DYNAMICS OF THE 3-*PRP* PLANAR PARALLEL ROBOT

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Recursive modelling for the kinematics and dynamics of the 3-*PRP* planar parallel robot is established in this paper. Three identical planar legs connecting to the moving platform are located in a vertical plane. Knowing the motion of the platform, we develop first the inverse kinematics and determine the positions, velocities and accelerations of the robot. Further, the principle of virtual work is used in the inverse dynamics problem. Several matrix equations offer iterative expressions and graphs for the powers of three prismatic actuators. The study of the dynamics of the parallel mechanisms is done mainly to solve successfully the control of the motion of such robotic systems.

LIST OF SYMBOLS

α_i $(i = A, B, C)$ – angles giving the position of the fixed base $A_0 B_0 C_0$

$\vec{\varepsilon}_{k,k-1}$ – relative angular acceleration of T_k rigid body

$\vec{\varepsilon}_{k0}$ – absolute angular acceleration of T_k

$\tilde{\varepsilon}_{k,k-1}$ – skew-symmetric matrix associated to $\vec{\varepsilon}_{k,k-1}$

f_{10}^A, f_{10}^B, f_{10}^C – forces of the three prismatic actuators

ϕ – orientation angle of the frame $Gx_G y_G z_G$

$\varphi_{k,k-1}$ – relative rotation angle of T_k

$\vec{\gamma}_{k,k-1}^A$ – relative acceleration of the joint's centre A_k

J_1, J_2 – two Jacobian matrices of the manipulator

l – edge length of the triangular moving platform

λ_{10}^i – displacements of the three sliders A_1, B_1, C_1

m_k, $\hat{I}_k$ – mass and symmetric matrix of the tensor of inertia of T_k about the frame $x_k y_k z_k$

$\vec{\omega}_{k,k-1}$ – relative angular velocity of T_k

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$\vec{\omega}_{k0}$ – absolute angular velocity of T_k
$\tilde{\omega}_{k,k-1}$ – skew-symmetric matrix associated to $\vec{\omega}_{k,k-1}$
p_{10}^{i} – powers required by the prismatic actuators
$q_{k,k-1}$ – orthogonal transformation matrix
R – general rotation matrix of the moving platform
$\vec{r}_{k,k-1}^{A}$ – relative position vector of the centre A_k
$\vec{r}_{00}^{i}$ – position vectors of the fixed summits A_0, B_0, C_0
$\vec{r}_{k}^{C}$ – position vector of the mass centre of T_k
$\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ – three orthogonal unit vectors
$\vec{v}_{k,k-1}^{A}$ – relative velocity of the joint's centre A_k
x_0^G, y_0^G – coordinates of centre G of the moving platform.

1. INTRODUCTION

Compared with the serial manipulators, potential advantages of the parallel architectures are higher kinematical precision, lighter weight, better stiffness, greater load bearing, stabile capacity and suitable positional actuator arrangements. However, they present limited workspace and complicated singularities [1].

Over the past decades, parallel manipulators have received an increasing amount of attention from researches and industries. The level of accuracy and precision required in performing a certain task is essential.

Considerable efforts have been devoted to the kinematics and dynamic analysis of fully parallel manipulators. Among these, the class of manipulators known as Stewart-Gough platform has received great attention (Stewart [2]; Merlet [3]; Parenti-Castelli and Di Gregorio [4]). They are used in flight simulators and more recently for Parallel Kinematics Machines. The prototype of Delta parallel robot (Clavel [5]; Tsai and Stamper [6]; Staicu and Carp-Ciocardia [7]) as well as the Star parallel manipulator (Hervé and Sparacino [8]) are equipped with three motors, which train on the mobile platform in a three-degree-of-freedom general translation motion. Angeles, Gosselin, Gagné and Wang [9–11] analysed the kinematics, dynamics and singularity loci of Agile Wrist spherical robot with three actuators.

A mechanism is said to be a *planar robot* if all the moving links in the mechanism perform the planar motions. For a planar mechanism, the loci of all points in all links can be drawn conveniently on a plane. In a planar linkage, the axes of all revolute joints must be normal to the plane of motion, while the direction of translation of a prismatic joint must be parallel to the plane of motion.

In their paper, Aradyfio and Qiao [12] examine the inverse kinematics solution for the three different 3-DOF planar parallel robots. Gosselin and Angeles [13] and Pennock and Kassner [14] each present a kinematical study of a planar parallel robot, where a moving platform is connected to a fixed base by three links, each leg consisting of two binary links and three parallel revolute joints. Sefrioui and Gosselin [15] give an interesting numerical solution in the inverse and direct kinematics of this kind of planar robot. Mohammadi-Daniali et al. [16] present a study of velocity relationships and singular conditions for general planar parallel robots.

Merlet [17] solved the forward pose kinematics problem for a broad class of planar parallel manipulators. Williams et al. [18] analysed the dynamics and the control of a planar three-degree-of-freedom parallel manipulator at Ohio University, while Yang et al. [19] concentrate on the singularity analysis of a class of 3-*RRR* planar parallel robots developed in their laboratory. Bonev, Zlatanov and Gosselin [20] describe several types of singular configurations by studying the direct kinematics model of a 3-*RPR* planar parallel robot with actuated base joints. Mohammadi-Daniali et al. [21] analysed the kinematics of a planar 3-DOF parallel manipulator using the three *PRP* legs, where the three revolute joint axes are perpendicular to the plane of motion while the prismatic joint axes lie on the same plane.

