A GENERALIZATION OF THE GIRSANOV THEOREMS ON THE HILBERT SPACE

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This paper gives a generalization of the Girsanov theorems on the separable Hilbert space. And we also give a characterization on the Hilbert space of the Wiener processes, using the quadratic variance process.

1. INTRODUCTION

Let $(\Omega, \Im, \mathbf{P})$ be a complete probability space, H a separable Hilbert space with the norm induced by the scalar product $\langle \cdot , \cdot \rangle$, $\| x \|^2 = \langle x , x \rangle$, $(\forall) x \in H$ and $W : I \times \Omega \to H$, $I = [0, T] \subset \mathbf{R}_+$ a Wiener process with $Q \in L_1(H)$ as the covariance operator and with normal filtration $\{ \Im_t \}_{t \in I}$; $W(0) = 0$, W has continuous paths, independent increases and $\mathbf{E}\{ W(t) \} = 0$, $\operatorname{cov} W(t) = t\,Q$. On the Hilbert space H we consider an ortonormat basis $\{ e_n \}_{n=1}^{\infty}$. In this case $W(t)$ has the next form: $W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \beta_n(t) e_n$, where $\beta_n(t) = \frac{1}{\sqrt{\lambda_n}} \langle W(t), e_n \rangle$, $n = 1, 2, \ldots$ are independent brownian real processes on $(\Omega , \Im , \mathbf{P})$ and $\lambda_n > 0$ with the property $Q e_n = \lambda_n e_n$, $n = 1, 2, \ldots$. [5]. It is considered a continuous stochastic process $X(t)$, $X : I \times \Omega \to H$, $X_t(\omega)$ continuous on I, $(\forall) \omega \in \Omega$ and a partition $\Delta \, on \, I$, $\Delta : t_0 = 0 < t_1 < t_2 < \ldots < t_k = T$, $\delta = \max\limits_{i = \overline{0 , k}} (t_{i+1} - t_i)$. We de fine the stochastic process $\langle X \rangle_t^{\Delta}$ using the formula: $\langle X \rangle_t^{\Delta} = \sum_{i=0}^{k-1} \| X(t \wedge t_{i+1}) - X(t \wedge t_i) \|^2$, where $t \wedge s = \min \{t , s\}$.

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Definition 1.1. t is called a *quadratic variance process* of the process $X(t)$, the application $\langle X\rangle_t$, where

$$\langle X\rangle_t = \lim_{\delta \to 0} \langle X\rangle_t^{\Delta}. \tag{1.1}$$

Observation 1.1.

(*i*) If $\{X(t)\}$ is with a finite variance then $\langle X\rangle_t = 0$.

(*ii*) The process $\langle X\rangle_t$ is continuous.

(*iii*) Let $\{M_j\}_j$ be a sequence of continuous martingal process, square integrable. Then M_j has a mean square convergence to a continuous martingal M, if and only if $\langle M_j - M\rangle_T$ has a mean convergence to zero;

(*iv*) $\langle W\rangle_t = tQ$, for the Wiener process.

Theorem 1.1 (The Girsanov Theorems) [4].

(I) *Let* $X(t) \in \mathbf{R}^n$ *be an Ito process:* $\mathrm{d}X(t) = f(t,\omega)\mathrm{d}t + \mathrm{d}B(t)$, $t \le T$, $X(0) = 0$ *where* $T < \infty$, $B(t)$ *is a n – dimensional Brownian motion, and*

$$M(t) = \exp\left(-\int_0^t f(s,\ \omega)\mathrm{d}B(s) - \frac{1}{2}\int_0^t f(s,\omega) f(s,\omega)^* \mathrm{d}s \right), t \le T \cdot$$

We suppose that $f(s,\omega)$ verifies the Novikov condition

$$\mathbf{E}\left\{ \exp\left(\frac{1}{2}\int_0^T f(s,\ \omega)\ f(s,\ \omega)^* \mathrm{d}s \right) \right\} < \infty. \tag{1.2}$$

We define the measure $\tilde{\mathbf{P}}$ *on* $(\Omega\ ,\ \Im_T)$ *by* $\mathrm{d}\tilde{\mathbf{P}}(\omega) = M(T)\mathrm{d}\mathbf{P}(\omega)$. *Then the* $X(t)$ *process is a Brownian motion with respect to the measure* $\widetilde{\mathbf{P}}$, *for* $t < T$.

