PERIODIC SOLUTION FOR DIFFERENTIAL EQUATIONS WITH SMALL PARAMETER

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There is considered a class of non-linear differential equations and there is analysed the existence of the periodical solutions and the stability of this solutions for the case when the linear part is admitting periodical solutions. This means the determination of a fundamental system of periodical solutions for the equation corresponding to the variation equation.

1. INTRODUCTION

The systems of non-linear differential equations with small parameter represent the mathematical model for a class of dynamic phenomena with periodical solutions. For non-periodical solutions of linearized systems, the serial developments related to a small parameter of the unknown functions mean there determination by iterative procedure. So, if the variations system corresponding to the generating solution has no other periodical solution then the trivial solution, the non-linear system with small parameter is admitting a unique periodical solution continuously dependent on the small parameter. It there periodical solution for the variations system, the existence of the non-linear periodical solutions of the system can’t be demonstrated, because this requires solving of an algebraic system with singular matrix. Taking this into account, this paper is going to propose a model stating the existence of periodical solutions for a non-linear differential equations class.

2. NON-LINEAR SYSTEMS WITH SMALL PARAMETER

Taking into account the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Z(x, \varepsilon) \tag{1}$$

where $Z(x, \varepsilon)$ is continuously differentiable and has the expression:

$$Z(x, \varepsilon) = X_0(x) + \varepsilon X_1(x, \varepsilon). \tag{2}$$

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Assume that for the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X_0(x) \tag{3}$$

exist the periodical solution $x = u(t)$.

We denote:

$$X_0(t) = \frac{X_0(x_0(t))}{\|X_0(x_0(t))\|}. \tag{4}$$

If X is proofing a local Lipschitz solution, the curve $x = \overline{X}(t)$ will not cover the entire unit sphere, thus there exist a unit vector e_1 so that $X(t) + e_1 \neq 0$ for each t.

Starting from the constant vector e_1 there will be constructed an orthogonal and normalized vector system $e_1, e_2, e_3, \ldots\ldots\ldots e_n$.

Because:

$$\cos\theta_i = \frac{(X_0(t), e_i)}{\|\overline{X}_0\|\|e_i\|}, \tag{5}$$

$$\cos(\overline{X}_0, e_i) = \cos\theta_1 \neq \cos\pi \neq -1, \tag{6}$$

there might formed the vectors:

$$\xi_\gamma = e_\gamma - \frac{\cos\theta_\gamma}{1 + \cos\theta_1}. \tag{7}$$

The vectors $(\overline{X}(t), \xi_2, \xi_3 \ldots\ldots \xi_n)$ are generating an orthogonal normalized system. Employing this system we are performing the change of variable given by:

$$x = u(\theta) + S(\theta)y, \tag{8}$$

where:

$$\begin{aligned} S(\theta) &= (\xi_2(\theta), \xi_3(\theta), \ldots\ldots \xi_n(\theta)), \\ y &= (y^1, y^2, \ldots\ldots y^{n-1}), \\ u(\theta) &= (u^1(\theta), u^2(\theta), \ldots.. u^n(\theta))^{\mathrm{T}}. \end{aligned} \tag{9}$$

Taking into account (3), by the change of variable (8), system (1) will became:

$$X_0[u(\theta)]\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}y + S(\theta)\frac{\mathrm{d}y}{\mathrm{d}t} = Z[u(\theta) + S(\theta)y, \varepsilon]. \tag{10}$$

Employing the orthogonality relation, multiplying equation (10) with $X_0^*[u(\theta)]$ will result:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{X_0^*\left[u(\theta)\right]Z\left[u(\theta)+S(\theta)y,\varepsilon\right]}{\left\|X_0\left[u(\theta)\right]\right\|^2+X_0^*\left[u(\theta)\right]\dfrac{\mathrm{d}S(\theta)}{d\theta}y}=\Theta(\theta,y,\varepsilon). \tag{11}$$