A recursive method is developed in the present paper for deriving the inverse dynamics of the 3-*PRP* planar parallel robot in a numerically efficient way.

2. KINEMATICS ANALYSIS

Having a closed-loop structure, the planar 3-*PRP* parallel robot is a special symmetrical mechanism composed of three planar kinematical chains with identical topology, all connecting the fixed base to the moving platform. The points A_0, B_0, C_0 represent the summits of a fixed triangular base and other three points define the geometry of the moving platform. Each leg consists of two links, with one revolute and two prismatic joints. The parallel mechanism with seven links $(T_k, k = 1, 2, ..., 7)$ consists of three revolute and six prismatic joints (Fig. 1). Grübler mobility equation predicts that the device has certainly three degrees of freedom.

In the actuation scheme *PRP* each prismatic joint is an actively controlled prismatic cylinder. Thus, all prismatic actuators can be installed on the fixed base.

For the purpose of analysis, we attach a Cartesian frame $x_0y_0z_0(T_0)$ to the fixed base with its origin located at triangle centre O, the z_0 axis perpendicular to the base and the x_0 axis pointing along the C_0B_0 direction. Another mobile reference frame $x_Gy_Gz_G$ is attached to the moving platform. The origin of this coordinate central system is located just at the centre G of the moving triangle (Fig. 2).

In what follows we will represent the intermediate reference systems by only two axes so as is used in many robotics papers [1, 3, 9]. It is noted that the relative translation $\lambda_{k,k-1}$ and the rotation angle $\varphi_{k,k-1}$ always point along or about the direction of z_k axis.

We consider that the moving platform is initially located at a *central configuration*, where the platform is not rotated with respect to the fixed base and the mass centre G is at the origin O of fixed frame.

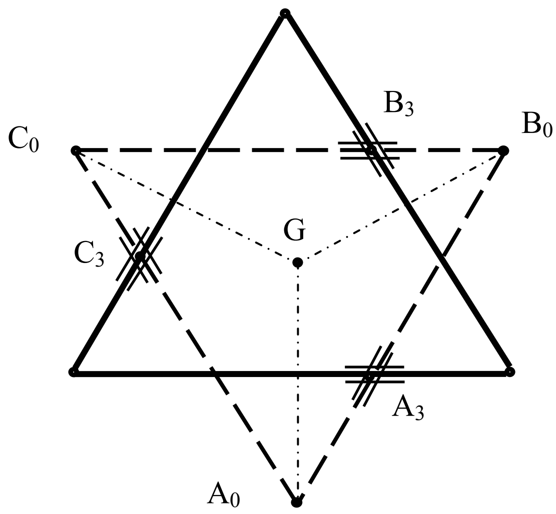

Fig. 1 – The 3-*PRP* planar parallel robot.

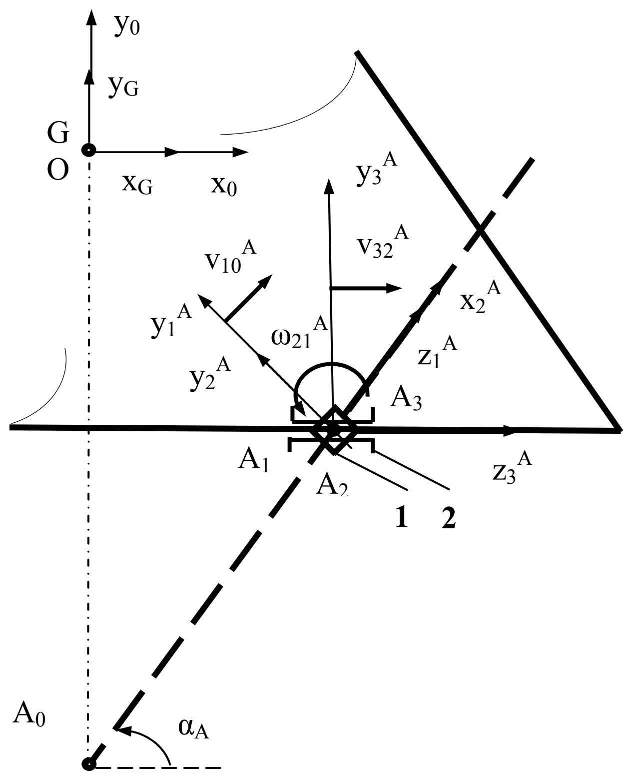

Fig. 2 – Kinematical scheme of first leg A of the mechanism.

One of three active legs (for example leg A) consists of a prismatic joint, which is as well as a piston **1** of mass m_1 linked at the $x_1^A y_1^A z_1^A$ frame, having a rectilinear motion of displacement λ_{10}^A, velocity $v_{10}^A = \dot{\lambda}_{10}^A$ and acceleration $\gamma_{10}^A = \ddot{\lambda}_{10}^A$. Second element of the leg is a rigid body **2** linked at the $x_2^A y_2^A z_2^A$ frame, having a relative rotation about z_2^A axis with the angle φ_{21}^A, velocity $\omega_{21}^A = \dot{\varphi}_{21}^A$ and acceleration $\varepsilon_{21}^A = \ddot{\varphi}_{21}^A$. It has the mass m_2 and tensor of inertia $\hat{I}_2$ with respect to T_2^A frame. Finally, a prismatic joint is introduced at a planar moving platform as an equilateral triangle with the edge $l = l_0\sqrt{3}$, mass m_3 and inertia tensor $\hat{I}_3$ with respect to A_3, which translate relatively with the displacement λ_{32}^A and the velocity $v_{32}^A = \dot{\lambda}_{32}^A$ along z_3^A axis.

