PROPERTIES OF THE ENTROPY AND INFORMATIONAL ENERGY FOR THE SOLUTIONS OF BLACK-SCHOLES EQUATIONS

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This paper studies the properties of the solutions of some stochastic differential equations and Black-Scholes type equations, in the *n*-dimensional case $\mathbf{R}^n$, from the point of view of informational energy and entropy associated and the relations that have been determined among these measurements are also presented. The characteristics of some solutions for Black-Scholes equations are presented as well using the maximum entropy criterion.

1. INTRODUCTION

Let $(\Omega\,,\,\Im\,,\,\mathbf{P})$ be a complete probability space, $n \in \mathbf{N}^*$, $I \subset \mathbf{R}_+$, $I = [0\,,\,T]$, $T > 0$ and a stochastic process $X : I \times \Omega \to \mathbf{R}^n$. On a linear euclidian space $\mathbf{R}^n$ we consider a scalar product $\langle \cdot\,,\,\cdot \rangle$ and an induced norm denoted by $\|\cdot\| = \langle \cdot\,,\,\cdot \rangle^{1/2}$. We introduce the linear space of a positive and symmetrical operator $L_s^+(\mathbf{R}^n)$:

$$L_s^+(\mathbf{R}^n) = \left\{ Q \in L_s^+(\mathbf{R}^n) \,\middle|\, Q = Q^T \text{ și } \langle Qx, x\rangle > 0\,,\, (\forall)\, x \in \mathbf{R}^n \right\}, \qquad (1.1)$$

where by Q^T we have denoted the adjoint matrix Q. On this space we introduce the trace norm $\| Q \|_1 = Tr\{ Q \}$.

We consider a Wiener process $W : I \times \Omega \to \mathbf{R}^n$ with the normal filtration $\{ \Im_t \}_{t \geq 0}$, $\Im_t \subset \Im$, $(\forall)\, t \in I$ and the covariance operator $Q \in L_s^+(\mathbf{R}^n)$, cov $W(t) = t\,Q$. For the stochastic process $X(t)$ we introduce the following notions: $\mathbf{E}\{ X(t) \}$ – the mean of the process;

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$$\mathbf{E}\{X(t)\} = \int_{\Omega} X\mathrm{d}P = \int_{\mathbf{R}^n} x\mathrm{d}\mu(x) = \int_{\mathbf{R}^n} xf(x)\mathrm{d}x \tag{1.2}$$

$f(x, t) = f(x)$ – the density function of the process $(x \in \mathbf{R}^n)$;

$\operatorname{cov} X(t) = \mathbf{E}\{X(t) \cdot X(t)^T\}$ – the covariance operator;

$\mu(X) = \mathbf{P} \circ X^{-1}$ – the law of the process.

Let $X : \Omega \to \mathbf{R}^n$ be a random variable with the density function $f : \mathbf{R}^n \to \mathbf{R}$.

Definition 1.1. It is known to be called the informational energy of a random variable X the expression:

$$\varepsilon(X) = \varepsilon(f) = \int_{\mathbf{R}^n} f^2(x_1, x_2, ..., x_n)\mathrm{d}x_1\mathrm{d}x_2...\mathrm{d}x_n . \tag{1.3}$$

Definition 1.2. The entropy of a random variable X is defined to be:

$$H(X) = \int_{\mathbf{R}^n} f(x_1, x_2, ..., x_n) \cdot \log_2 f(x_1, x_2, ..., x_n) \ \mathrm{d}x_1\mathrm{d}x_2...\mathrm{d}x_n . \tag{1.4}$$

Let the differential stochastic equation be like:

$$\mathrm{d}X(t) = r(t) \cdot X(t)\mathrm{d}t + \sigma(t)X(t)\mathrm{d}W(t), \ X(0) = X^0, \tag{1.5}$$

where: $r : [0, T] \to L\ (\mathbf{R}^n),\ r(t) = \operatorname{diag}\{r_1(t),\ r_2(t),\ ...,\ r_n(t)\}$, is a diagonalizable matrix, $\sigma : [0, T] \times \mathbf{R}^n \to L_s^+(\mathbf{R}^n)$,

$\sigma(t)X(t) = \left(X_i(t)\sigma_{ij}(t)\right)_{1 \le i\ ,\ j \le n}$ and $X^0 \in \mathbf{R}^n$, X^0 is nonrandom.

