ON THE MODAL STRAIN ENERGY APPROACH*

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In this paper, the modal strain energy distribution for a transversely isotropic thick-walled toroidal shell and free boundary is considered. For initial data in the form of a sine-wave, we obtain the same result as in .the Fermi-Pasta and Ulam problem, *i.e.* the energy does not spread throughout all the normal modes but remains in the initial mode (85%) and a few nearby modes. This result is very important for the treatment of damping, because the distribution of the strain energy in the initial mode can be directly used in the definition of the best treatment areas.

1. THE FERMI-PASTA-ULAM PROBLEM

In 1955, Fermi, Pasta and Ulam [1–3] were investigated the behaviour of a dynamical system consisting of N identical particles of unit mass on a line with fixed end points with forces acting between nearest neighbours. The nonlinearity was introduced as a perturbation. In the absence of such perturbations, the energy in each of the normal modes of the linear system would be constant. They expected that the nonlinear interactions between the modes would lead to the energy of system being evenly distributed throughout all of the modes. This is a result of the much known theorem of equipartition theorem. The results obtained by Fermi, Pasta and Ulam have contradicted this theorem. It is known that the unexpected nature of these results have stimulated the development of the soliton theory [4–7], because a little mathematics was needed to obtain the Korteweg and de Vries equation (KdV) for the motion of such a system. From KdV equation it results *that the energy does not spread throughout all the normal modes but remains in the initial mode and a few nearby modes, for initial data in the form of a sine-wave.* Over a large number of oscillations, the energy in each normal mode is seen to be almost periodic in time, with no loss of energy to higher modes as time increases. The explanation of this periodicity stimulated a lot of studies [8–11].

In this paper, we analyse the modal strain energy distribution for a thick-walled toroidal shell with free boundary conditions. The shell is assumed to be transversely isotropic with respect to the toroidal radial direction. We obtain the same result as Fermi, Pasta and Ulam, *i.e.* the energy does not spread throughout

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all the normal modes but remains in the initial mode and a few nearby modes, for initial data in the form of a sine-wave. The motivation of this chooses consists is the control of vibrations of toroidal pipelines [12, 13]. The oil, gas, chemical and petrol-chemical industries operate pipelines, often exposed to vibrations. For the passive vibrations control, the constrained viscoelastic layer damping treatment is an efficient method used for pipelines.

2. THE TOROIDAL SHELL

Let us choose the toroidal coordinate system (r,θ,φ), where φ is the angle measured from the y-axis (Fig. 2.1). By denoting with u_r, u_θ, u_φ the displacements and R the radius relative to the center of the torus, the vector of position of a point is defined by

$$\mathbf{r} = (R + r\cos\theta)\sin\varphi\mathbf{i} + (R + r\cos\theta)\cos\varphi\mathbf{j} + r\sin\theta\mathbf{k}\,. \tag{2.1}$$

The strain components can be written as [14]

$$\begin{aligned}
\varepsilon_{rr} &= s_1 = u_{r,r}\,,\\
\varepsilon_{\theta\theta} &= s_2 = r^{-1}u_{\theta,\theta} + r^{-1}u_r\,,\\
\varepsilon_{\varphi\varphi} &= s_3 = (R + r\cos\theta)^{-1}(u_{\varphi,\varphi} + u_r\cos\theta - u_\theta\sin\theta)\,,\\
\varepsilon_{\theta\varphi} &= s_4 = r^{-1}u_{\varphi,\theta} + (R + r\cos\theta)^{-1}(u_{\theta,\varphi} + u_\varphi\sin\theta)\,,\\
\varepsilon_{r\varphi} &= s_5 = u_{\varphi,r} + (R + r\cos\theta)^{-1}(u_{r,\varphi} - u_\varphi\cos\theta)\,,\\
\varepsilon_{r\theta} &= s_6 = u_{\theta,r} - r^{-1}u_\theta + r^{-1}u_{r,\theta}\,.
\end{aligned} \tag{2.2}$$