At the central configuration, we also consider that all legs are symmetrically extended and that the angles of orientation of three edges of fixed platform are given by

$$\alpha_A = \frac{\pi}{3},\ \alpha_B = \pi,\ \alpha_C = -\frac{\pi}{3}. \quad (1)$$

In the following, we apply the method of successive displacements to geometric analysis of closed-loop chains and we note that a joint variable is the displacement required to move a link from the initial location to the actual position. If every link is connected to least two other links, the chain forms one or more independent closed-loops.

The variable angle $\varphi^i_{k,k-1}$ of rotation about the joint axis z^i_k is the parameter needed to bring the next link from a reference configuration to the next configuration. We call the matrix $q^{\varphi}_{k,k-1}$, for example, the orthogonal transformation 3×3 matrix of relative rotation with the angle $\varphi^i_{k,k-1}$ of link T^i_k around z^i_k.

In the study of the kinematics of robot manipulators, we are interested in deriving a matrix equation relating the location of an arbitrary T^i_k body to the joint variables. When the change of coordinates is successively considered, the corresponding matrices are multiplied. So, starting from the reference origin O and pursuing the three legs $OA_0A_1A_2A_3$, $OB_0B_1B_2B_3$, $OC_0C_1C_2C_3$, we obtain the following transformation matrices [22]:

$$\begin{gathered} q_{10} = \theta_1\theta^i_{\alpha},\ q_{21} = q^{\varphi}_{21}\theta^{\mathrm{T}}_1,\ q_{32} = \theta_1\theta_2 \\ (q = a,\ b,\ c),\ (i = A,\ B,\ C) \end{gathered}, \quad (2)$$

where

$$\theta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ \theta_2 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},\ \theta^i_{\alpha} = \begin{bmatrix} \cos\alpha_i & \sin\alpha_i & 0 \\ -\sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$q^{\varphi}_{k,k-1} = \begin{bmatrix} \cos\varphi^i_{k,k-1} & \sin\varphi^i_{k,k-1} & 0 \\ -\sin\varphi^i_{k,k-1} & \cos\varphi^i_{k,k-1} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ q_{k0} = \prod_{s=1}^{k} q_{k-s+1,k-s}\ (k = 1, 2, 3). \quad (3)$$

Three displacements $\lambda^A_{10}, \lambda^B_{10}, \lambda^C_{10}$ of the active links are the joint variables that give the input vector $\vec{\lambda}_{10} = [\lambda^A_{10}\ \ \lambda^B_{10}\ \ \lambda^C_{10}]^{\mathrm{T}}$ of the instantaneous position of the mechanism. But, in the inverse geometric problem, we can consider that the position of the mechanism is completely given by the coordinates x^G_0, y^G_0 of the mass centre G of the moving platform and the orientation angle ϕ of the movable frame $x_G y_G z_G$. The orthogonal rotation matrix of the moving platform from $x_0 y_0 z_0$ to $x_G y_G z_G$ reference system is

$$R = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Further, we suppose that the position vector $\vec{r}_0^{\,G} = [x_0^G \;\; y_0^G \;\; 0]^{\mathrm{T}}$ of G centre and the ϕ orientation angle, which are expressed by following analytical functions

$$\frac{x_0^G}{x_0^{G*}} = \frac{y_0^G}{y_0^{G*}} = \frac{\phi}{\phi^*} = 1 - \cos\frac{\pi}{3} t\,, \tag{5}$$

can describe the general absolute motion of the moving platform in its *vertical plane*. The values $2x_0^{G*}$, $2y_0^{G*}$, $2\phi^*$ denote the final position of the moving platform.

The conditions concerning the absolute orientation of the moving platform are expressed by three identities

$$\overset{\circ}{q}{}_{30}^{\mathrm{T}} q_{30} = R \;\; (q = a,\, b,\, c), \tag{6}$$

where the resulting matrix q_{30} is obtained by multiplying three basic matrices

$$q_{30} = q_{32} q_{21} q_{10}\,,\;\; \overset{\circ}{q}_{30} = q_{30}(t=0) = \theta_1 \theta_2 \theta_\alpha^i\,,\;\; (i = A, B, C)\,. \tag{7}$$

From these conditions one obtains the first relations between the angles of rotation

$$\varphi_{21}^A = \varphi_{21}^B = \varphi_{21}^C = \phi\,. \tag{8}$$

Six independent variables $\lambda_{10}^A, \lambda_{32}^A, \lambda_{10}^B, \lambda_{32}^B, \lambda_{10}^C, \lambda_{32}^C$ will be determined by several vector-loop equations as follows

$$\vec{r}_{10}^{\,i} + \sum_{k=1}^{2} q_{k0}^{\mathrm{T}} \;\; \vec{r}_{k+1,\,k}^{\,i} + q_{30}^{\mathrm{T}} \;\; \vec{r}_3^{\,Gi} = \vec{r}_0^{\,G} \;\;\; (q = a, b, c) \;\; (i = A, B, C)\,, \tag{9}$$