(II) *Let* $X(t) \in \mathbf{R}^n$ *be an Ito process:*

$$\mathrm{d}X(t) = b(t,\omega)\mathrm{d}t + \sigma(t,\omega)\mathrm{d}B(t),\ t \le T$$

$b(t\ ,\ \omega) \in \mathbf{R}^n, \sigma(t\ ,\ \omega) \in L(\mathbf{R}^n\ ,\ \mathbf{R}^n)$. We *suppose that the integrable processes* $a(t\ ,\ \omega)$ *and* $f(t\ ,\ \omega)$ *exist such that:* $\sigma(t,\omega)f(t,\omega) = b(t,\omega) - a(t,\omega)$ *and* $f(t\ ,\ \omega)$ *verifies Novikov's condition. Then* $\tilde{B}(t) := \int_0^t f(t,\omega)\mathrm{d}s + B(t)$, $t \le T$ *is a Brownian motion with respect to the measure* $\widetilde{\mathbf{P}}$, *and the process* $X(t)$ *verifies the equation:*

$$\mathrm{d}X(t) = a(t,\ \omega)X(t)\mathrm{d}t + \sigma(t,\ \omega)\mathrm{d}\tilde{B}(t).$$

(III) *Let the processes* $X(t)$, $Y(t) \in \mathbf{R}^n$ *such that* $X(0) = Y(0) = x$ *and*

$$\mathrm{d}X(t) = b\big(X(t)\big)\mathrm{d}t + \sigma\big(X(t)\big)\mathrm{d}B(t),\ t \le T$$

$$\mathrm{d}Y(t) = \big[\gamma(t,\omega) + b\big(Y(t)\big)\big]\mathrm{d}t + \sigma\big(Y(t)\big)\mathrm{d}B(t),\ t \le T,$$

where $b \in L(\mathbf{R}^n)$, $\sigma \in L(\mathbf{R}^n, L(\mathbf{R}^n))$ *and* $\gamma(t,\omega)$ *is an integrable process.*

If it exists an integrable $f(t,\ \omega)$ *witch verifies the Novikov condition and* $\sigma\big(Y(t)\big)f(t,\omega) = \gamma(t,\omega)$, *then*

$$\mathrm{d}Y(t) = b\big(Y(t)\big)\mathrm{d}t + \sigma\big(Y(t)\big)\mathrm{d}\tilde{B}(t).$$

More than that the low of the process $Y(t)$ is the same as the law of the process $X(t)$, $t \le T$. A generalization of this result it can be found in [5].

Proposition 1.1. *Let* $f(t,\omega)$ *be a predictibile, square integrable process, such that:*

$$\mathbf{E}\left\{\exp\left(\frac{1}{2}\int_0^T f(t,\ \omega)Qf(t,\ \omega)^*\mathrm{d}s\right)\right\} < \infty \tag{1.3}$$

Then the stochastic process $M(t)$, *defined by the relation:*

$$M(t) = \exp\left\{\int_0^t f(s,\omega)\ \mathrm{d}W(s) - \frac{1}{2}\int_0^t f(s,\omega)Qf^*(s,\omega)\mathrm{d}s\right\} \tag{1.4}$$

is a martingal with the mean $\mathbf{E}\{M(t)\} = 1$.

