THIN FILMS FLOW DRIVEN BY GRAVITY AND A SURFACE TENSION GRADIENT

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The flow of a thin layer on a horizontal plate in the lubrication approximation is considered. The flow is driven simultaneously by gravity and some gradients of surface tension. These gradients imply a non-zero tangential stress boundary condition (Marangoni effect).

1. INTRODUCTION

Surface tension is a very important mechanism for small scale flows such as paint films, the motion of a contact lens on the eyeball or various wetting or coating flows.

For example, to model paint films or foams, it may be important to take surface tension gradients into account, giving rise to a different extra term in evolution equations. Such gradients give rise to the so-called Marangoni flows, unexpectedly, and they have been found to dominate many zero-gravity fluid dynamics experiments carried out in space, in particular those concerned with crystal growth. The ability of the surface tension to vary spatially is also a crucial ingredient for the fluid to be able to form a foam. It is also believed to be the mechanism responsible for the ripples that are often observed on solvent-based paint films.

2. PROBLEM FORMULATION

Consider the flow of a thin layer of an incompressible Newtonian fluid with constant density ρ and constant viscosity μ down a horizontal plate.

The flow is driven simultaneously by gravity and a surface tension gradient $\Sigma = \partial\sigma / \partial x$.

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We choose Cartesian axes *Oxyz,* with the x axis in the direction of flow and the z axis transverse to the direction of flow (Fig. 1) and so the velocity is given by $\boldsymbol{u} = u(x,z,t)\boldsymbol{i} + w(x,z,t)\boldsymbol{k}$.

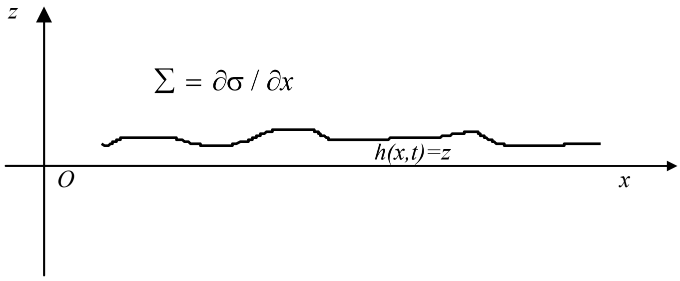

Fig. 1 – The geometry of problem.

In the thin film approximation the Navier-Stokes equations read

$$0 = (-1/\rho)\partial p / \partial x + \nu(\partial^2 u / \partial z^2) + f_1, \tag{1}$$

$$0 = (-1/\rho)\partial p / \partial z + f_2, \tag{2}$$

where $f_1 = 0$, $f_2 = -\boldsymbol{g}$. Here $z = h(x, t)$ is the unknown equation of the free-surface, p is the pressure in the fluid, $\boldsymbol{g}$ is the gravitational acceleration, $\mu = \rho\nu$ is the coefficient of the dynamic viscosity, $\boldsymbol{u}$ and p depend on $\boldsymbol{x} = (x, y, z, t)$, where t is the time and z points upwards.

Moreover, the non-slip condition must be satisfied

$$\boldsymbol{u} = 0\text{, at } z = 0. \tag{3}$$

The normal stress and tangential stress (shear) at the free surface $z = h(x,t)$ imply the following two boundary conditions

$$\boldsymbol{p} = p_0\text{, at } z = h(x, t), \tag{4}$$

where p_0 is the atmospheric pressure, and

$$\mu(\partial u / \partial z) = \Sigma\text{, at } z = h(x, t). \tag{5}$$

The condition (5) is the Levich-Aris boundary condition.

In addition to the dynamic boundary conditions, on the free surface we impose the kinematic boundary condition

$$w = \partial h / \partial t + u\partial h / \partial x,\ \text{at } z = h(x, t). \tag{6}$$

Integrating (1) under the stress-free boundary condition we obtain

$$u = (\rho g / 2\mu)\partial\ h / \partial\ x \cdot z^2 + (\Sigma / \mu - \rho g / 2\nu \cdot \partial\ h / \partial\ x \cdot h)z, \tag{7}$$

$$w = (-\rho g / 6\mu)\partial^2 h / \partial\ x^2 \cdot z^3 + (\rho g / 2\mu)z^2[\partial^2 h / \partial\ x^2 \cdot h + (\partial\ h / \partial\ x)^2]. \tag{8}$$