Similarly, by multiplication of (10) with $\xi_\mu^*(\theta)$, there will be obtained:

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}t}=\xi_\mu^*(\theta)Z\left[u(\theta)+S(\theta)y,\varepsilon\right]-\Theta(\mu,y,\varepsilon)\xi_\mu^*(\theta)\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}y=y^\mu(\theta,y,\varepsilon). \tag{12}$$

Because we have:

$$\begin{gathered}Z\left[u(\theta),0\right]=X_0\left[u(\theta)\right],\\ \frac{\partial Z\left[u(\theta),0\right]}{\partial x}=\frac{\partial X_0\left[u(\theta),0\right]}{\partial x}=A(\theta)\end{gathered} \tag{13}$$

by a Taylor serial development, the expression of will became:

$$Z[u(\theta)+S(\theta)y,\varepsilon]=X_0[u(\theta)]+A(\theta)S(\theta)y+\varepsilon\frac{\partial Z[u(\theta),0]}{\partial\varepsilon}+0(|y|+|\varepsilon|). \tag{14}$$

From (12) and (14) will result the transformed system, called also variation system:

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}t}=\frac{\xi_\mu(\theta)AS(\theta)y+\varepsilon\xi_\mu^*(\theta)X\left[u(\theta)+S(\theta)y,0\right]}{\Theta(\theta,y,\varepsilon)}-\xi_\mu^*(\theta)\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}y, \tag{15}$$

which might be written:

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=Y_0(y,\theta)+\varepsilon Y_1(y,\theta,\varepsilon), \tag{16}$$

where Y_0 and Y_1 are periodical reported to θ, having the period ω and admits continuos partial derivatives.

For $\varepsilon=0$ there will be obtained the generating system:

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=Y_0(y,\theta), \tag{17}$$

which is admitting the periodical solution $y_0(\theta)=0$.

For the future developments it will be necessary to use:

THEOREM 1. *If the variation system:*

$$\frac{\mathrm{d}v}{\mathrm{d}\theta}=\frac{\partial}{\partial y}\left[Y_0\left(y_0(\theta),\theta\right)\right]v \tag{18}$$

have periodic solutions with period ω, *than the adjoint system*:

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t}=-\bar{v}\frac{\partial}{\partial y}\left[Y_0\left(y_0(\theta),\theta\right)\right]v \tag{19}$$

will admit the same number of linear independent solutions as system (18).

Considering the vectorial space of the variation system solution (18), there will exist a base of it in the orthogonal normalized system so that any element of the solution space might be written as a linear combination of the vectors in the base.

So we will have:

$$v(\theta) = \overline{p}(\theta)\overline{X}[u(\theta)] + \sum_{\gamma=2}^{n} p^{\gamma}(\theta)\xi_{\gamma}(\theta), \tag{20}$$

where $p^{\gamma}\left(\gamma = 2,\ 3, ...\right)$ are the normal components of the variations. There will be easy to demonstrate [2] following.

Proposition. *The normal components* p^{γ} *on the variations are proofing the normal variations system (15).*

To establish the periodicity condition of the non-linear system (16) there will be formulated [2]:

THEOREM 2. *The necessary and sufficient condition that the system (16) shall admit periodical solutions is to be fulfilled the orthogonality conditions:*

$$P \equiv \int_{0}^{\omega} W(s) Y_1\left[y_0(s), s, 0\right] \mathrm{d}s = 0\,, \tag{21}$$

$W(s)$ *being the matrix which has as lines the periodical solutions of the adjoint system of the variation system (18).*

3. SMALL PARAMETER NON-LINEAR DIFFERENTIAL EQUATIONS

Let's be the small parameter non-linear differential equations:

$$\ddot{x} + ax + \varepsilon X_1(x, \varepsilon) = 0\,, \tag{22}$$

where $X_1(x, \varepsilon)$ is a non-linear term depending from the small parameter ε and $a > 0$. The fundamental matrix of the linearized system solutions is given by.