where

$$\vec{r}_{10}^{\,i} = \vec{r}_{00}^{\,i} + (l_0/\sqrt{3} + \lambda_{10}^i) q_{10}^{\mathrm{T}} \vec{u}_3\,,\;\; \vec{r}_{00}^{\,A} = l_0 [0 \;\; -1 \;\; 0]^{\mathrm{T}}$$

$$\vec{r}_{00}^{\,B} = 0.5 l_0 [\sqrt{3} \;\; 1 \;\; 0]^{\mathrm{T}},\;\; \vec{r}_{00}^{\,C} = 0.5 l_0 [-\sqrt{3} \;\; 1 \;\; 0]^{\mathrm{T}}$$

$$\vec{r}_{21}^{\,i} = \vec{0},\;\; \vec{r}_{32}^{\,i} = \lambda_{32}^i q_{32}^{\mathrm{T}} \vec{u}_3\,,\;\; \vec{r}_3^{\,Gi} = 0.5 l_0 [0 \;\; 1 \;\; -1/\sqrt{3}]^{\mathrm{T}}$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tilde{u}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

Actually, these vector equations mean that

$$(\frac{l_0}{\sqrt{3}} + \lambda_{10}^i)\cos\alpha_i + \lambda_{32}^i \cos(\phi - \frac{\pi}{3} + \alpha_i) = x_0^G - x_{00}^i - \frac{l_0}{2\sqrt{3}}\cos(\phi + \alpha_i) + \frac{l_0}{2}\sin(\phi + \alpha_i)$$

$$(\frac{l_0}{\sqrt{3}} + \lambda_{10}^i)\sin\alpha_i + \lambda_{32}^i \sin(\phi - \frac{\pi}{3} + \alpha_i) = y_0^G - y_{00}^i - \frac{l_0}{2\sqrt{3}}\sin(\phi + \alpha_i) - \frac{l_0}{2}\cos(\phi + \alpha_i) \tag{11}$$

$$(i = A,\ B,\ C).$$

We develop the inverse kinematics problem and determine the velocities and accelerations of the manipulator, supposing that the planar motion of the moving platform is known. First, we compute the linear and angular velocities of each leg in terms of the angular velocity $\vec{\omega}_0^G = \dot{\phi}\vec{u}_3$ and the centre's velocity $\vec{v}_0^G = \dot{\vec{r}}_0^G$ of the moving platform.

The rotations of the compounding elements of each leg (for example the leg *A*) are characterized by recursive relations of following skew-symmetric matrices

$$\tilde{\omega}_{k0}^A = a_{k,k-1}\tilde{\omega}_{k-1,0}^A a_{k,k-1}^{\mathrm{T}} + \omega_{k,k-1}^A \tilde{u}_3,\ \omega_{k,k-1}^A = \dot{\varphi}_{k,k-1}^A \quad (k = 1,2,3), \tag{12}$$

which are *associated* to the absolute angular velocities

$$\vec{\omega}_{10}^A = \vec{0},\ \vec{\omega}_{20}^A = a_{21}\vec{\omega}_{10}^A + \vec{\omega}_{21}^A = \dot{\phi}\vec{u}_3,\ \vec{\omega}_{30}^A = a_{32}\vec{\omega}_{20}^A + \vec{\omega}_{32}^A = \dot{\phi}\vec{u}_3 \tag{13}$$

Following relations give the velocities $\vec{v}_{k0}^A$ of joints A_k

$$\vec{v}_{10}^A = \dot{\lambda}_{10}^A \vec{u}_3,\ \vec{v}_{21}^A = \vec{0},\ \vec{v}_{32}^A = \dot{\lambda}_{32}^A \vec{u}_3,\ \vec{v}_{k0}^A = a_{k,k-1}\vec{v}_{k-1,0}^A + a_{k,k-1}\tilde{\omega}_{k-1,0}^A \vec{r}_{k,k-1}^A + v_{k,k-1}^A \vec{u}_3. \tag{14}$$

Equations of geometrical constraints (8) and (9) when differentiated with respect to time lead to the following *matrix conditions of connectivity* [23]

$$v_{10}^A \vec{u}_j^{\mathrm{T}} a_{10}^{\mathrm{T}} \vec{u}_3 + v_{32}^A \vec{u}_j^{\mathrm{T}} a_{30}^{\mathrm{T}} \vec{u}_3 = \vec{u}_j^{\mathrm{T}} \dot{\vec{r}}_0^G - \omega_{21}^A \vec{u}_j^{\mathrm{T}} \{\lambda_{32}^A a_{20}^{\mathrm{T}} \tilde{u}_3 a_{32}^{\mathrm{T}} \vec{u}_3 + a_{20}^{\mathrm{T}} \tilde{u}_3 a_{32}^{\mathrm{T}} \vec{r}_3^{GA}\}$$

$$\omega_{21}^A = \dot{\phi} \quad (j = 1, 2), \tag{15}$$

where $\tilde{u}_3$ is a skew-symmetric matrix associated to unit vector $\vec{u}_3$ pointing in the positive direction of z_k axis. From these equations, we obtain the relative velocities v_{10}^A, ω_{21}^A, v_{32}^A as functions of angular velocity of the platform and velocity of mass centre G. But, the conditions (15) give the *complete* Jacobian matrix of the manipulator. This matrix is a fundamental element for the analysis of the robot workspace and the particular configurations of singularities where the manipulator becomes uncontrollable.

