

# A GENERALIZATION OF THE GIRSANOV THEOREMS ON THE HILBERT SPACE

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This paper gives a generalization of the Girsanov theorems on the separable Hilbert space. And we also give a characterization on the Hilbert space of the Wiener processes, using the quadratic variance process.

## 1. INTRODUCTION

Let  $(\Omega, \mathfrak{F}, \mathbf{P})$  be a complete probability space,  $H$  a separable Hilbert space with the norm induced by the scalar product  $\langle \cdot, \cdot \rangle$ ,  $\|x\|^2 = \langle x, x \rangle$ ,  $(\forall)x \in H$  and  $W : I \times \Omega \rightarrow H$ ,  $I = [0, T] \subset \mathbf{R}_+$  a Wiener process with  $Q \in L_1(H)$  as the covariance operator and with normal filtration  $\{\mathfrak{F}_t\}_{t \in I}$ ;  $W(0) = 0$ ,  $W$  has continuous paths, independent increases and  $\mathbf{E}\{W(t)\} = 0$ ,  $\text{cov } W(t) = tQ$ . On the Hilbert space  $H$  we consider an ortonormat basis  $\{e_n\}_{n=1}^\infty$ . In this case  $W(t)$  has the next form:

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \text{ where } \beta_n(t) = \frac{1}{\sqrt{\lambda_n}} \langle W(t), e_n \rangle, \quad n = 1, 2, \dots \text{ are}$$

independent brownian real processes on  $(\Omega, \mathfrak{F}, \mathbf{P})$  and  $\lambda_n > 0$  with the property  $Qe_n = \lambda_n e_n$ ,  $n = 1, 2, \dots$  [5]. It is considered a continuous stochastic process  $X(t)$ ,  $X : I \times \Omega \rightarrow H$ ,  $X_t(\omega)$  continuous on  $I$ ,  $(\forall)\omega \in \Omega$  and a partition  $\Delta$  on  $I$ ,  $\Delta : t_0 = 0 < t_1 < t_2 < \dots < t_k = T$ ,  $\delta = \max_{i=0, k} (t_{i+1} - t_i)$ . We de fine the

$$\text{stochastic process } \langle X \rangle_t^\Delta \text{ using the formula: } \langle X \rangle_t^\Delta = \sum_{i=0}^{k-1} \|X(t \wedge t_{i+1}) - X(t \wedge t_i)\|^2,$$

where  $t \wedge s = \min \{t, s\}$ .

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*Definition 1.1.*  $t$  is called a *quadratic variance process* of the process  $X(t)$ , the application  $\langle X \rangle_t$ , where

$$\langle X \rangle_t = \lim_{\delta \rightarrow 0} \langle X \rangle_t^\Delta. \quad (1.1)$$

*Observation 1.1.*

(i) If  $\{X(t)\}$  is with a finite variance then  $\langle X \rangle_t = 0$ .

(ii) The process  $\langle X \rangle_t$  is continuous.

(iii) Let  $\{M_j\}_j$  be a sequence of continuous martingal process, square integrable. Then  $M_j$  has a mean square convergence to a continuous martingal  $M$ , if and only if  $\langle M_j - M \rangle_T$  has a mean convergence to zero;

(iv)  $\langle W \rangle_t = tQ$ , for the Wiener process.

**Theorem 1.1** (The Girsanov Theorems) [4].

(I) Let  $X(t) \in \mathbf{R}^n$  be an Ito process:  $dX(t) = f(t, \omega)dt + dB(t)$ ,  $t \leq T$ ,  $X(0) = 0$  where  $T < \infty$ ,  $B(t)$  is a  $n$ -dimensional Brownian motion, and

$$M(t) = \exp\left(-\int_0^t f(s, \omega)dB(s) - \frac{1}{2}\int_0^t f(s, \omega)f(s, \omega)^* ds\right), t \leq T.$$

We suppose that  $f(s, \omega)$  verifies the Novikov condition

$$\mathbf{E}\left\{\exp\left(\frac{1}{2}\int_0^T f(s, \omega)f(s, \omega)^* ds\right)\right\} < \infty. \quad (1.2)$$

We define the measure  $\tilde{\mathbf{P}}$  on  $(\Omega, \mathfrak{F}_T)$  by  $d\tilde{\mathbf{P}}(\omega) = M(T)d\mathbf{P}(\omega)$ . Then the  $X(t)$  process is a Brownian motion with respect to the measure  $\tilde{\mathbf{P}}$ , for  $t < T$ .