$$\overline{x} = \begin{pmatrix} \overline{x}_{11} & \overline{x}_{12} \\ \overline{x}_{12} & \overline{x}_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{a}\sin\sqrt{at} & -\sqrt{a}\cos\sqrt{at} \\ -a\cos\sqrt{at} & a\sin\sqrt{at} \end{pmatrix}. \tag{23}$$

Because the manifold of the linearized equation solutions is a linear space, we will take as vectors of the base the vectors of co-ordinates:

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{24}$$

A periodical solution of the linearized equations (22) might be written:

$$u(t,\alpha) = \alpha\begin{pmatrix}\sin\sqrt{a}t\\ \sqrt{a}\cos\sqrt{a}t\end{pmatrix} = \alpha\left(\sin\sqrt{a}te_1 + \sqrt{a}\cos\sqrt{a}te_2\right). \tag{25}$$

It will result:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = X_0\left[u(t,\alpha)\right]. \tag{26}$$

Taking into account (4) we have:

$$\overline{X}_0\left[u(t,\alpha)\right] = \frac{\cos\sqrt{a}te_1 - \sqrt{a}\sin\sqrt{a}te_2}{\sqrt{\cos^2\sqrt{a}t + a\sin^2\sqrt{a}t}}. \tag{27}$$

Using the relation (5) there will be obtained:

$$\begin{aligned}\cos\theta_1 &= \left(\overline{X}_0, e_1\right) = \frac{\cos\sqrt{a}t}{\sqrt{\cos^2\sqrt{a}t + a\sin^2\sqrt{a}t}}\\ \cos\theta_2 &= \left(\overline{X}_0, e_2\right) = \frac{\sqrt{a}\sin\sqrt{a}t}{\sqrt{\cos^2\sqrt{a}t + a\sin^2\sqrt{a}t}}\end{aligned}, \tag{28}$$

so that (7) will became:

$$\xi_2(\theta) = \begin{pmatrix}\dfrac{\sqrt{a}\sin\sqrt{a}\theta}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}}\\ \dfrac{\cos\sqrt{a}\theta}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}}\end{pmatrix} = S(\theta). \tag{29}$$

Performing the calculation:

$$X_0^*\left[u(\theta)\right]\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta}y = \frac{\alpha a\sqrt{a}t}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}}y_2,$$

$$\xi_2^*(\theta)\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta} = 0, \tag{30}$$

$$X_0^*\left[u(\theta)\right]AS(\theta)y = \alpha\sqrt{a}\,\frac{a^2\sin^2\sqrt{a}\theta + \cos^2\sqrt{a}\theta}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}}y_2.$$

Because a is a parameter which has positive values we don't restrict it's generality if we will take $a = \mathrm{ctg}^2\theta$ for which the expression (11) is given by:

$$\Theta(\theta, y, \varepsilon) = 1 + \frac{X_0^*[u(\theta)]X_0[u(\theta) + S(\theta)y]}{\alpha^2 a\left(\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta\right) + \dfrac{\alpha\sqrt{a}}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}}\, y_2} \tag{31}$$

from which by Taylor development there will be obtained:

$$\frac{1}{\Theta(\theta, y, \varepsilon)} = 1 - \varepsilon\frac{X_0^*[u(\theta)]X_0[u(\theta) + S(\theta)y, 0]}{\alpha^2 a\left(\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta\right) + \dfrac{\alpha a\sqrt{a}}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}}} + 0(\varepsilon) \tag{32}$$