Finally, using the continuity equation in the thin-film approximation and the kinematic boundary condition leads the evolution equation for $z = h(x, t)$

$$\rho g / 3\mu \cdot \partial\ (h^3 \cdot \partial\ h / \partial\ x) / \partial\ x = h_t + \Sigma / \mu \cdot h \cdot \partial\ h / \partial\ x\,. \tag{9}$$

At this point, it is convenient to introduce apropriate nondimensional variables defined by

$$h^* = h / h_0,\ x^* = x / L,\ t^* = tU / h_0,\ \Sigma^* = (L\,\Sigma) / \sigma_0\,, \tag{10}$$

where the velocity U and the length scale L are characteristic quantities of the problem. Assume that $\delta = h_0 / L << 1$, where h_0 is the characteristic length for the film thickness.

Then convert the equation (9) into nondimensional form in terms of the nondimensional variables h^*, x^*, t^*. For the sake of simpler notation we drop the "star". In this way the equation (9) becomes

$$\mathrm{Bo} \cdot \partial\ (h^3 \cdot \partial\ h / \partial\ x) / \partial\ x = h_t + \mathrm{Ca}\,\Sigma \cdot h \cdot \partial\ h / \partial\ x\,, \tag{11}$$

where $\mathrm{Bo} = \delta^4(\rho g L^2)/(3\mu U)$ is the Bond number and $\mathrm{Ca} = (L\sigma_0 / \mu U)\delta^2$ is the cappilarity number.

For the nonlinear equation (11), we can find the wave solution

$$h = Y(x - ct)\,, \tag{12}$$

which Y satisfies the ordinary differential equation

$$\mathrm{Bo}(Y^3 Y'' + 3 \cdot Y^2 Y'^2) = -cY' + \mathrm{Ca}\,\Sigma\, Y Y'\,. \tag{13}$$

Here the prime denotes the differentiation with respect to $T = x - ct$.

Let us write (13) as the system of two ordinary differential equations

$$Y' = X, \tag{14}$$

$$X' = -c / \mathrm{Bo} \cdot X / Y^3 + \mathrm{Ca}\,\Sigma / \mathrm{Bo} \cdot X / Y^2 - 3X^2 / Y. \tag{15}$$

It generates a dynamical system all equilibria of which are nonhyperbolic. Eliminating T between the two equations (14–15) we obtain the ordinary differential equation

$$dX/dY = -c/\mathrm{Bo}\cdot 1/Y^3 + \mathrm{Ca}\Sigma/\mathrm{Bo}\cdot 1/Y^2 - 3\cdot X/Y\,, \tag{16}$$

the solution of which reads

$$X(Y) = c^2/(2\mathrm{BoCa}\Sigma)\cdot 1/Y^3 - c/(\mathrm{Bo}Y^2) + \mathrm{Ca}\Sigma/(2\mathrm{Bo}Y)\,. \tag{17}$$

and its Taylor expanding about $c/\mathrm{Ca}\Sigma$ is

$$X(Y) = (\mathrm{Ca})^4\Sigma^4/(2\mathrm{Bo}c^3)(Y - c/\mathrm{Ca}\Sigma)^2\,. \tag{18}$$

The phase portrait is shown in Fig. 2.