(II) Let  $X(t) \in \mathbf{R}^n$  be an Ito process:

$$dX(t) = b(t, \omega)dt + \sigma(t, \omega)dB(t), t \leq T$$

$b(t, \omega) \in \mathbf{R}^n$ ,  $\sigma(t, \omega) \in L(\mathbf{R}^n, \mathbf{R}^n)$ . We suppose that the integrable processes  $a(t, \omega)$  and  $f(t, \omega)$  exist such that:  $\sigma(t, \omega)f(t, \omega) = b(t, \omega) - a(t, \omega)$  and  $f(t, \omega)$  verifies Novikov's condition. Then  $\tilde{B}(t) := \int_0^t f(t, \omega)ds + B(t)$ ,  $t \leq T$

is a Brownian motion with respect to the measure  $\tilde{\mathbf{P}}$ , and the process  $X(t)$  verifies the equation:

$$dX(t) = a(t, \omega)X(t)dt + \sigma(t, \omega)d\tilde{B}(t).$$

(III) Let the processes  $X(t), Y(t) \in \mathbf{R}^n$  such that  $X(0) = Y(0) = x$  and

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \quad t \leq T$$

$$dY(t) = [\gamma(t, \omega) + b(Y(t))]dt + \sigma(Y(t))dB(t), \quad t \leq T,$$

where  $b \in L(\mathbf{R}^n)$ ,  $\sigma \in L(\mathbf{R}^n, L(\mathbf{R}^n))$  and  $\gamma(t, \omega)$  is an integrable process.

If it exists an integrable  $f(t, \omega)$  witch verifies the Novikov condition and  $\sigma(Y(t))f(t, \omega) = \gamma(t, \omega)$ , then

$$dY(t) = b(Y(t))dt + \sigma(Y(t))d\tilde{B}(t).$$

More than that the law of the process  $Y(t)$  is the same as the law of the process  $X(t)$ ,  $t \leq T$ . A generalization of this result it can be found in [5].

**Proposition 1.1.** Let  $f(t, \omega)$  be a predictable, square integrable process, such that:

$$\mathbf{E} \left\{ \exp \left( \frac{1}{2} \int_0^T f(t, \omega) Q f(t, \omega)^* ds \right) \right\} < \infty \quad (1.3)$$

Then the stochastic process  $M(t)$ , defined by the relation:

$$M(t) = \exp \left\{ \int_0^t f(s, \omega) dW(s) - \frac{1}{2} \int_0^t f(s, \omega) Q f^*(s, \omega) ds \right\} \quad (1.4)$$

is a martingal with the mean  $\mathbf{E} \{M(t)\} = 1$ .

**Proof:** The process  $\{M(t)\}$  verifies the differential stochastic equation:

$$dM(t) = -f(t, \omega) M(t) dW(t), \quad M(0) = 1.$$

Let  $M(t) = \sum_{n=1}^{\infty} M_n(t) e_n$ . Then

$$dM_n(t) = -\langle f(t, \omega) M(t), e_n \rangle \sqrt{\lambda_n} dB_n(t)$$

where  $M_n(t)$  are real stochastic processes,  $n \in \mathbf{N}^*$ . Because of these the problem is reduced to the real case, which is the equation:

$$dM(t) = -\sqrt{\lambda} f(t, \omega) M(t) dB(t) \Leftrightarrow \frac{dM(t)}{M(t)} = -f(t, \omega) \sqrt{\lambda} dB(t),$$

where  $M(t)$  and  $f(t, \omega)$  are real processes, and  $B(t)$  is a real Brownian motion.