Proof: The process $\{M(t)\}$ verifies the differential stochastic equation:

$$\mathrm{d}M(t) = -f(t,\ \omega)\ \ M(t)\mathrm{d}W(t),\quad M(0) = 1.$$

Let $M(t) = \sum_{n=1}^{\infty} M_n(t)e_n$. Then

$$\mathrm{d}M_n(t) = -\big\langle f(t,\omega)M(t), e_n\big\rangle\sqrt{\lambda_n}\mathrm{d}B_n(t)$$

where $M_n(t)$ are real stochastic processes, $n \in \mathbf{N}^*$. Because of these the problem is reduced to the real case, which is the equation:

$$\mathrm{d}M(t) = -\sqrt{\lambda} f(t,\omega) M(t) \mathrm{d}B(t) \Leftrightarrow \frac{dM(t)}{M(t)} = -f(t,\omega)\sqrt{\lambda}\mathrm{d}B(t),$$

where $M(t)$ and $f(t\ ,\ \omega)$ are real processes, and $B(t)$ is a real Brownian motion.

From Ito's lemma which is applied to the function $g : [0\ , T] \times \mathbf{R} \to \mathbf{R}$, $g\ \ (t,\ x) = \ln x$ it results that

$$\mathrm{d}g(t, M(t)) = \frac{\mathrm{d}M(t)}{M(t)} - \frac{1}{2} f^2(t,\omega)\lambda \mathrm{d}t \ \Rightarrow$$

$$M(t) = \exp\left(-\int_0^t f(s,\omega)\mathrm{d}W(s) - \frac{1}{2}\int_0^t \lambda f^2(s,\omega)\mathrm{d}s\right).$$

So:

$$\langle M(t),\ e_n\rangle = M_n(t) = \exp\left\{-\int_0^t \langle (s,\omega), e_n\rangle \sqrt{\lambda_n}\mathrm{d}B_n(s) -\right.$$

$$\left. -\ \frac{1}{2}\int_0^t \lambda_n \langle f(s,\ \omega),\ e_n\rangle\ \ \langle f(s,\ \omega),\ e_n\rangle \mathrm{d}s\right\}.$$

Because $M(t) = \sum_{n=1}^{\infty} \langle M(t), e_n\rangle e_n$ we obtain:

$$M(t) = \exp\left\{-\int_0^t f(s,\ \omega)\mathrm{d}W(s) - \frac{1}{2}\int_0^t f(s,\ \omega) Qf(s,\ \omega)^* \mathrm{d}s\right\}.$$

Forward we will proof that $\mathbf{E}\{M(t)\} = 1$. The integral equation associated to the $M(t)$ process is $M(t) = M(0) - \int_0^t f(s,\ \omega) M(s)\mathrm{d}W(s)$. We take the expectation in this relation and we gain $\mathbf{E}\{M(t)\} = 1$. Let f be a predictable square integrable process. Then it exist a sequence of elementary processes $\{f_i\}$ such that $\mathbf{E}\left\{\int_0^T |f - f_i\ |^2 \mathrm{d}s\right\} \to 0$. The case of elementary processes is reduced to the case of constant processes using the relation:

$$f\ = \sum_{j=0}^{k-1} a_k\ \chi(t_j\ \ , t_{j+1}), a_k\ \in \mathbf{R}.$$

It result that

$$M(t) = \exp\left\{-\int_0^t f(s,\ \omega)\mathrm{d}W(s) - \frac{1}{2}\int_0^t f^2(s,\ \omega)\lambda \mathrm{d}s\right\} =$$
$$= \exp\left\{\sum_{j=0}^{k-1}\left[-a_j W(t_{j+1}) - \frac{1}{2}a_j^2\lambda t_{j+1}\right] + \left[-a_j W(t_j) - \frac{1}{2}a_j^2\lambda t_j\right]\right\} =$$
$$= \prod_{j=0}^{k-1}\exp\left(-a_j W(t_{j+1}) - \frac{1}{2}a_j^2\lambda t_{j+1}\right)\cdot\exp\left(-a_j W(t_j) - \frac{1}{2}a_j^2\lambda t_j\right).$$