By rearranging, the derivatives with respect to time of the six constraint equations (11) lead to the matrix equation

$$J_1 \dot{\vec{\lambda}}_{10} = J_2 [\dot{x}_0^G \quad \dot{y}_0^G \quad \dot{\phi}]^{\mathrm{T}} \tag{16}$$

for the planar robot with prismatic actuators.

The matrices J_1 and J_2 are the inverse and forward Jacobian of the manipulator and can be expressed as

$$J_1 = \mathrm{diag}\{\delta_A \; \delta_B \; \delta_C\}$$

$$J_2 = \begin{bmatrix} \beta_1^A & \beta_2^A & \beta_3^A \\ \beta_1^B & \beta_2^B & \beta_3^B \\ \beta_1^C & \beta_2^C & \beta_3^C \end{bmatrix}, \tag{17}$$

with

$$\delta_i = \sin(\phi - \frac{\pi}{3}) \quad (i = A,\ B,\ C)$$

$$\beta_1^i = \sin(\phi - \frac{\pi}{3} + \alpha_i)$$

$$\beta_2^i = -\cos(\phi - \frac{\pi}{3} + \alpha_i) \tag{18}$$

$$\beta_3^i = r\left[(x_0^G - x_{00}^i - \frac{l_0}{\sqrt{3}}\cos\alpha_i)\cos(\phi - \frac{\pi}{3} + \alpha_i) + \right.$$
$$\left. +(y_0^G - y_{00}^i - \frac{l_0}{\sqrt{3}}\sin\alpha_i)\sin(\phi - \frac{\pi}{3} + \alpha_i) - \lambda_{10}^i \cos(\phi - \frac{\pi}{3}) \right].$$

The three kinds of singularities of the three closed-loop kinematical chains can easily be determined through the analysis of two Jacobian matrices J_1 and J_2 [24, 25].

Now, let us assume that the robot has a *virtual motion* determined first by three linear velocities $v_{10a}^{Av} = 1$, $v_{10a}^{Bv} = 0$, $v_{10a}^{Cv} = 0$. The characteristic virtual velocities are expressed as functions of the position of the mechanism by the general kinematical constraints equations (15). These virtual velocities are required into the computation of the virtual work of all forces applied to the elements of the robot.

As for the relative accelerations γ_{10}^A, ε_{21}^A, γ_{32}^A of the robot, connectivity conditions are obtained by the time derivative of equations in (15), which are [26]

$$\gamma_{10}^A \vec{u}_j^{\mathrm{T}} a_{10}^{\mathrm{T}} \vec{u}_3 + \gamma_{32}^A \vec{u}_j^{\mathrm{T}} a_{30}^{\mathrm{T}} \vec{u}_3 = \vec{u}_j^{\mathrm{T}} \ddot{\vec{r}}_0^G - \omega_{21}^A \omega_{21}^A \vec{u}_j^{\mathrm{T}} \{a_{20}^{\mathrm{T}} \tilde{u}_3 \tilde{u}_3 a_{32}^{\mathrm{T}} \vec{u}_3 + a_{20}^{\mathrm{T}} \tilde{u}_3 \tilde{u}_3 a_{32}^{\mathrm{T}} \vec{r}_3^{GA}\} -$$

$$-\varepsilon_{21}^A \vec{u}_j^{\mathrm{T}} \{\lambda_{32}^A a_{20}^{\mathrm{T}} \tilde{u}_3 a_{32}^{\mathrm{T}} \vec{u}_3 + a_{20}^{\mathrm{T}} \tilde{u}_3 a_{32}^{\mathrm{T}} \vec{r}_3^{GA}\} - 2\omega_{21}^A v_{32}^A \vec{u}_j^{\mathrm{T}} a_{20}^{\mathrm{T}} \tilde{u}_3 a_{32}^{\mathrm{T}} \vec{u}_3 ,$$

$$\varepsilon_{21}^A = \ddot{\phi} \quad (j = 1, 2). \tag{19}$$

The formulations in (15) and (19) are for A only and they also apply to the legs B and C, if the superscript A is replaced by either B or C.

The following recursive relations give the angular accelerations $\vec{\varepsilon}_{k0}^A$ and the accelerations $\vec{\gamma}_{k0}^A$ of joints A_k

$$\vec{\gamma}_{10}^A = \ddot{\lambda}_{10}^A \vec{u}_3, \ \vec{\gamma}_{21}^A = \vec{0}, \ \vec{\gamma}_{32}^A = \ddot{\lambda}_{32}^A \vec{u}_3$$

$$\vec{\varepsilon}_{10}^A = \vec{0}, \ \vec{\varepsilon}_{21}^A = \ddot{\phi} \vec{u}_3, \ \vec{\varepsilon}_{32}^A = \vec{0}$$

$$\vec{\varepsilon}_{k0}^A = a_{k,k-1} \vec{\varepsilon}_{k-1,0}^A + \varepsilon_{k,k-1}^A \vec{u}_3 + \omega_{k,k-1}^A a_{k,k-1} \tilde{\omega}_{k-1,0}^A a_{k,k-1}^{\mathrm{T}} \vec{u}_3$$