From Ito's lemma which is applied to the function  $g : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$ ,  $g(t, x) = \ln x$  it results that

$$\begin{aligned} dg(t, M(t)) &= \frac{dM(t)}{M(t)} - \frac{1}{2} f^2(t, \omega) \lambda dt \Rightarrow \\ M(t) &= \exp \left( -\int_0^t f(s, \omega) dW(s) - \frac{1}{2} \int_0^t \lambda f^2(s, \omega) ds \right). \end{aligned}$$

So:

$$\begin{aligned} \langle M(t), e_n \rangle = M_n(t) &= \exp \left\{ -\int_0^t \langle f(s, \omega), e_n \rangle \sqrt{\lambda_n} dB_n(s) - \right. \\ &\left. - \frac{1}{2} \int_0^t \lambda_n \langle f(s, \omega), e_n \rangle \langle f(s, \omega), e_n \rangle ds \right\}. \end{aligned}$$

Because  $M(t) = \sum_{n=1}^{\infty} \langle M(t), e_n \rangle e_n$  we obtain:

$$M(t) = \exp \left\{ -\int_0^t f(s, \omega) dW(s) - \frac{1}{2} \int_0^t f(s, \omega) Q f(s, \omega)^* ds \right\}.$$

Forward we will proof that  $\mathbf{E} \{M(t)\} = 1$ . The integral equation associated to the  $M(t)$  process is  $M(t) = M(0) - \int_0^t f(s, \omega) M(s) dW(s)$ . We take the expectation in this relation and we gain  $\mathbf{E} \{M(t)\} = 1$ . Let  $f$  be a predictable square integrable process. Then it exist a sequence of elementary processes  $\{f_i\}$  such that  $\mathbf{E} \left\{ \int_0^T |f - f_i|^2 ds \right\} \rightarrow 0$ . The case of elementary processes is reduced to the case of constant processes using the relation:

$$f = \sum_{j=0}^{k-1} a_k \chi(t_j, t_{j+1}), a_k \in \mathbf{R}.$$

It result that

$$\begin{aligned} M(t) &= \exp \left\{ -\int_0^t f(s, \omega) dW(s) - \frac{1}{2} \int_0^t f^2(s, \omega) \lambda ds \right\} = \\ &= \exp \left\{ \sum_{j=0}^{k-1} \left[ -a_j W(t_{j+1}) - \frac{1}{2} a_j^2 \lambda t_{j+1} \right] + \left[ -a_j W(t_j) - \frac{1}{2} a_j^2 \lambda t_j \right] \right\} = \\ &= \prod_{j=0}^{k-1} \exp \left( -a_j W(t_{j+1}) - \frac{1}{2} a_j^2 \lambda t_{j+1} \right) \cdot \exp \left( -a_j W(t_j) - \frac{1}{2} a_j^2 \lambda t_j \right). \end{aligned}$$

In this way the general case is reduced to the study of processes which have the following form :  $M(t) = \exp \left( -a W(t) - \frac{1}{2} a^2 \lambda t \right)$ . We proof that  $M(t)$  is a

$\mathfrak{F}_t$  - martingal by reducing them to the constant real processes. We consider constant processes like  $f(s, \omega) = a$ . Then  $\mathbf{E} \left\{ \exp \left[ -a W(t) - \frac{1}{2} a^2 \lambda t \right] \middle| \mathfrak{F}_s \right\} =$

$$= \exp \left\{ -a W(s) - \frac{1}{2} a^2 \lambda s \right\} \mathbf{E} \left\{ \exp \left[ -a(W(t) - W(s)) \right] \middle| \mathfrak{F}_s \right\}$$

because  $W(s)$  is  $\mathfrak{F}_s$  - measurable and

$$\mathbf{E} \left\{ \exp \left[ -a(W(t) - W(s)) \right] \middle| \mathfrak{F}_s \right\} = \mathbf{E} \left\{ \exp \left[ -a(W(t) - W(s)) \right] \right\} = \exp \left( \frac{1}{2} a^2 \lambda (t - s) \right)$$

because  $W(t) - W(s)$  is  $\mathfrak{F}_s$  - independent. It results:

$$\mathbf{E} \left\{ \exp \left[ -a W(t) - \frac{1}{2} a^2 \lambda t \right] \middle| \mathfrak{F}_s \right\} = \exp \left[ -a W(s) - \frac{1}{2} a^2 \lambda s \right].$$

Because  $M(t) = \sum_{n=1}^{\infty} M_n(t) e_n$ , where  $M_n(t)$  are real square integrable processes, it results that  $M_n(t)$  are real martingales and

$$\mathbf{E} \left\{ M(t) \middle| \mathfrak{F}_s \right\} = \sum_{n=1}^{\infty} \mathbf{E} \left\{ M_n(t) \middle| \mathfrak{F}_s \right\} e_n = \sum_{n=1}^{\infty} \mathbf{E} \left\{ M_n(s) \right\} e_n = \mathbf{E} \left\{ M(s) \right\},$$

for every predictable processes  $f(s, \omega)$   $H$ - valued.