The infinitesimal generator associate to the process $\{X(t)\}_{t \in I}$ is defined by the relations:

$$D(A) = \left\{ g : I \times \mathbf{R}^n \to \mathbf{R} \;\middle|\; (\exists) \lim_{t \to 0} \frac{\mathbf{E}\{g(X(t))\} - g(X^0)}{t},\ (\forall)\ X^0 \in \mathbf{R}^n \right\},$$

$$Ag(x) = \left\langle r(t)x, \frac{\partial g}{\partial x} \right\rangle + \frac{1}{2}\operatorname{Tr}\left\{ \frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\sigma(x)Q[\sigma(x)]^* \right\} \tag{1.6}$$

with the property:

$C_0^2 \subset D\ (A)$; $C_0^2 = \left\{ g : \mathbf{R}^n \to \mathbf{R} \;\middle|\; g \in C^2, g \text{ has a compact support} \right\}$.

2. BLACK-SCHOLES EQUATIONS AND THEIR SOLUTIONS

On a market without arbitrage we consider an asset whose price $X(t)$, $t \in I = [0\ ,\ T]$, $T > 0$ verifies a differential stochastic equation:

$$\mathrm{d}X(t) = r(t)\cdot X(t)\mathrm{d}t + \sigma(t)\ X(t)\mathrm{d}W(t),\ X(0) = X. \tag{2.1}$$

Then $X(t)$ is an Ito process, whose component *i*, $X_i(t)$, is given by:

$$\begin{cases} \mathrm{d}X_i(t) = r_i(t)X_i(t)\mathrm{d}t + X_i(t)\langle\sigma_i(t),\ \mathrm{d}W(t)\rangle \\ X_i(0) = X_i^0,\ i = \overline{1,n} \end{cases}, \tag{2.2}$$

where $\sigma_i(t)$ represents the row vector *i* of the matrix $\sigma(t)$. The solution of this equation is written like this:

$$X_i(t) = X_i^0 \exp\left(\int_0^t r_i(s)\mathrm{d}s + \int_0^t \langle\sigma_i(s),\ \mathrm{d}W(s)\rangle - \frac{1}{2}\int_0^t \sigma_i(s)Q\sigma_i^*(s)\mathrm{d}s\right),\ i = \overline{1,\ n}\ \cdot \tag{2.3}$$

For obtaining the Black-Scholes equation we write the Kolmogorov "backward" equation perturbed with the operator $K \in L\left(C_0^2\right)$, $K\left(g(x)\right) = \bar{r}(t)\cdot g(x)$:

$$\begin{cases} \dfrac{\partial V}{\partial t} = AV - KV \\ V(0,x) = \varphi(x) \end{cases}, \tag{2.4}$$

where $\varphi \in B_b\left(\mathbf{R}^n\ ,\mathbf{R}\right)$ is the space of the real bounded functions Borel measurable.

The solution (2.5) is given by the formula:

$$V(t,x) = \mathbf{E}\left\{\varphi\left(X(t)\right)\ \cdot\ \exp\left(-\int_0^t \bar{r}(s)\mathrm{d}s\right)\right\},\ \bar{r}(s) = \frac{1}{n}\sum_{i=1}^n r_i(s). \tag{2.5}$$

