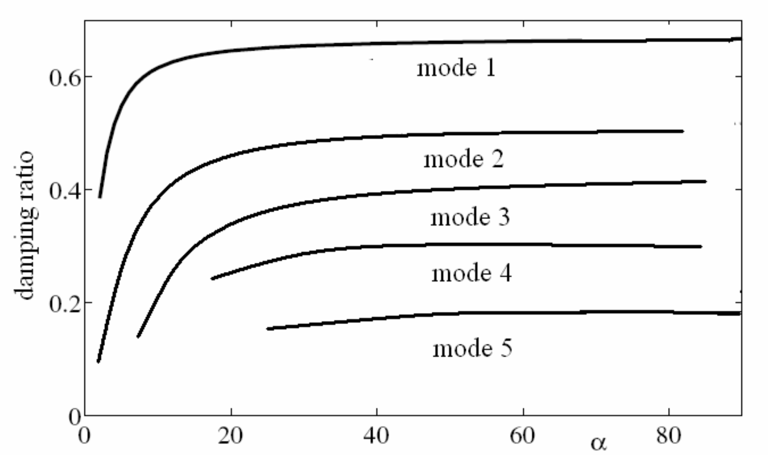

Fig. 4.2 – The dependence of the damping ration on the parameter α for the first five modes.

The influence of the length parameter α a in the spatial kernel function on the damping ratios of the first five modes of the beam with two auxetic patchs, is shown in Fig. 4.2, for a time hysteresis parameter of $\mu = \infty$. The damping ratios are very sensitive for small values of α and approach steady values for large α, with a fixed damping coefficient $H_0 = 2$.

5. CONCLUSIONS

The Euler-Bernoulli beams with nonlocal damping are able to dissipate in its auxetic patchs a great fraction of the kinetic energy initially acquired. The beams with auxetic patchs are described with the nonlocal theory of damping, in which the damping force is obtained as a weighted average of the velocity field over the spatial domain, determined by a kernel function based on distance measures.

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