Taking into account that

$$AS(\theta) = \begin{pmatrix} \dfrac{\cos\sqrt{a}\theta}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}} \\ \dfrac{-\sqrt{a}\sin\sqrt{a}\theta}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}} \end{pmatrix},$$

$$\xi_2(\theta) AS(\theta) = \frac{\sqrt{a}(1-a)\sin\sqrt{a}\theta\cos\sqrt{a}\theta}{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}, \tag{33}$$

$$\xi_2^*(\theta)\frac{\mathrm{d}S(\theta)}{\mathrm{d}\theta} = 0,$$

the transformed system (15) will became:

$$\begin{aligned} \frac{\mathrm{d}y_2}{\mathrm{d}\theta} &= \frac{\sqrt{a}(1-a)\sin\sqrt{a}\theta\cos\sqrt{a}\theta}{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}\, y_2 + \\ &+ \varepsilon\left\{\xi_2^*(\theta) - K(\theta, y_2)X^*\left[u(\theta)\right]\right\} X_0\left[u(\theta) + S(\theta)y, 0\right], \end{aligned} \tag{34}$$

where it was denoted:

$$K(\theta, y_2) = \frac{\sqrt{a}(1-a)\sin\sqrt{a}\theta\cos\sqrt{a}\theta y_2}{\left[\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta\right]}\cdot$$

$$\cdot\frac{1}{\left[\alpha^2 a\left(\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta\right) + \dfrac{\alpha\sqrt{a}}{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}\, y_2\right]} \tag{35}$$

As stated by the proposition in the previous chapter, the normal variation equation (34) accepts the normal components of the variation system obtained by linearization of the equation (22).

Developing the solutions of this system after the vectors $\left(\overline{X}_0, \xi_2\right)$ we have:

$$\begin{aligned}&\sqrt{a}\sin\sqrt{a}\theta e_1 + a\cos\sqrt{a}\theta e_2 = p_1^1\overline{X}_0 + p_1^2\xi_2 - \\ &-\sqrt{a}\cos\sqrt{a}\theta e_1 + a\sin\sqrt{a}\theta e_2 = p_2^1\overline{X}_0 + p_2^2\xi_2,\end{aligned} \tag{36}$$

from where it follows:

$$\begin{aligned}&p_1^1 = \frac{\sqrt{a}\,(1-a)\sin\sqrt{a}\theta\cos\sqrt{a}\theta}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}},\\[2ex] &p_2^1 = -\sqrt{a}\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta},\\[2ex] &p_1^2 = \frac{a}{\sqrt{\cos^2\sqrt{a}\theta + a\sin^2\sqrt{a}\theta}},\\[2ex] &p_2^2 = 0.\end{aligned} \tag{37}$$

The variation system solutions are p_1^2, p_2^2.

As stated by a well known [2], if the variation system, corresponding to a non-linear system, has periodic solution only the trivial solution the non-linear system is have periodic solution.

Thus we shall analyse the case $p_1^2 = 0$.

Because the periodical solution of the adjoin system is $\left(p_1^2\right)^{-1}$ it follows:

$$y_0(\theta) = y_2^{-1}(\theta) = \sqrt{\cos^2\sqrt{a}\theta + a\sin\sqrt{a}\theta}\,. \tag{38}$$

Regarding (29) for the periodical solution (38) we have:

$$S(\theta)y = \begin{pmatrix}\sqrt{a}\sin\sqrt{a}\theta\\ \sqrt{a}\cos\sqrt{a}\theta\end{pmatrix}, \tag{39}$$

so that

$$u(\theta) + S(\theta)y = \begin{pmatrix}\left(\alpha + \sqrt{a}\right)\sin\sqrt{a}\theta\\ \sqrt{a}(\alpha+1)\cos\sqrt{a}\theta\end{pmatrix}. \tag{40}$$

Further there is verified the necessary periodicity condition of the analysed non-linear equations solution.