In this way the general case is reduced to the study of processes which have the following form : $M(t) = \exp\left(-aW(t) - \frac{1}{2}a^2\lambda t\right)$. We proof that $M(t)$ is a $\Im_t$ – martingal by reducing them to the constant real processes. We consider constant processes like $f(s\ ,\ \omega) = a$. Then $\mathbf{E}\left\{\exp\left[-aW(t) - \frac{1}{2}a^2\lambda t\right] | \Im_s\right\} =$

$$= \exp\left\{-aW(s) - \frac{1}{2}a^2\lambda t\right\}\mathbf{E}\{\exp[-a(W(t) - W(s))] | \Im_s\}$$

because $W(s)$ is $\Im_s$ – measurable and

$$\mathbf{E}\{\exp[-a(W(t) - W(s))] | \Im_s\} = \mathbf{E}\{\exp[-a(W(t) - W(s))]\} = \exp\left(\frac{1}{2}a^2\lambda(t-s)\right)$$

because $W(t) - W(s)$ is $\Im_s$ – independent. It results:

$$\mathbf{E}\left\{\exp\left[-aW(t) - \frac{1}{2}a^2\lambda t\right] | \Im_s\right\} = \exp\left[-aW(s) - \frac{1}{2}a^2\lambda s\right].$$

Because $M(t) = \sum_{n=1}^{\infty} M_n(t) e_n$, where $M_n(t)$ are real square integrable processes, it results that $M_n(t)$ are real martingales and

$$\mathbf{E}\{M(t) | \Im_s\} = \sum_{n=1}^{\infty}\mathbf{E}\{M_n(t) | \Im_s\} e_n = \sum_{n=1}^{\infty}\mathbf{E}\{M_n(s)\} e_n = \mathbf{E}\{M(s)\},$$

for every predictable processes $f(s,\ \omega)$ H – valued.

Theorem 1.2 (In the finite case we have this result in [2]). *Let* $\left\{X\left(\int t\right)\right\}_{t\in I}$ *be a continuous process*, $\Im_t$ *– adapted with the mean* 0 (zero) *and* *the covariance*, $\operatorname{cov}\{X(s),\ X(t)\} = Q(s,\ t)$. *Then the following sentences are* equivalent:

(i) $X(t)$ *is a* $\Im_t$ *Wiener process*;

(ii) $X_n(t)$ *is martingal square integrable*, $n = 1, 2, \ldots$ *and* $\langle X \rangle_t = tQ,\ Q \in L^1(H)$;

(iii) $\exp\left\{-\int_0^t f(s,\ \omega)\mathrm{d}X(s) - \frac{1}{2}\int_0^t f(s,\ \omega)Qf(s,\ \omega)^* \mathrm{d}s\right\}$ *is martingal, with a unity mean, for every f – square integrable predictable process*;

(iv) $\{X(t)\}_{t \in I}$ *is a gaussian process such that* $X(t) - X(s)$ is $\Im_s$ – *independent for every* $0 < s < t$.

Proof: The operator $Q(s,\ t)$ is defined by the relation

$$Q(s,\ t) = \mathbf{E}\{X(t) \otimes X(s)\},$$

and in the case of a Wiener process the relation $Q(s,\ t) = Q(t-s)$ exists, where $0 < s < t$ and $Q \in L_1(H)$.

$(i) \Rightarrow (ii)$ let $0 < s < t$. $X(t) - X(s)$ are $\Im_s$ – independent and

$\mathbf{E}\{X(t) - X(s)\ |\Im(s)\} = 0 \Rightarrow \mathbf{E}\{X_n(t) - X_n(s)\} = 0 \Rightarrow \mathbf{E}\{X_n(t)|\ \Im(s)\} = X_n(s)$, $\langle X \rangle_t = t\,Q$ (we used the observation 1.1);

$(i) \Rightarrow (iii)$ using the proposition 1.1;

$(i) \Leftrightarrow (iv)$ it is obtained from the definition of a Wiener process;

$(ii) \Rightarrow (i)$ let $\langle X \rangle_t = t\,Q$ and $X_n(t)$ be martingal square integrable.