$$\tilde{\omega}_{k0}^A \tilde{\omega}_{k0}^A + \tilde{\varepsilon}_{k0}^A = a_{k,k-1} \left(\tilde{\omega}_{k-1,0}^A \tilde{\omega}_{k-1,0}^A + \tilde{\varepsilon}_{k-1,0}^A \right) a_{k,k-1}^{\mathrm{T}} +$$

$$+\omega_{k,k-1}^A \omega_{k,k-1}^A \tilde{u}_3 \tilde{u}_3 + \varepsilon_{k,k-1}^A \tilde{u}_3 + 2\omega_{k,k-1}^A a_{k,k-1} \tilde{\omega}_{k-1,0}^A a_{k,k-1}^{\mathrm{T}} \tilde{u}_3 \tag{20}$$

$$\vec{\gamma}_{k0}^A = a_{k,k-1} \vec{\gamma}_{k-1,0}^A + a_{k,k-1} (\tilde{\omega}_{k-1,0}^A \tilde{\omega}_{k-1,0}^A + \tilde{\varepsilon}_{k-1,0}^A) \vec{r}_{k,k-1}^A +$$

$$+2 v_{k,k-1}^A a_{k,k-1} \tilde{\omega}_{k-1,0}^A a_{k,k-1}^{\mathrm{T}} \vec{u}_3 + \gamma_{k,k-1}^A \vec{u}_3 \quad (k = 1, \ 2, \ 3).$$

The matrix relations (19) and (20) will be further used for the computation of the wrench of the inertia forces for every rigid body of the robot.

For simulation purposes let us consider a planar robot which has the following characteristics:

$$x_0^{G*} = 0.025 \ \text{m} , \ y_0^{G*} = 0.025 \ \text{m} , \ \phi^* = \frac{\pi}{12}$$

$$l_0 = OA_0 = OB_0 = OC_0 = 0.3 \ \text{m}, \ l = l_0 \sqrt{3}$$

$$m_1 = 1 \ \text{kg}, \ m_2 = 1.5 \ \text{kg}, \ m_3 = 3 \ \text{kg}, \ \Delta t = 3 \ \text{s}. \tag{21}$$

A program which implements the suggested algorithm is developed in MATLAB to solve first the inverse kinematics of the planar <u>*P*</u>*RP* parallel robot. For illustration, it is assumed that for a period of three second the platform starts at rest from a central configuration and rotates or moves along two orthogonal directions. A numerical study of the robot kinematics is carried out by computation of the input displacements λ_{10}^A, λ_{10}^B, λ_{10}^C, for example, of three prismatic actuators.

Following examples are solved to illustrate the simulation. For the first example we consider the *rotation motion* of the moving platform about z_0 axis with variable angular acceleration while all the other positional parameters are held equal to zero. As can be seen from Fig. 3 it is remarked that during the rotation of the platform, all displacements of the actuators are identically distributed.

In a second example, the supposed motion of the platform is a *translation* on horizontal x_0 axis (Fig. 4).

Concerning the comparison in the case when the centre G moves along a *rectilinear trajectory* on y_0 axis without any rotation of the platform, we remark that the distribution of displacement calculated by the program and depicted in Fig. 5 is the same, at any instant, for two of three actuators.

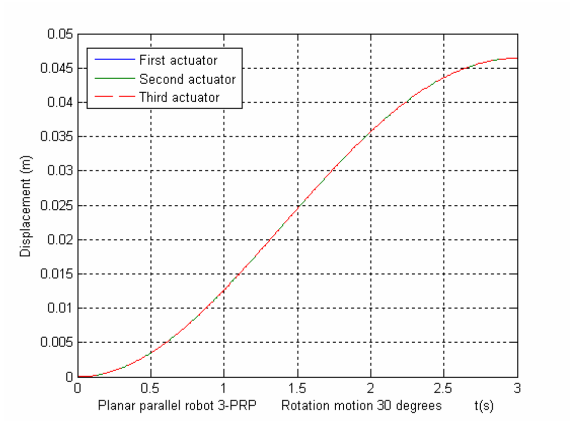

Fig. 3 – Displacements λ_{10}^{A}, λ_{10}^{B}, λ_{10}^{C}.

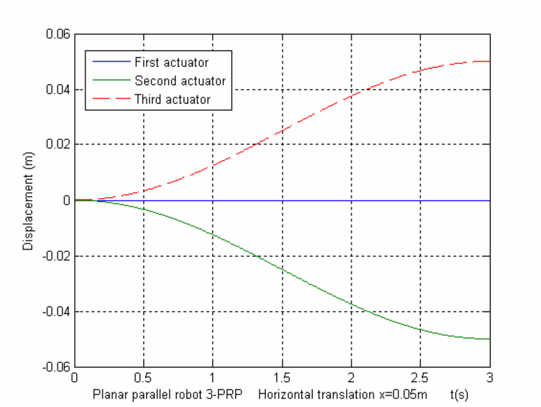

Fig. 4 – Displacements λ_{10}^{A}, λ_{10}^{B}, λ_{10}^{C}.

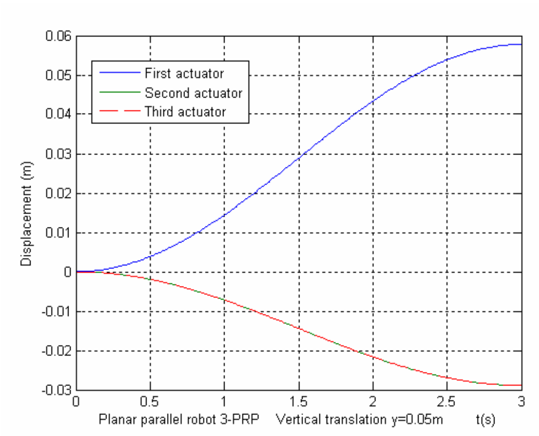

Fig. 5 – Displacements λ_{10}^{A}, λ_{10}^{B}, λ_{10}^{C}.