3. THE INFORMATIONAL ENERGY AND ENTROPY ASSOCIATED TO THE SOLUTIONS OF THE STOCHASTIC DIFFERENTIAL EQUATIONS

Let $t \in I$, fixed, $\mathbf{E}\{X(t)\} = m$, $m \in \mathbf{R}^n$, $\mathrm{cov}\{X(t)\} = Q$, $Q \in L_S^+\left(\mathbf{R}^n\right)$. Because of $Q \in L_S^+\left(\mathbf{R}^n\right)$, it results that the operator Q is diagonalizable. So, it exist an invertible matrix P and a diagonalizable matrix D, $P\ ,D\ \in\ L\left(\mathbf{R}^n\right)$ such that:

$$D = PQP^{-1},\ D = \operatorname{diag}\{\lambda_1 , \lambda_2 , \dots , \lambda_n\},\ \lambda_i > 0 ,\ i = \overline{1 , n}.$$

In this way we reduce the general case to the case of diagonalizable matrix $Q = D$, by changing the canonic basis with the orthonormat basis of the eigenvector of the matrix Q.

Proposition 3.1. *The informational energy for* $X(t)$ *is*:

$$\varepsilon(X) = \frac{1}{\left(2\sqrt{\pi}\right)^n \cdot (\det Q)^{1/2}} \quad \text{or}$$
$$\varepsilon(X) = \frac{1}{\left(2\sqrt{\pi}\right)^n \cdot (\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n)^{1/2}}. \tag{3.1}$$

Proposition 3.2. *The entropy associated to* $X(t)$ *is*:

$$H(X(t)) = \frac{1}{2} \cdot \log_2 (2\pi e)^n \cdot \det Q. \tag{3.2}$$

The proof of these results can be found in [6].

4. THE RELATION BETWEEN ENTROPY AND INFORMATIONAL ENERGY WITH BLACK-SCHOLES SOLUTION

We denote by $\|\cdot\|_2$ the norm of the Hilbert space, $L^2(\mathbf{R}^n , \mathbf{R})$ for the square integrable real functions: $\| f \|_2 = \left(\int_{R^n} f^2 \, \mathrm{d}x \right)^{1/2} < \infty$.

Proposition 4.1. *If* $X(t)$ *is the solution of the equation* (2.1), $V(t , X^0)$ *is the solution of the Black-Scholes equation given by (2.4), then between the Black-Scholes solution and the informational energy of a random variable at the moment* $t \in [0 , T]$ *take places the relation*:

$$\left|V(t, X^0)\right| \le \left\| \varphi(x) \mathrm{e}^{-\int_0^t \bar{r}(s)\mathrm{d}s} \right\|_2 \cdot \left(\varepsilon(X(t))\right)^{1/2}. \tag{4.1}$$

Proof: We use the Schwartz inequalities:

$$\int_{\mathbf{R}^n} f \cdot g \, \mathrm{d}x \le \left(\int_{\mathbf{R}^n} f^2 \, \mathrm{d}x \right)^{1/2} \cdot \left(\int_{\mathbf{R}^n} g^2 \, \mathrm{d}x \right)^{1/2}.$$

Then

$$V\left(t,X^0\right)=\mathbf{E}\left\{\varphi\left(X\left(t,X^0\right)\right)\cdot \mathrm{e}^{-\int_0^t \bar{r}(s)\mathrm{d}s}\right\} = \int_{\mathbf{R}^n} \varphi(x)\cdot \mathrm{e}^{-\int_0^t \bar{r}(s)\mathrm{d}s}\cdot f(x)\mathrm{d}x \le$$

$$\le\left(\int_{\mathbf{R}^n}\left(\varphi(x)\cdot \mathrm{e}^{-\int_0^t \bar{r}(s)\,\mathrm{d}s}\right)^2\mathrm{d}x\right)^{1/2}\cdot\left(\int_{\mathbf{R}^n} f^2(x)\mathrm{d}x\right)^{1/2}=$$

$$=\left\|\varphi\mathrm{e}^{-\int_0^t \bar{r}(s)\,\mathrm{d}s}\right\|_2\cdot\left(\varepsilon\left(X(t)\right)\right)^{1/2},$$

where $\|\cdot\|_2$ is the norm.