Then $\langle X_n \rangle_t = \lambda_n t$, $t \in I$. Using Levy's theorem [5], on **R**, we obtain that X_n is a real Wiener process which has $\Im_s$ – independent increases. It results that:

$$X(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, B_n(t) e_n\ ,\ t \in [0\ ,\ T],$$

where $B_n(t) = \dfrac{1}{\sqrt{\lambda_n}} X_n(t)$ are independent standard Wiener processes.

$(iii) \Rightarrow (iv)$ we observe that $\mathbf{E}\{\exp[aW(t)]\} = \mathbf{E}\{\exp[-aW(t)]\}$. And the following result takes place:

[1] Let μ be the associate measure for $X(t)$. Then the μ measure is gaussian if and only if his characteristic function approves:

$$\mathbf{E}\left\{\exp\left[i\left\langle a\ ,\ W(t)\right\rangle\right]\right\} = \exp\left\{\ i\left\langle m\ ,\ a\right\rangle - \frac{1}{2}\left\langle Qa\ ,\ a\right\rangle\right\},$$

where m is the mean, and Q – the covariance.

Let $M\left(t\right) = \exp\left\{\ -\left\langle a\ ,\ X\left(t\right)\right\rangle - \frac{1}{2}\,a\,Q\,a^{*}\,t\right\}$ be $\Im_t$ – martingal with $\mathbf{E}\left\{M\left(t\right)\right\} = 1$. It results that $\mathbf{E}\left\{\exp -\left\langle a\ ,\ X\left(t\right)\right\rangle\right\} = \exp\left\{\ \frac{1}{2}\,a\,Q\,a^{*}\,t\right\}$ and

$$\mathbf{E}\left\{\exp i\left\langle a\ ,\ X(t)\right\rangle\right\} = \mathbf{E}\left\{\ \exp\sum_{n=1}^{\infty} i\,a_n\ X_n(t)e_n\right\} = \prod_{n=1}^{\infty}\mathbf{E}\left\{\exp\ i\,a_n\ X_n(t)e_n\right\} =$$

$$= \prod_{n=1}^{\infty}\exp\left\{\ -\frac{1}{2}\lambda_n\ a_n^2\ t\,e_n\right\} = \exp\left(\sum_{n=1}^{\infty} -\frac{1}{2}\lambda_n\ a_n^2\ e_n\,t\right) = \exp\left\{-\frac{1}{2}\,a\,Q\,a^{*}(t)\right\} =$$

$$= \exp\left\{\ -\frac{1}{2}\left\langle Qa\ ,\ a\right\rangle t\right\}.$$

In this way we have proved that $\left\{X\left(t\right)\right\}$ is a gaussian process with the zero mean and the covariance Q. Knowing that $M\left(t\right)$ is martingal, with the unite mean we see that $\exp\left\{-\left\langle a\ ,\ X(t)\right\rangle\right\}$ is martingal, $(\forall)\,a \in H$ and

$$\mathbf{E}\left\{\exp\left[-\left\langle a, X(t) - X(s)\right\rangle\right]\middle|\ \Im_s\right\} = \exp\left\{-\frac{1}{2}\,a\,Q\,a^{*}(t-s)\right\},\ (\forall)\,a \in H\,.$$

So $X(t) - X(s)$ is $\Im_s$ – independent and in this way we have finished the proof.

2. THE GIRSANOV THEOREMS ON THE SEPARABLE HILBERT SPACE

Theorem 2.1 (The first Girsanov Theorem). *Let* $X\left(t\right)$ *be a stochastic process* $X : \left[0\ ,\ T\right]\times\Omega \to H$ *with the following form*:

$$\begin{cases} \mathrm{d}X\left(t\right) = Qf\left(t,\ \omega\right)\mathrm{d}t + \mathrm{d}W\left(t\right) \\ X\left(0\right) = 0,\ t \in \left[0,\ T\right], \end{cases} \tag{2.1}$$

where W *is Q-Wiener process, and* $f\left(\cdot\ ,\ \cdot\right)$ *is a square integrable predictable process, and the process*

$$M\left(t\right) = \exp\left(-\int_0^t f\left(s,\ \omega\right)\mathrm{d}W\left(s\right)\ -\ \frac{1}{2}\int_0^t f\left(s,\ \omega\right)Qf\left(s,\ \omega\right)^{*}\mathrm{d}s\right)$$