**Theorem 1.2** (In the finite case we have this result in [2]). *Let  $\left\{ X \left( \int t \right) \right\}_{t \in I}$  be a continuous process,  $\mathfrak{F}_t$  - adapted with the mean 0 (zero) and*

the covariance,  $\text{cov}\{X(s), X(t)\} = Q(s, t)$ . Then the following sentences are equivalent:

- (i)  $X(t)$  is a  $\mathfrak{F}_t$  Wiener process;
- (ii)  $X_n(t)$  is martingal square integrable,  $n = 1, 2, \dots$  and  $\langle X \rangle_t = tQ, Q \in L^1(H)$ ;
- (iii)  $\exp\left\{-\int_0^t f(s, \omega) dX(s) - \frac{1}{2} \int_0^t f(s, \omega) Qf^*(s, \omega) ds\right\}$  is martingal, with a unity mean, for every  $f$ -square integrable predictable process;
- (iv)  $\{X(t)\}_{t \in I}$  is a gaussian process such that  $X(t) - X(s)$  is  $\mathfrak{F}_s$  - independent for every  $0 < s < t$ .

**Proof:** The operator  $Q(s, t)$  is defined by the relation

$$Q(s, t) = \mathbf{E}\{X(t) \otimes X(s)\},$$

and in the case of a Wiener process the relation  $Q(s, t) = Q(t-s)$  exists, where  $0 < s < t$  and  $Q \in L_1(H)$ .

(i)  $\Rightarrow$  (ii) let  $0 < s < t$ .  $X(t) - X(s)$  are  $\mathfrak{F}_s$  - independent and

$\mathbf{E}\{X(t) - X(s) \mid \mathfrak{F}(s)\} = 0 \Rightarrow \mathbf{E}\{X_n(t) - X_n(s)\} = 0 \Rightarrow \mathbf{E}\{X_n(t) \mid \mathfrak{F}(s)\} = X_n(s)$ ,  
 $\langle X \rangle_t = tQ$  (we used the observation 1.1);

(i)  $\Rightarrow$  (iii) using the proposition 1.1;

(i)  $\Leftrightarrow$  (iv) it is obtained from the definition of a Wiener process;

(ii)  $\Rightarrow$  (i) let  $\langle X \rangle_t = tQ$  and  $X_n(t)$  be martingal square integrable.

Then  $\langle X_n \rangle_t = \lambda_n t$ ,  $t \in I$ . Using Levy's theorem [5], on  $\mathbf{R}$ , we obtain that  $X_n$  is a real Wiener process which has  $\mathfrak{F}_s$  - independent increases. It results that:

$$X(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} B_n(t) e_n, t \in [0, T],$$

where  $B_n(t) = \frac{1}{\sqrt{\lambda_n}} X_n(t)$  are independent standard Wiener processes.

(iii)  $\Rightarrow$  (iv) we observe that  $\mathbf{E}\{\exp[aW(t)]\} = \mathbf{E}\{\exp[-aW(t)]\}$ . And the following result takes place:

[1] Let  $\mu$  be the associate measure for  $X(t)$ . Then the  $\mu$  measure is gaussian if and only if his characteristic function approves:

$$\mathbf{E} \left\{ \exp [i \langle a, W(t) \rangle] \right\} = \exp \left\{ i \langle m, a \rangle - \frac{1}{2} \langle Qa, a \rangle \right\},$$

where  $m$  is the mean, and  $Q$  – the covariance.