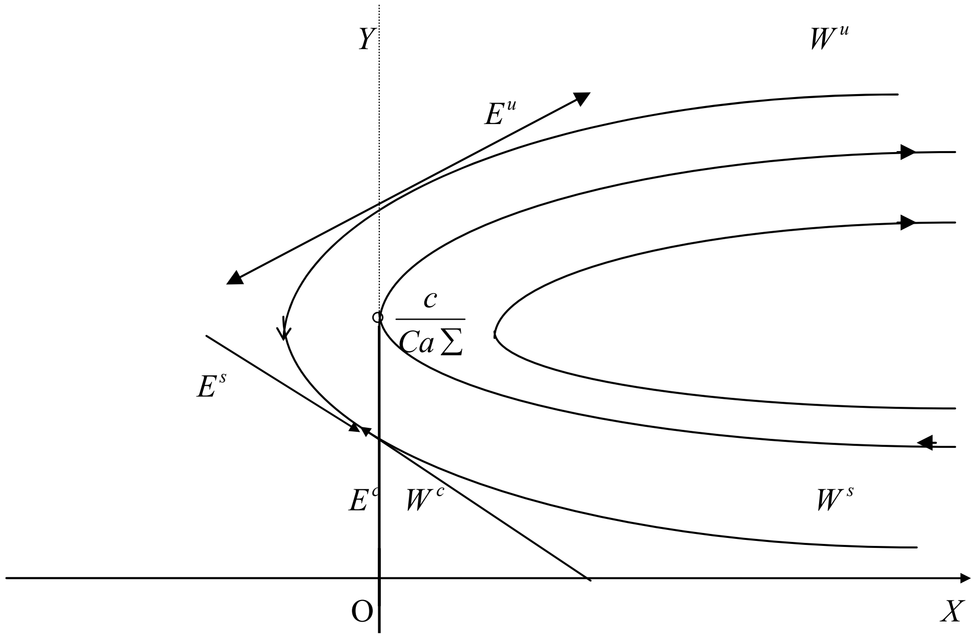

Fig. 2 – Phase portrait.

Here W^s, W^u, W^c are the invariant stable, unstable and center manifolds and E^s, E^u, E^c are the corresponding invariant subspaces for the linearized system.

3. THE SPECIAL CASE Ca = 1

In the case Ca = 1, the system (14–15) reduces to

$$Y' = X\,, \tag{19}$$

$$X^{'} = -c/\mathrm{Bo}\cdot X/Y^{3} + \Sigma/\mathrm{Bo}\cdot X/Y^{2} - 3X^{2}/Y. \tag{20}$$

and, similarly as in Section 2, it implies

$$\mathrm{d}X/\mathrm{d}Y = -c/\mathrm{Bo}\cdot 1/Y^{3} + \Sigma/\mathrm{Bo}\cdot 1/Y^{2} - 3\cdot X/Y\,, \tag{21}$$

The solution of this ordinary differential equation reads

$$X(Y) = (c^{2}/2\mathrm{Bo}\Sigma)\cdot 1/Y^{3} - c/\mathrm{Bo}\cdot 1/Y^{2} + \Sigma/(2\mathrm{Bo}Y)\,, \tag{22}$$

such that its Taylor expanding about c/Σ is

$$X(Y) = (\Sigma^{4}/2\mathrm{Bo}c^{3})\cdot(Y - c/\Sigma)^{2}\,. \tag{21}$$

And the phase portrait is shown in Fig. 3

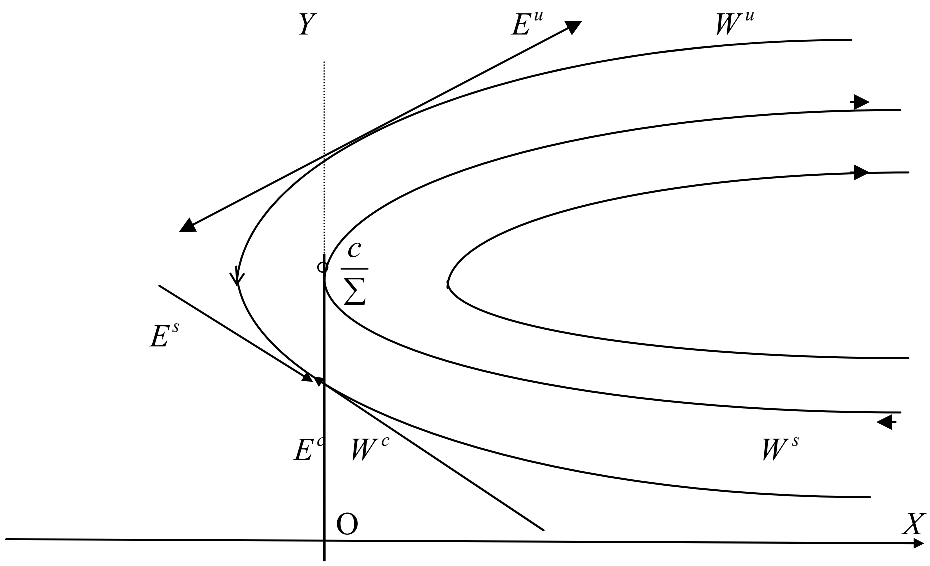

Fig. 3 – Phase portrait for Ca = 1.

Let us remark that the flows with thickness smaller then $Y = c/\Sigma$ are stable.

4. CONCLUSION

In lubrication approximation we have considered the flow of a thin layer on a horizontal plate. The flow is driven simultaneously by gravity and some gradients of surface tension. These gradients imply a non-zero tangential stress boundary condition (Marangoni effect).

This study shows us that there are stable stationary solutions in the flows of the thin layer.

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