Proposition 4.2. *If $X(t)$ is the solution of the differential stochastic equation (2.1), $V\left(t\,,\,X^0\right)$ is the solution of Black-Scholes equation given by (2.4), then between the Black Scholes solution and the entropy of a random variable at the moment $t\in[0\,,\,T]$ take places the relation:*

$$V\left(t,\ X^0\right)\ge \exp\left(-H\left(X(t)\right)\right)\cdot\exp\left(\int_{\mathbf{R}^n} h(x)\mathrm{d}x\right), \tag{4.2}$$

where $h(x)=\ln\left(\dfrac{\varphi(x)}{f\ (x)}\cdot \mathrm{e}^{-\int_0^t \bar{r}(s)\,\mathrm{d}s}\right)\cdot f(x).$

Proof: Let $(X\,,\,\Im\,,\,\mu)$ measurable space such that $\mu(x)=1$ and $f\in L_1^+(X\,,\,\mathbf{R})$. Then $\int_X \ln f(x)\,\mathrm{d}\mu(x)\le \ln\left(\int_X f(x)\,\mathrm{d}\mu(x)\right)$. The proof of this result can be found in [5]. So for $(\Omega,\ \Im\ ,P)$,

$$V\left(t,X^0\right)=\int_\Omega \varphi\left(X\left(t,X^0\right)\right)\cdot \mathrm{e}^{-\int_0^t \bar{r}(s)\mathrm{d}s}\mathrm{d}P,\quad\text{where}\quad \varphi>0$$

we have:

$$\ln\left(V\left(t,X^0\right)\right)\ge\int_\Omega \ln\left[\varphi\left(X\left(t,X^0\right)\right)\mathrm{e}^{-\int_0^t \bar{r}(s)\mathrm{d}s}\right]\cdot\,\mathrm{d}P(\omega)=$$

$$= \int_{\mathbf{R}^n} \ln\left(\varphi(x)\mathrm{e}^{-\int_0^t \bar{r}(s)\mathrm{d}s}\right) f(x)\mathrm{d}x = \int_{\mathbf{R}^n} f(x)\ln f(x)\mathrm{d}x + \int_{\mathbf{R}^n} h(x)\mathrm{d}x$$

$$\Rightarrow V\left(t,\ X^0\right) \geq \mathrm{e}^{-H(X(t))} \cdot \exp\int_{\mathbf{R}^n} h(x)\mathrm{d}x .$$

5. OBSERVATIONS ABOUT MAXIMUM ENTROPY CRITERION

We consider a real process $\{X(t)\}_{t\in I}$ which verifies the differential stochastic equation:

$$\mathrm{d}X(t) = r(t)X(t)\mathrm{d}t + \sigma(t)X(t)\mathrm{d}B(t),\ X(s) = y\ 0<s<t<T, \qquad (5.1)$$

where r , $\sigma : [0, T] \to \mathbf{R}$, B is a real Wiener process with the variance $\lambda > 0$, and $y \in \mathbf{R}$.

Observation 5.1:

a) The solution of the equation (5.1) is:

$$X(T) = y\exp\left(\int_s^T r(t)\mathrm{d}t - \frac{\lambda}{2}\int_s^T \sigma^2(t)\mathrm{d}t + \int_s^T \sigma(t)\mathrm{d}B(t)\right). \qquad (5.2)$$

b) The mean and the covariance of $X(T)$ are:

$$m = y\exp\int_s^T r(t)\mathrm{d}t \ \text{ and } \ g = m^2\left(\exp\left(\lambda\int_s^T \sigma^2(t)\mathrm{d}t\right) - 1\right). \qquad (5.3)$$

Using the maximum entropy criterion we obtain the problem of optimization with respect to the volatility $\sigma(\cdot)$ of the process $X(t)$ on $[0, T]$:

$$\max\int_0^\infty -f(x)\ln f(x)\mathrm{d}x, \qquad (5.4)$$

where $f(x)$ is the function of density associated to the process $X(t)$.