Let  $M(t) = \exp \left\{ -\langle a, X(t) \rangle - \frac{1}{2} a Q a^* t \right\}$  be  $\mathfrak{F}_t$  – martingal with  $\mathbf{E}\{M(t)\} = 1$ . It results that  $\mathbf{E} \left\{ \exp -\langle a, X(t) \rangle \right\} = \exp \left\{ -\frac{1}{2} a Q a^* t \right\}$  and

$$\begin{aligned} \mathbf{E} \left\{ \exp i \langle a, X(t) \rangle \right\} &= \mathbf{E} \left\{ \exp \sum_{n=1}^{\infty} i a_n X_n(t) e_n \right\} = \prod_{n=1}^{\infty} \mathbf{E} \left\{ \exp i a_n X_n(t) e_n \right\} = \\ &= \prod_{n=1}^{\infty} \exp \left\{ -\frac{1}{2} \lambda_n a_n^2 t e_n \right\} = \exp \left( \sum_{n=1}^{\infty} -\frac{1}{2} \lambda_n a_n^2 e_n t \right) = \exp \left\{ -\frac{1}{2} a Q a^* (t) \right\} = \\ &= \exp \left\{ -\frac{1}{2} \langle Qa, a \rangle t \right\}. \end{aligned}$$

In this way we have proved that  $\{X(t)\}$  is a gaussian process with the zero mean and the covariance  $Q$ . Knowing that  $M(t)$  is martingal, with the unite mean we see that  $\exp \{-\langle a, X(t) \rangle\}$  is martingal,  $(\forall) a \in H$  and

$$\mathbf{E} \left\{ \exp [-\langle a, X(t) - X(s) \rangle] \middle| \mathfrak{F}_s \right\} = \exp \left\{ -\frac{1}{2} a Q a^* (t - s) \right\}, \quad (\forall) a \in H.$$

So  $X(t) - X(s)$  is  $\mathfrak{F}_s$  – independent and in this way we have finished the proof.

## 2. THE GIRSANOV THEOREMS ON THE SEPARABLE HILBERT SPACE

**Theorem 2.1** (The first Girsanov Theorem). *Let  $X(t)$  be a stochastic process  $X : [0, T] \times \Omega \rightarrow H$  with the following form:*

$$\begin{cases} dX(t) = Qf(t, \omega) dt + dW(t) \\ X(0) = 0, t \in [0, T], \end{cases} \quad (2.1)$$

where  $W$  is  $Q$ -Wiener process, and  $f(\cdot, \cdot)$  is a square integrable predictable process, and the process

$$M(t) = \exp \left( -\int_0^t f(s, \omega) dW(s) - \frac{1}{2} \int_0^t f(s, \omega) Q f(s, \omega)^* ds \right)$$

with the property:  $\mathbf{E} \left\{ \exp \frac{1}{2} \int_0^T f(s, \omega) Q f(s, \omega)^* ds \right\} < \infty$ . We define the measure  $\tilde{\mathbf{P}}$  on  $(\Omega, \mathfrak{F}_T)$  by:  $d\tilde{\mathbf{P}}(\omega) = M(T) d\mathbf{P}(\omega)$ .

Then  $\{X(t)\}$  is a Wiener process related to the measure  $\tilde{\mathbf{P}}$ .

**Proof:** We write the integral equation, which is associated to the differential equation:

$$X(t) = Q \int_0^t f(s, \omega) ds + W(t).$$

In this way, related to the measure  $\mathbf{P}$  the following relations are proven to be true:

$$\mathbf{E} \{X(t)\} = Q \int_0^t \mathbf{E} \{f(s, \omega)\} ds;$$

$$M(T) d\mathbf{P} = M(t) d\mathbf{P} \text{ pe } \mathfrak{F}_t, t \leq T; X(0) = 0;$$

if  $f(s, \omega) = f(s)$  is a non-random function, then  $\text{cov } X(t) = Qt$ .

From Proposition 1.1 we know that  $M(T)$  is  $\mathfrak{F}_t$  - martingal and  $\mathbf{E} \{M(t)\} = 1$ . It results that  $\tilde{\mathbf{P}}(\Omega) = 1$ , so  $\tilde{\mathbf{P}}$  is a probability measure.

We proof that  $X(s)$  is a Wiener process related to the  $\tilde{\mathbf{P}}$  measure used in the first theorem, so:  $X(t)$  is a Wiener process related to the  $\tilde{\mathbf{P}}$  measure  $\Leftrightarrow$

$$\Leftrightarrow N(t) = \exp \left\{ - \int_0^t g(s, \omega) dX(s) - \frac{1}{2} \int_0^t g(s, \omega) Q g(s, \omega)^* ds \right\},$$

is  $\mathfrak{F}_t$  - martingal related to the  $\tilde{\mathbf{P}}$  measure (where  $g$  - a predictable square integrable process)  $\Leftrightarrow N(t)M(t)$  - a martingal related to the  $\mathbf{P}$  measure.