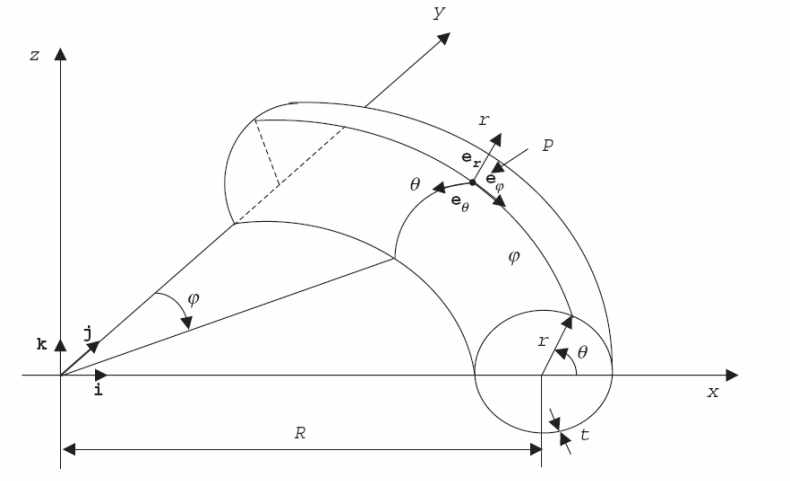

Fig. 2.1 – The toroidal system of coordinates.

When the poling axis corresponds to the φ toroidal axis, the material matrix $\mathbf{C}$ is defined as

$$\mathbf{C}=\begin{bmatrix} C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\ C_{23} & C_{33} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}, \tag{2.3}$$

with $C_{44}=(C_{33}-C_{23})/2$. By writing

$$\begin{aligned} u_r &= U(r,\theta)\cos m\varphi\cos\omega t\,, \\ u_\theta &= V(r,\theta)\cos m\varphi\cos\omega t\,, \\ u_\varphi &= W(r,\theta)\sin m\varphi\cos\omega t\,, \end{aligned} \tag{2.4}$$

where m is the circumferential wave number and ω is the circular frequency. The governing equation for free vibration takes the form

$$\mathbf{KU}-\omega^2\mathbf{MU}=0\,, \tag{2.5}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices respectively, and $\mathbf{U}$ is the eigenvector of nodal displacements. The strain energy of the i th-mode is written as

$$E=\phi_i^T\mathbf{K}\phi_i\,, \tag{2.6}$$

where ϕ_i is the i th modal shape vector. The non-dimensional natural frequencies Ω are calculated with respect to the outside radius of the toroidal cross-section a, the density of material ρ and the circular frequency ω

$$\Omega=\omega a\sqrt{\frac{\rho}{C_{55}}}\,,\ R=\alpha a\,,\ b=\lambda a\,,\ \beta=(C_{33}-C_{23})/2C_{55}\,, \tag{2.7}$$

where R and b arew non-dimensional quantities and numbers $\alpha>1$, $\lambda<1$. For $\beta=1$ the material is isotropic. For crystals, β varies from 0.372 to 1.892 [15].

3. RESULTS AND CONCLUSIONS

For transversely isotropic cadmium toroidal shell, the matrix (2.3) becomes (multiplied by 10^9 N/m^2 [14]) and $\beta=0.372$

$$\mathbf{C}=\begin{bmatrix} 40.8 & 35.4 & 29 & 0 & 0 & 0 \\ 35.4 & 40.8 & 29 & 0 & 0 & 0 \\ 29 & 29 & 52.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.26 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.26 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.7 \end{bmatrix}. \tag{3.1}$$

The natural frequencies of the shell are calculated from (2.7). In Table 3.1, the first five natural frequencies are displayed for shell poled relative to φ and r in the (r,θ,φ) system of coordinates, and $R=2$, $b=0.5$. The first value corresponds to the case of poled φ, and he second to r.

Calculation of the energy for the five modes yields to the conclusion that the energy does not spread throughout all the normal modes but remains in the initial mode (85%) and a few nearby modes, for initial data in the form of a sine-wave. This result is very important for the treatment of damping, because the distribution of the strain energy in the initial mode can be directly used in the definition of the best treatment areas. The best treatment location areas on the shells can be possibly identified, based only on the distribution of the modal strain energy in the first mode.

Table 3.1

Natural frequencies for for transversely isotropic cadmium toroidal shell

Mode	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	0.44	0.58	0.25	0.53	0.83	1.09
	0.73	1.06	0.45	1.04	1.63	2.18
2	0.73	0.79	0.33	0.72	1.09	1.41
	1.10	1.29	0.55	1.23	1.84	2.40
3	1.33	1.45	0.92	1.28	1.65	1.91
	2,38	2.29	0.69	2.28	2.67	3.11
4	1.48	1.69	1.08	1.38	1.66	1.97
	2.41	2.33	1.84	2.47	2.87	3.27
5	2.20	2.14	1.68	1.91	2.12	2.36
	2.46	2.69	2.36	2.57	3.08	3.59