with the property: $\mathbf{E}\left\{\exp\frac{1}{2}\int_0^T f(s,\ \omega)Qf(s,\ \omega)^* \mathrm{d}s\right\} < \infty$. *We define the measure* $\widetilde{\mathbf{P}}$ *on* $(\Omega\ ,\ \Im_T)$ *by:* $\mathrm{d}\tilde{\mathbf{P}}(\omega) = M(T)\mathrm{d}\mathbf{P}(\omega)$.

Then $\{X(t)\}$ *is a Wiener process related to the measure* $\widetilde{\mathbf{P}}$.

Proof: We write the integral equation, which is associated to the differential equation:

$$X(t) = Q\int_0^t f(s,\ \omega)\mathrm{d}s + W(t).$$

In this way, related to the measure $\mathbf{P}$ the following relations are proven to be true:

$$\mathbf{E}\{X(t)\} = Q\int_0^t \mathbf{E}\{f(s,\ \omega)\}\mathrm{d}s\,;$$

$$M(T)\mathrm{d}\mathbf{P} = M(t)\mathrm{d}\mathbf{P} \ \ pe\ \Im_t,\ t \le T\,;\ X(0) = 0\,;$$

if $f(s\ ,\ \omega) = f(s)$ is a non-random function, then $\operatorname{cov} X(t) = Q\,t$.

From Proposition 1.1 we know that $M(T)$ is $\Im_t$ – martingal and $\mathbf{E}\{M(t)\} = 1$. It results that $\widetilde{\mathbf{P}}(\Omega) = 1$, so $\widetilde{\mathbf{P}}$ is a probability measure.

We proof that $X(s)$ is a Wiener process related to the $\widetilde{\mathbf{P}}$ measure used in the first theorem, so: $X(t)$ is a Wiener process related to the $\widetilde{\mathbf{P}}$ measure $\Leftrightarrow$

$$\Leftrightarrow N(t) = \exp\left\{-\int_0^t g(s,\ \omega)\mathrm{d}X(s) - \frac{1}{2}\int_0^t g(s,\ \omega)Qg(s,\ \omega)^* \mathrm{d}s\right\},$$

is $\Im_t$ – martingal related to the $\widetilde{\mathbf{P}}$ measure (where g – a predictable square integrable process) $\Leftrightarrow N(t)M(t)$ – a martingal related to the $\mathbf{P}$ measure.

We used the relations: $\tilde{\mathbf{E}}\{X(t)\} = 0$ and $\tilde{\operatorname{cov}}\{X(t)\} = tQ$ (related to the $\widetilde{\mathbf{P}}$ measure); $\widetilde{\mathbf{E}}\{N(t)\} = \mathbf{E}\{N(t)M(t)\}$.

We solve $N(t)M(t)$:

$$N(t)M(t) = \exp\left\{-\int_0^t \langle g(s,\ \omega),\ Qf(s,\ \omega)\rangle \mathrm{d}s - \int_0^t g(s,\ \omega)\mathrm{d}W(s)\ -\right.$$

$$\left.-\frac{1}{2}\int_0^t \langle Qg(s,\omega),\ g(s,\omega)\rangle \mathrm{d}s - \int_0^t f(s,\ \omega)\mathrm{d}W(s) - \frac{1}{2}\int_0^t \langle Qf(s,\ \omega),\ f(s,\ \omega)\rangle \mathrm{d}s\right\} =$$

$$= \exp\left\{-\int_0^t h(s,\ \omega)\mathrm{d}W(s) - \frac{1}{2}\int_0^t h(s,\ \omega)Qh(s,\ \omega)^* \mathrm{d}s\right\},$$

where $h(s,\ \omega) = (s,\ \omega) + g(s,\ \omega)$ and we also used the relation:

$$\frac{1}{2}\langle Q(x+y), x+y\rangle = \frac{1}{2}\langle Q\,x, x\rangle + \langle Q\,x, y\rangle + \frac{1}{2}\langle Q\,y, y\rangle,\ (\forall)\,x,\ y \in H$$

(because $Q \in L_1(H)$, Q is positive and selfadjoint).