We used the relations:  $\tilde{\mathbf{E}} \{X(t)\} = 0$  and  $\text{c}\tilde{\text{ov}} \{X(t)\} = tQ$  (related to the  $\tilde{\mathbf{P}}$  measure);  $\tilde{\mathbf{E}} \{N(t)\} = \mathbf{E} \{N(t)M(t)\}$ .

We solve  $N(t)M(t)$ :

$$\begin{aligned} N(t)M(t) &= \exp \left\{ - \int_0^t \langle g(s, \omega), Q f(s, \omega) \rangle ds - \int_0^t g(s, \omega) dW(s) - \right. \\ &\quad \left. - \frac{1}{2} \int_0^t \langle Q g(s, \omega), g(s, \omega) \rangle ds - \int_0^t f(s, \omega) dW(s) - \frac{1}{2} \int_0^t \langle Q f(s, \omega), f(s, \omega) \rangle ds \right\} = \\ &= \exp \left\{ - \int_0^t h(s, \omega) dW(s) - \frac{1}{2} \int_0^t h(s, \omega) Q h(s, \omega)^* ds \right\}, \end{aligned}$$



where  $h(s, \omega) = (s, \omega) + g(s, \omega)$  and we also used the relation:

$$\frac{1}{2} \langle Q(x+y), x+y \rangle = \frac{1}{2} \langle Qx, x \rangle + \langle Qx, y \rangle + \frac{1}{2} \langle Qy, y \rangle, (\forall) x, y \in H$$

(because  $Q \in L_1(H)$ ,  $Q$  is positive and selfadjoint).

Using Theorem 1.2 for the Wiener process  $W(t)$ , the predictable process  $h(s, \omega)$  and the  $\mathbf{P}$  measure we obtain that  $N(t)M(t)$  is a  $\mathfrak{F}_t$ -martingal.

**Theorem 2.2** (The second Girsanov Theorem). *Let  $X(t)$  be an Ito process,  $X : [0, T] \times \Omega \rightarrow H$  which has the form:*

$$\begin{cases} dX(t) = b(t, \omega)dt + \sigma(t, \omega)dW(t), & 0 \leq t \leq T \\ X(0) = x, & x \in H, \end{cases} \quad (2.2)$$

where  $b(t, \omega)$  is a  $\mathfrak{F}_t$  adapted process with the property

$$\mathbf{P} \left\{ \int_0^t \|b(s, \omega)\| ds < \infty, (\forall) t \in I \right\} = 1 \text{ and } \sigma(s, \omega) \text{ is a predictable process,}$$

$\mathfrak{F}_t$ -measurable and

$$\mathbf{P} \left\{ \int_0^t \|\sigma(s, \omega)\|^2 ds < \infty, (\forall) t \in I \right\} = 1.$$

We suppose that the predictable processes  $\tilde{W}(t) = Q \int_0^t f(s, \omega) ds + W(t)$  exist, and they are  $\mathfrak{F}_t$ -measurable and square integrable on  $I$  such that:

$$b(t, \omega) - a(t, \omega) = \sigma(t, \omega) Q f(t, \omega), (\forall) t \in I$$

and the process  $f(t, \omega)$  verifies the Novikov condition:

$$\mathbf{E} \left\{ \exp \left( \frac{1}{2} \int_0^T f(t, \omega) Q f(t, \omega)^* dt \right) \right\} < \infty. \quad (2.3)$$

If we consider  $M(t)$ ,  $\tilde{\mathbf{P}}$ ,  $\tilde{W}(t)$  as in the Theorem 2.1

$$\tilde{W}(t) = Q \int_0^t f(s, \omega) ds + W(t),$$

then the differentiable equation associated to the  $X(t)$  process related with the  $\tilde{\mathbf{P}}$  measure and the Wiener process  $\tilde{W}(t)$  is:

$$\begin{cases} dX(t) = a(t, \omega)dt + \sigma(t, \omega)d\hat{W}(t), t \in [0, T] \\ X(0) = 0. \end{cases} \quad (2.4)$$

**Proof:** From the relation:  $dW(t) = d\tilde{W}(t) - Qf(t, \omega)dt$  and by putting it in (2.2) we get:  $dX(t) = a(t, \omega)dt + \sigma(t, \omega)d\tilde{W}(t)$ .