3. DYNAMICS EQUATIONS

Dynamic analysis of parallel robots is complicated because the existence of a spatial kinematical structure, which possesses a large number of passive degrees of freedom, dominance of the inertial forces, frictional and gravitational components and by the problem linked to real-time control in the inverse dynamics.

Three different methods could lead to the same results concerning the input forces or active torques. The first one is the Newton-Euler approach, which consists to apply the free-body diagram (FBD) procedure for each body where all joint forces and moments are unknown [27, 28, 29, 30]. The second method is based on the Lagrange formalism, which introduces scalar multipliers for each kinematical closure equation [31, 32]. The third method for the dynamics analysis is based on the fundamental principle of virtual work [1, 9, 22, 33, 34].

In the context of the real-time control, neglecting the frictions forces and considering the gravitational effects, the relevant objective of the dynamics is to determine the input forces or the active torques, which must be exerted by the actuators in order to produce a given trajectory of the effectors. Knowing the position and kinematics state of each link as well as the external forces acting on the robot, the torques of actuators or the active forces required in a given motion of the moving platform can be recursively formulated based on the principle of virtual work.

Three independent mechanical systems acting along the planar directions z_1^A, z_1^B, z_1^C with the forces $\vec{f}_{10}^{A} = f_{10}^{A}\vec{u}_3$, $\vec{f}_{10}^{B} = f_{10}^{B}\vec{u}_3$, $\vec{f}_{10}^{C} = f_{10}^{C}\vec{u}_3$ can control the motion of the moving platform. The parallel robot can artificially be transformed in a set of three open chains C_i $(i = A, B, C)$ subject to the constraints. This is possible by cutting each joint for moving platform, and takes its effect into account by introducing the corresponding constraint conditions. The first and more complicated open tree system includes the acting link and could comprise the moving platform.

The force of inertia of an arbitrary rigid body T_k^A, for example,

$$\vec{f}_{k0}^{inA} = -m_k^A \left[\vec{\gamma}_{k0}^A + \left(\tilde{\omega}_{k0}^A \tilde{\omega}_{k0}^A + \tilde{\varepsilon}_{k0}^A \right) \ \vec{r}_k^{CA} \right] \tag{22}$$

and the resulting moment of the forces of inertia

$$\vec{m}_{k0}^{inA} = - \ [m_k^A \tilde{r}_k^{CA} \vec{\gamma}_{k0}^A + \hat{I}_k^A \vec{\varepsilon}_{k0}^A + \tilde{\omega}_{k0}^A \hat{I}_k^A \vec{\omega}_{k0}^A], \tag{23}$$

are determined with respect to the centre of joint A_k. On the other hand, the wrench of two vectors $\vec{f}_k^{*A}$ and $\vec{m}_k^{*A}$ evaluates the influence of the action of the weight $m_k^A \vec{g}$ and of other external and internal forces applied to the same element T_k^A of the manipulator, for example:

$$\vec{f}_k^{*A} = 9.81 m_k^A a_{k0} \vec{u}_2, \ \ \vec{m}_k^{*A} = 9.81 m_k^A \tilde{r}_k^{CA} a_{k0} \vec{u}_2 \quad (k = 1,2,...,6). \tag{24}$$

The fundamental principle of the virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism. During three independent virtual motions of the robot, the virtual displacements and velocities should be compatible to the motions imposed by all kinematical constraints at a given instant in time. By intermediate of the complete Jacobian matrix expressed by the conditions of connectivity (15), the absolute virtual velocities associated with all moving links are related to a set of independent *relative virtual velocities* $\vec{\omega}_{k,k-1}^v = \omega_{k,k-1}^v \vec{u}_3$. The virtual work contributed by the input forces, the inertia forces and moments of inertia forces and by the wrench of known external forces can be written in a compact form, based on the relative virtual angular velocities $\omega_{k,k-1}^v$ only.

Assuming that frictional forces at the joints are negligible, the virtual work produced by all forces of constraint at the joints is zero. Applying *the fundamental equations of the parallel robots dynamics* [35, 36], the following compact matrix relation results

$$f_{10}^A = \vec{u}_3^{\mathrm{T}} \{ \vec{f}_1^A + \omega_{21a}^{Av} \vec{m}_2^A + v_{32a}^{Av} \vec{f}_3^A + \omega_{21a}^{Bv} \vec{m}_2^B + \omega_{21a}^{Cv} \vec{m}_2^C \} \tag{25}$$

for the force of first fixed *prismatic actuator*.