4. THE EXISTENCE OF PERIODICAL SOLUTIONS FOR NON-LINEAR EQUATIONS CLASS

The non-linear term of the normal equation variation (34) is written:

$$Y_1(y,\theta,\varepsilon)=\{\xi_2^*(\theta)-K(\theta,y_2)X_0^*[u(\theta)]\}X_0[u(\theta)+S(\theta)y,0] \quad (41)$$

or explicit:

$$Y_1(y,\theta,\varepsilon)=\left[\frac{\sqrt{a}\sin\sqrt{a}\theta}{\cos^2\sqrt{a}\theta+a\sin^2\sqrt{a}\theta}-\alpha\sqrt{a}\left(\cos\sqrt{a}\theta\right)K(\theta,y_0)\right]X_0^1[u(\theta)+S(\theta)y,0]+$$

$$+\left[\frac{\cos\sqrt{a}\theta}{\cos^2\sqrt{a}\theta+a\sin^2\sqrt{a}\theta}+\alpha a\left(\sin\sqrt{a}\theta\right)K(\theta,y_0)\right]X_0^2[u(\theta)+S(\theta)y,0], \quad (42)$$

where $X_0^i(i=1,2)$ are the components of x_0.

Taking into account (40) we have:

$$\begin{aligned} X_0^1\left[u(\theta)+S(\theta)y,0\right]&=\left(\alpha\sqrt{a}+1\right)\cos\sqrt{a}\theta,\\ X_0^2\left[u(\theta)+S(\theta)y,0\right]&=-a\left(\alpha+\sqrt{a}\right)\sin\sqrt{a}\theta.\end{aligned} \quad (43)$$

For the solution (38) determined for the linear part of the normal variation we will obtain:

$$K(\theta,y_0)=\frac{\sqrt{a}(1-a)\sin\sqrt{a}\theta\cos\sqrt{a}\theta}{\sqrt{\cos^2\sqrt{a}\theta+a\sin^2\sqrt{a}\theta}}\frac{1}{\alpha^2a\left(\cos^2\sqrt{a}\theta+a\sin^2\sqrt{a}\theta\right)+\alpha a\sqrt{a}} \quad (44)$$

and replacing the expression of the non-linear term $Y_1(y_0,\theta,\varepsilon)$ given by (42), the relation (21) will became:

$$P(\alpha)\equiv\int_0^{2\pi}\sqrt{a}(1-a)\sin\sqrt{a}\theta\cos\sqrt{a}\theta\frac{\cos^2\sqrt{a}\theta+a^2\sin^2\sqrt{a}\theta-a}{\alpha\left(\cos^2\sqrt{a}\theta+a\sin^2\sqrt{a}\theta\right)-\sqrt{a}}\mathrm{d}\theta\,. \quad (45)$$

There will be easy find out that the integral (45) is null and thus, as stated by theorem 2, being fulfilled the orthogonality condition (21), is resulting the existence of the periodical solution of the analysed non-linear differential equations class. So it might be expressed:

THEOREM 3. *If the linearized equation (34) exist a periodic solution family* $y_0(\theta,\alpha)$, *having the period* 2π, *than for each value* α_0 *so that:*

$$P(\alpha_0)=0,\quad \frac{\det\partial P(\alpha)}{\partial\alpha}\Big|_{\alpha=\alpha_0}\neq 0\,, \quad (46)$$

where $P(\alpha)$ *is given by (45), there exists a periodic solution* $y_0(\theta, \alpha)$ *of the non-linear equation (34), with the period* 2π *and thus:*

$$\lim_{\varepsilon \to 0} y(\theta, \varepsilon) = y_0(t, \alpha_0). \tag{47}$$

Together with the existence of the periodical solution for non-linear equation with a small parameter, this theorem is stating the property given the relation between the non-linear equation solution and the generating equation solution.

5. CONCLUSIONS

The performed study shows an analysis method regarding the existence of periodical solutions for small parameter differential equations, as a particular case of the non-linear differential equations systems. The perturbation method for the periodical solutions determination is using the developments after a small parameter annihilating the non periodical terms [1], [2], [5].

As a consequence, there is presumed the existence of the periodical solutions without demonstrating it. Further not always is possible to develop the coefficient "a" of the linear part of the equation after the small parameter, *e.g.* the equation of Poll [1] which is belonging to the non-linear equation class taken into consideration. The present paper was searching an answer for that purpose.

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