*Observation 2.1.* A consequence of this result is that the  $\sigma(t, \omega)$  is invertible and  $f(t, \omega)$  is unique  $f(t, \omega) = Q^{-1}\sigma^{-1}(t, \omega) [b(t, \omega) - a(t, \omega)]$ .

**Theorem 2.3** (The third Girsanov Theorem). *We consider the processes  $X(t), Y(t), X : I \times \Omega \rightarrow H$  and  $Y : I \times \Omega \rightarrow H$  which have the form:*

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dW(t), t \in I \\ X(0) = x, x \in H \end{cases} \quad (2.5)$$

$$\begin{cases} dY(t) = [c(t, \omega) + b(Y(t))]dt + \sigma(Y(t))dW(t), t \in I \\ Y(0) = x, \end{cases} \quad (2.6)$$

where the operators  $b \in L(H)$ ,  $\sigma \in L(H, L(H))$  are verifying the Lipschitz conditions:  $b, \sigma$  are measurable and

$$\|b(x)\| + \|\sigma(x)\| \leq k(1 + \|x\|), x \in H, k > 0,$$

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \leq k\|x - y\|, x, y \in H.$$

Further more  $c(t, \omega)$  is a predictable square integrable process,  $\mathfrak{F}_t$  - measurable. We suppose that exist one process,  $f(t, \omega)$ , predictable, square integrable,  $\mathfrak{F}_t$  - measurable which verify the Novikov relation (2.3) such that:  $c(t, \omega) = \sigma(Y(t))Qf(t, \omega)$ . We define  $M(t), \tilde{\mathbf{P}}, \tilde{W}(t)$  like in the third theorem. Then

$$dY(t) = b(Y(t))dt + \sigma(Y(t))d\tilde{W}(t) \quad (2.7)$$

and the  $X(t), Y(t)$  processes have the same distribution.

**Proof:** We apply Theorem 2.2 in the case below:  $\sigma(t, \omega) = \sigma(X(t))$ ,  $b(t, \omega) = c(t, \omega) + b(Y(t))$  and  $a(t, \omega) = b(Y(t))$ .

### 3. APPLICATION: THE BLACK-SCHOLES EQUATION

We consider the equation of an active with the form [3]:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t),$$

where  $\mu$  is the mean of the process  $X(t)$ , and  $\sigma$  his volatility. We consider a rate  $r > 0$ . The Black-Scholes equation has the next form:

$$\begin{cases} \frac{\partial}{\partial t}V(t, x) + rX(t) \frac{\partial}{\partial x}V(t, x) + \frac{1}{2}\sigma^2 X(t)^2 \frac{\partial^2 V}{\partial X^2}(t, x) = rX(t) \\ V(t, x) = g(x), g \text{ is a } C^2 \text{ function.} \end{cases} .$$

Using the Girsanov theorems for transforming the process  $X(t)$  in a geometrical Brownian motion process which will generate the solution of the Black-Scholes equation like this:

$$\frac{dX(t)}{X(t)} = \mu dt + \sigma dW(t) \Leftrightarrow d(\ln S(t)) = \mu dt + \sigma dW(t)$$

and

$$b(t, \omega) = \mu, \quad \sigma(t, \omega) = \sigma \text{ and } f(t, \omega) = \frac{1}{\lambda \sigma} (\mu - r)$$

with the property:  $\mathbf{E} \left\{ \exp \frac{1}{2} \left( \frac{\mu - r}{\lambda \sigma} \right)^2 T \right\} < \infty$ .  $M(t)$  is choosing to be like:

$$M(t) = \exp \left( -\frac{(\mu - r)^2}{2 \sigma^2} t - \int_0^t \frac{\mu - r}{\lambda \sigma} dw(t) \right)$$

and  $d\tilde{W}(t) = dW(t) + f(t, \omega)dt$ .

So we obtain the equation:

$$d\tilde{X}(t) = r\tilde{X}(t)dt + \sigma\tilde{X}(t)d\tilde{W}(t)$$

and the Black Scholes solution is written like this:

$$V(t, x) = \mathbf{E} \left\{ e^{-rt} g(\tilde{X}(t)) \right\} .$$

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