Using Theorem 1.2 for the Wiener process $W(t)$, the predictable process $h(s\ ,\ \omega)$ and the **P** measure we obtain that $N(t)M(t)$ is a $\Im_t$ – martingal.

Theorem 2.2 (The second Girsanov Theorem). *Let* $X(t)$ *be an Ito process,* $X : [0\ ,\ T] \times \Omega \to\ H$ *which has the form*:

$$\begin{cases} \mathrm{d}X(t) = b(t,\ \omega)\mathrm{d}t + \sigma(t,\ \omega)\mathrm{d}W(t),\ 0 \le t \le T \\ X(0) = x,\ x \in H, \end{cases} \tag{2.2}$$

where $b(t\ ,\ \omega)$ is a $\Im_t$ *adapted process with the property* $\mathbf{P}\left\{\int_0^t \left\|b(s,\ \omega)\right\|\mathrm{d}s < \infty,\ (\forall)\ t \in I\right\} = 1$ *and* $\sigma(s,\ \omega)$ *is a predictable process,* $\Im_t$ *– measurable and*

$$\mathbf{P}\left\{\int_0^t \left\|\sigma(s,\ \omega)\right\|^2 \mathrm{d}s < \infty,\ (\forall)\ t \in I\right\} = 1\,.$$

We suppose that the predictable processes $\tilde{W}(t) = Q\int_0^t f(s\ ,\omega)\mathrm{d}s + W(t)$ exist, and they are $\Im_t$ – measurable and square integrable on *I* such that:

$$b(t,\ \omega) - a(t,\ \omega) = \sigma(t,\ \omega)\ \ Qf(t,\ \omega),\ (\forall)t \in I$$

and the process $f(t,\ \omega)$ verifies the Novikov condition:

$$\mathbf{E}\left\{\exp\left(\ \frac{1}{2}\int_0^T f(t\ ,\ \omega)Qf(t,\ \omega)^* \mathrm{d}t\right)\right\} < \infty\,. \tag{2.3}$$

If we consider $M(t)$, $\tilde{\mathbf{P}}$, $\tilde{W}(t)$ as in the Theorem 2.1

$$\tilde{W}(t) = Q\int_0^t f(s,\omega)\mathrm{d}s + W(t),$$

then the differentiable equation associated to the $X\ (t)$ process related with the $\tilde{\mathbf{P}}$ measure and the Wiener process $\tilde{W}(t)$ is:

$$\begin{cases} \mathrm{d}X(t) = a(t,\ \omega)\mathrm{d}t + \sigma(t,\ \omega)\mathrm{d}\hat{W}(t),\ t \in [0,\ T] \\ X(0) = 0. \end{cases} \tag{2.4}$$

Proof: From the relation: $\mathrm{d}W(t) = \mathrm{d}\tilde{W}(t) - Qf(t,\ \omega)\mathrm{d}t$ and by putting it in (2.2) we get: $\mathrm{d}X(t) = a(t,\ \omega)\mathrm{d}t + \sigma(t,\ \omega)\mathrm{d}\tilde{W}(t)$.

Observation 2.1. A consequence of this result is that the $\sigma(t,\ \omega)$ is invertible and $f(t,\ \omega)$ is unique $f(t,\ \omega) = Q^{-1}\sigma^{-1}(t,\ \omega)\ \left[b(t,\ \omega) - a(t,\ \omega)\right]$.