Observation 5.2. An example for determine the minimum of entropy can be the next problem:

For obtaining a system well organized we determine a value as small as possible for the entropy, so for $\delta > 0$ chosen to be the smallest positive value we have the equality $H(X(T)) = \delta$, from where results:

$$\int_s^T \sigma^2(t)\,\mathrm{d}t = \frac{1}{\lambda}\ln\left[\frac{1}{2\pi e}\exp\left(2\delta - 2\int_s^T r(t)\,\mathrm{d}t - 1\right) + 1\right] \qquad (5.5)$$

Observation 5.3 (an application to the maximum entropy criterion). For the maximum entropy criterion we will use the Hamilton Jacobi Bellman equation, in the particular case $r(t) = r\,u(t)$, $\sigma(t) = \sigma\,u(t)$, $r, \sigma \in R$ and $u : [0, T] \to R$ is the control function with the property that is square integrable. Because the $\{X(t)\}_{t \in [0,T]}$ process is a gaussian one we suppose that the density function has the following form $f(x) = \frac{1}{\sqrt{2\pi q}}\exp\left(-\frac{(x-m)^2}{2q}\right)$, where m and q are given by the relation (5.3).

In this way the control problem is :

$$\begin{cases} \mathrm{d}X(t) = ru(t)X(t)\,\mathrm{d}t + \sigma u(t)X(t)\,\mathrm{d}B(t), \;\; X(s) = y \\ \max\limits_{u(\cdot)} \mathbf{E}\{-\ln f(X(T))\}. \end{cases} \qquad (5.6)$$

We determine a Markov control u^* with the property:

$\Phi(s, y) = \sup\{J(s, y, u) : u : [0, T] \to \mathbf{R} \;\; \textit{a control function}\}$,

where $J(s, y, u) = \mathbf{E}\{-\ln f(X(T))\}$.

We associate the operator:

$$L_u = \frac{\partial \Phi}{\partial s} + r\,u\,y\,\frac{\partial \Phi}{\partial y} + \frac{1}{2}\sigma^2 u^2 y^2 \frac{\partial^2 \Phi}{\partial y^2}$$

and the Hamilton Jacobi-Bellman equation

$$\sup_{u(\cdot)}\{(L_u\Phi)(s, y)\} = 0, \;\; \Phi(s, y) = -\ln f(y). \qquad (5.7)$$

So $\Phi(s, y) = \frac{1}{2}\ln 2\pi g + \frac{(y-m)^2}{2q}$ we obtain:

$$L_u\,\Phi(s, y) = 0 \;\; \Rightarrow \;\; u^* = \frac{(m - y)\,r}{y\,\sigma^2} \qquad (5.8)$$

and the partial differential equation for the Φ function is:

$$\frac{\partial \Phi}{\partial s} + \frac{r^2}{\sigma^2}(m - y)\frac{\partial \Phi}{\partial y} + \frac{1}{2}\frac{r^2}{\sigma^2}(m - y)^2 \frac{\partial^2 \Phi}{\partial y^2} = 0. \qquad (5.9)$$

This equation can be reduced to an Euler equation using a substitution like: $\Phi(s,y) = e^{\frac{r^2}{\sigma^2}s}\varphi(y)$.

Therefore the differential equation associates to φ is:

$$\varphi(y) + (m - y)\,\varphi'(y) + \frac{1}{2}(m - y)^2\,\varphi''(y) = 0. \qquad 5.10$$

The Euler substitution associated is $y - m = e^z$, then the equation in the z variable becomes:

$$\varphi''(z) + \varphi'(z) + 2\,\varphi(z) = 0, \qquad 5.11$$

whose solution is $\varphi(z) = c_1\sqrt{z}\,\cos\frac{\sqrt{7}}{2}z + c_2\sqrt{z}\,\sin\frac{\sqrt{7}}{2}z$, where c_1, c_2 real constants are.

Conclusion. For the control function u^* we obtain a function $\Phi(\cdot,\cdot)$ for which the entropy associated to the solution (5.2) is maxim.

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