Two recursive relations generate the vectors

$$\begin{aligned}\vec{f}_k^A &= \vec{f}_{k0}^A + a_{k+1,k}^{\mathrm{T}} \vec{f}_{k+1}^A \\ \vec{m}_k^A &= \vec{m}_{k0}^A + a_{k+1,k}^{\mathrm{T}} \vec{m}_{k+1}^A + \tilde{r}_{k+1,k}^A a_{k+1,k}^{\mathrm{T}} \vec{f}_{k+1}^A\end{aligned}, \tag{26}$$

where one denoted

$$\vec{f}_{k0}^A = -\vec{f}_{k0}^{inA} - \vec{f}_k^{*A}, \ \vec{m}_{k0}^A = -\vec{m}_{k0}^{inA} - \vec{m}_k^{*A}. \tag{27}$$

The relations (25) and (26) represent the *inverse dynamics model* of the 3-*PRP* planar parallel manipulator.

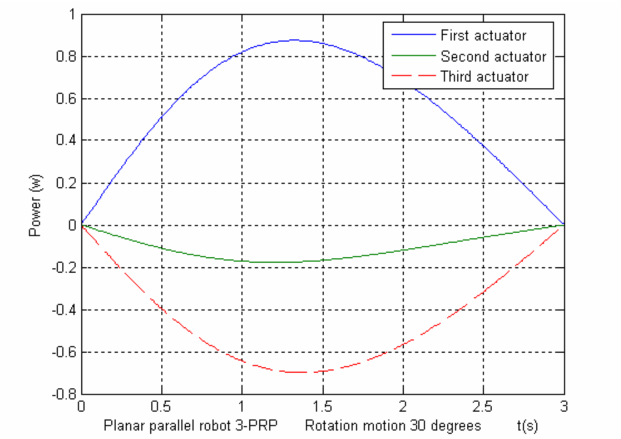

Fig. 6 – Powers $p_{10}^A, p_{10}^B, p_{10}^C$ of three actuators.

Compared with Tsai's analytical method based on the principle of virtual work [1, 6], the advantages of the present approach are the followings:

– Geometrical constraint relations, under matrix form, generate through successive recast the connectivity conditions that will supply all the relative velocities and relative accelerations, which characterize the independent kinematical chains.

– The accelerations of the mass centers, the angular accelerations and the twists of the inertia forces are expressed through matrix formulae, which contain the kinematical characteristics of the relative motion of the building elements of the manipulator.

– A single matrix relation supplies all the virtual velocities.

– The explicit dynamics equation represents a *definitive formula*, obtained by the transformation of the general expression of the virtual work where the relative virtual velocities only appear, generated by recursive relations.

– All intermediate analytical calculations were eliminated and the numerical computation is achieved through the numerical code, for each active force or torque applied by the driving system.

Assuming that there are no external forces and moments acting on the moving platform during its evolution, a numerical computation in the robot dynamics is developed, based on the determination of three active powers p_{10}^A, p_{10}^B, p_{10}^C.

During the *rotation motion* of the platform, the first actuator A only is working with positive values of the absorbed power (Fig. 6). If the platform accomplices a *translation motion* on the horizontal x_0 axis, a power distribution is plotted in Fig. 7.

The powers required by the actuators in the case of a *rectilinear translation* along y_0 axis are calculated by the program and depicted in Fig. 8.

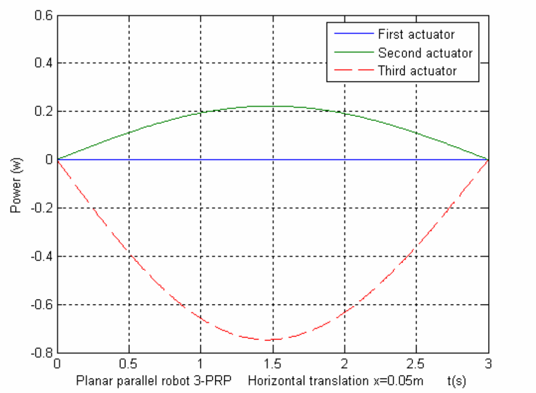

Fig. 7 – Powers $p_{10}^A, p_{10}^B, p_{10}^C$ of three actuators.

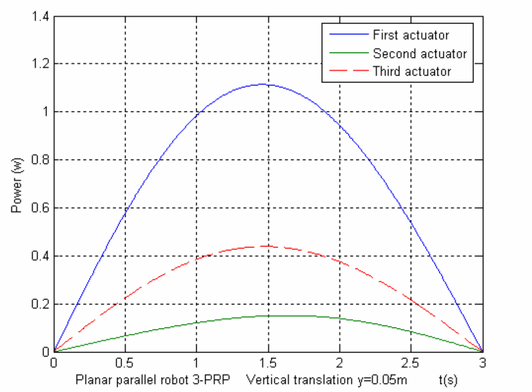

Fig. 8 – Powers $p_{10}^A, p_{10}^B, p_{10}^C$ of three actuators.

4. CONCLUSIONS

The number of relations given by the Lagrange formalism is equal to the total number of the position variables and multipliers inclusive. The commonly known Newton-Euler method, which takes into account the free-body-diagrams of the mechanism, leads to a system of equations with unknowns among which are also the connecting forces in the joints.

Within the inverse kinematics analysis some exact relations that give in real-time the position, velocity and acceleration of each element of the parallel robot have been established in present paper. The dynamics model takes into consideration the masses and forces of inertia introduced by all component elements of the parallel mechanism. The new approach based on the principle of virtual work can eliminate all forces of internal joints and establishes a direct determination of the time-history evolution of powers required by the actuators. Also, the method described above is quit available in forward and inverse mechanics of all serial or planar parallel mechanisms, the platform of which behaves in translation, rotation evolution or general 3-DOF motion.

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