Theorem 2.3 (The third Girsanov Theorem). *We consider the processes* $X(t),\ Y(t),\ X : I \times \Omega \to H$ *and* $Y : I \times \Omega \to H$ *which have the form*:

$$\begin{cases} \mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t),\ t \in I \\ X(0) = x,\ x \in H \end{cases} \tag{2.5}$$

$$\begin{cases} \mathrm{d}Y(t) = \left[c(t,\ \omega) + b(Y(t))\right]\mathrm{d}t + \sigma(Y(t))\mathrm{d}W(t),\ t \in I \\ Y(0) = x, \end{cases} \tag{2.6}$$

where the operators $b \in L(H),\ \sigma \in L\ (H,\ L(H))$ *are verifying the Lipschitz conditions:* b, σ *are measurable and*

$$\|b(x)\| + \|\sigma(x)\| \le k\ \ (1 + \|\ x\ \|),\ x \in H,\ k > 0,$$

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\ \| \le k\|\ x - y\|,\ x,\ y \in H.$$

Further more $c(t,\ \omega)$ is a predictable square integrable process, $\Im_t$ – measurable. We suppose that exist one process, $f(t,\ \omega)$, predictable, square integrable, $\Im_t$ – measurable which verify the Novikov relation (2.3) such that: $c(t,\ \omega) = \sigma(Y(t))Qf(t,\omega)$. We define $M(t)$, $\tilde{\mathbf{P}}$, $\tilde{W}(t)$ like in the third theorem. Then

$$\mathrm{d}Y(t) = b(Y(t))\mathrm{d}t + \sigma(Y(t))\mathrm{d}\tilde{W}(t) \tag{2.7}$$

and the $X(t),\ Y(t)$ processes have the same distribution.

Proof: We apply Theorem 2.2 in the case below: $\sigma(t\ ,\ \omega) = \sigma(\ X\ (t))$, $b(t\ ,\ \omega) = c(t\ ,\ \omega) + b(\ Y\ (t))$ and $a(t\ ,\ \omega) = b(\ Y\ (t))$.

3. APLICATION: THE BLACK-SCHOLES EQUATION

We consider the equation of an active with the form [3]:

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),$$

where μ is the mean of the process $X(t)$, and σ his volatility. We consider a rate $r > 0$. The Black-Scholes equation has the next form:

$$\begin{cases} \dfrac{\partial}{\partial t}V(t,\ x) + rX(t)\ \dfrac{\partial}{\partial x}V(t,\ x) + \dfrac{1}{2}\sigma^2 X(t)^2 \dfrac{\partial^2 V}{\partial X^2}(t,\ x) = rX(t) \\ V(t,\ x) = g(x),\ g \ \text{ is a } \ C^2 \ \text{ function.} \end{cases}.$$

Using the Girsanov theorems for transforming the process $X(t)$ in a geometrical Brownian motion process which will generate the solution of the Black-Scholes equation like this:

$$\frac{\mathrm{d}X(t)}{X(t)} = \mu\mathrm{d}t + \sigma\mathrm{d}W(t) \Leftrightarrow \mathrm{d}(\ln S(t)) = \mu\mathrm{d}t + \sigma\mathrm{d}W(t)$$

and

$$b(t\ ,\ \omega) = \mu\ ,\ \ \sigma(t\ ,\ \omega) = \sigma \ \text{ and } \ f(t\ ,\ \omega) = \frac{1}{\lambda\,\sigma}(\mu - r)$$

with the property: $\mathbf{E}\left\{ \exp\dfrac{1}{2}\left(\dfrac{\mu - r}{\lambda\,\sigma}\right)^2 T \right\} < \infty$. $M(t)$ is choosing to be like:

$$M(t) = \exp\left(-\frac{(\mu - r)^2}{2\ \sigma^2}t - \int_0^t \frac{\mu - r}{\lambda\ \sigma}\mathrm{d}w(t) \right)$$

and $\mathrm{d}\tilde{W}(t) = \mathrm{d}W(t) + f(t,\ \omega)\mathrm{d}t$.

So we obtain the equation:

$$\mathrm{d}\tilde{X}(t) = r\tilde{X}(t)\mathrm{d}t + \sigma\tilde{X}(t)\mathrm{d}\tilde{W}(t)$$

and the Black Scholes solution is written like this:

$$V(t,\ x) = \mathbf{E}\left\{ e^{-rt} g\left(\tilde{X}(t)\right)\right\}.$$

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