Fig. 3.1. shows: a) love mode (1,1), b) shell mode (1,2), c) bending mode (2,1), d) bending mode (2,2), and shell mode (2,2). Fig. 3.2 displays the variation with respect to time of the strain energy for $m\to\infty$. The value $E_{\max}$ corresponds to the energy of the first mode. The numerical experiments have shown that over large number of oscillations, the energy in each normal mode is seen to be almost periodic in time, with no loss of energy to higher modes as time increases. So, the energy $E_{\max}$ periodically reappears, the period being approximately 70 cycles. The main conclusion of this work is that the optimal treatment location does not depend

on the particular mode shape considered. Therefore, it is easy to design efficient partial viscoelastic treatments by applying to pipelines viscoelastic layers on the regions where the strain energy is higher. These regions can be chosen from information of only the first mode of vibrations.

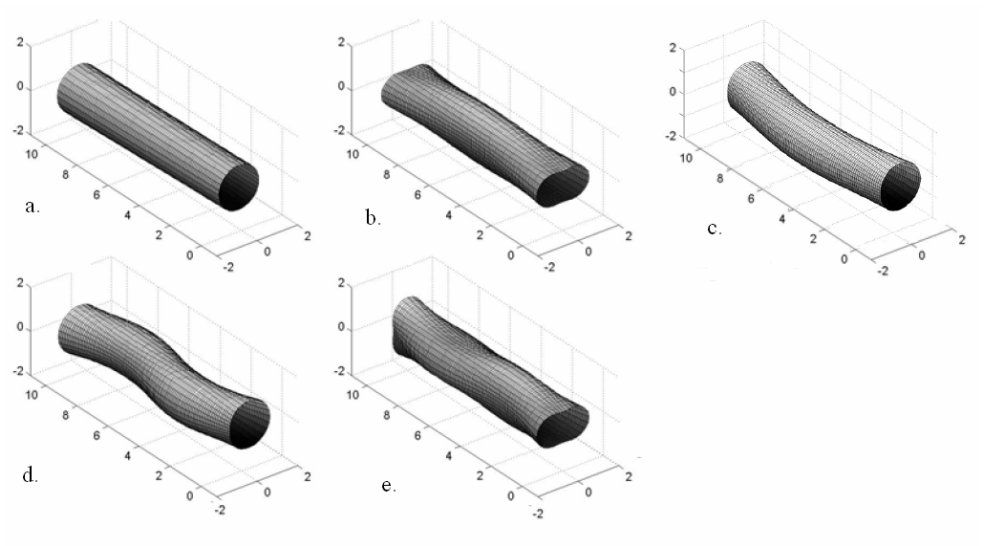

Fig. 3.1 – The five modes of toroidal shell poled relative to φ and r.

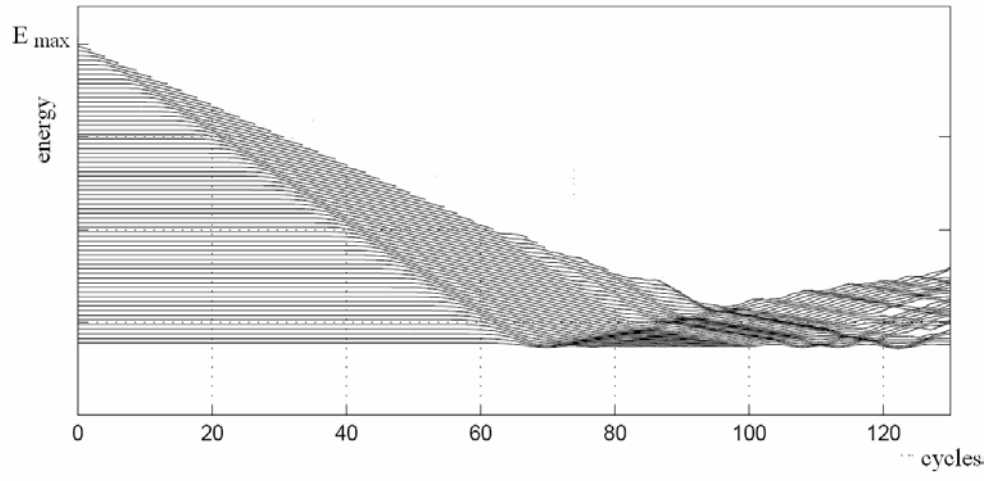

Fig. 3.2 – The plot of the energy function against time for $m \to \infty$.

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