INVERSE DYNAMICS OF THE CINCINNATI-MILACRON WRIST ROBOT

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Recursive matrix relations in dynamics of the Cincinnati-Milacron wrist robot are established in this paper. The prototype of this mechanism is a three-degrees-of-freedom spherical system with six moving links and three bevel gear pairs. Controlled by electric motors, three active elements of the robot have three independent rotations. Supposing that the position and the rotation motion of the end-effector are known, the inverse dynamic problem is solved using an approach based on the principle of virtual work, but the results have been verified in the framework of the Lagrange equations of second kind. Finally, some recursive matrix relations and some graphs for the torques and the powers of the actuators are obtained.

1. INTRODUCTION

The orienting mechanisms are incorporated in the structure of industrial robots and have two or three output rotations. Generally, these mechanical systems have conical and cylindrical toothed elements in their structure, while the input axes are parallel and the output axes are orthogonal. The three orientation motions are usually performed around the axes of a Cartesian orthogonal frame, having its axes linked to the last arm of the robot's positioning mechanism.

The industrial robots with orienting gear trains can perform several operations such as welding, flame cutting, spray painting, milling or assembling. Being comparatively simple and compact in size, the bevel-gear wrist mechanisms can be sealed in a metallic box that keeps the device of contamination. Furthermore, using bevel gear trains for power transmission, the actuators can be mounted remotely on the forearm, thereby reducing the weight and inertia of a robot manipulator. Planetary gear trains with three degrees of freedom are adopted as the design concept for robotic wrist (Hsieh and Sheu [1]; Paul and Stevenson [2]; Willis [3]; Ma and Gupta [4]; White [5]; Stackhouse [6]). Generally, the number of actuators is typically equal to the number of degrees of freedom such that even kinematical chain can be controlled at or near the fixed base.

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Rev. Roum. Sci. Techn. – Méc. Appl. Tome 55, No. 1, P. 73– 87, Bucarest, 2010

2. GEOMETRIC MODELLING

Recursive matrix relations for inverse kinematics of a Cincinnati-Milacron planetary bevel-gear mechanism, which has a non-symmetrical kinematical schema, are first developed in the paper. The end-effectors must rotate around three non-orthogonal axes so that the orienting wrist mechanism has three degrees of freedom. A matrix methodology for the kinematics analysis based on the concept of *fundamental circuit* of an open-loop chain is presented. This method involves the identification of all open-loop chains and the derivation of the geometric relationships between the orientation of the end-effector and the joint angles of compounding links, including the input actuator displacements [7, 8, 9].

Let $Ox_0y_0z_0$ be a fixed Cartesian frame, about which the oblique wrist mechanism moves (Fig. 1). The mechanism topology consists of six moving links, six revolute joints and three bevel gear pairs. Therefore, the wrist is a 3-DOF spherical mechanism, which has non-limited rotational ranges about the joint axes. There are two active gears $1b$, $1c$ of radii $r_1^B = r_1, r_1^C = r_5$, masses m_1^B, m_1^C and inertia tensors $\hat{J}_1^B, \hat{J}_1^C$, and three kinematical chains $0-1a-2a-3a-4a$, $0-1b-2b$ and $0-1c-2c$ are identified starting from the forearm and ending to a common element as end-effector $e=4a$. A significant feature of this wrist device is that there is no physical interference between all the links.

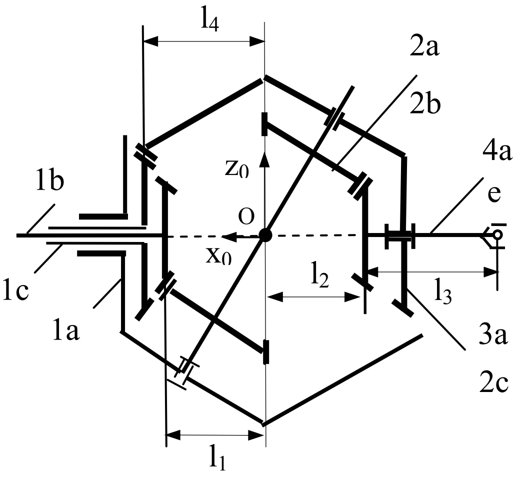

Fig. 1 – Cincinnati-Milacron wrist robot.

In the Cincinnati-Milacron wrist, the link $1a$ as a box of characteristic length l_0, mass m_1^A and tensor of inertia $\hat{J}_1^A$, one of the three driving parts of the spherical mechanism, serves as carrier for the $1b-2b$ and $1c-2c$ bevel gear pairs, while the link $3a$, of characteristic length r, mass m_3^A and tensor of inertia $\hat{J}_3^A$, serves as carrier for the $2a-4a$ bevel gear pair. The rotation of gear $1c$ is transmitted to link $3a$ via a special gear pair, while the rotation of second active gear $1b$ is transmitted to gear $4a$ through two coaxial gears.

The two gears of radii r_2, r_3, total mass m_2^A and total inertia tensor $\hat{J}_2^A$ are planet gears rigidly connected together by a single shaft of known length that is housed in both carriers $1a$ and $3a$. The axes of rotation of three motors are supported by bearings housed in the forearm.

Including the end-effector of length l_3 at the last gear $4a = e$ of radius $r_4^A = r_4$, we obtain an assembly of total mass m_4^A and total tensor of inertia $\hat{J}_4^A$ that is free to arbitrary undergo three concurrent rotations with respect to the common center O. Concerning the output motion of the end-effector, we remark that the wrist *roll motion* is achieved by rotating link $1a$ about z_1^A-axis while the *pitch motion* is accomplished by relative rotating link $3a$ with respect to link $1a$ about the z_3^A-axis. In the followings, we apply the method of *successive displacements* to geometric analysis of closed-loop chains and we note that a joint variable is the displacement required to move a link from the initial location to the actual position. If every link is connected to at least two other links, the chain forms one or more independent closed-loops.

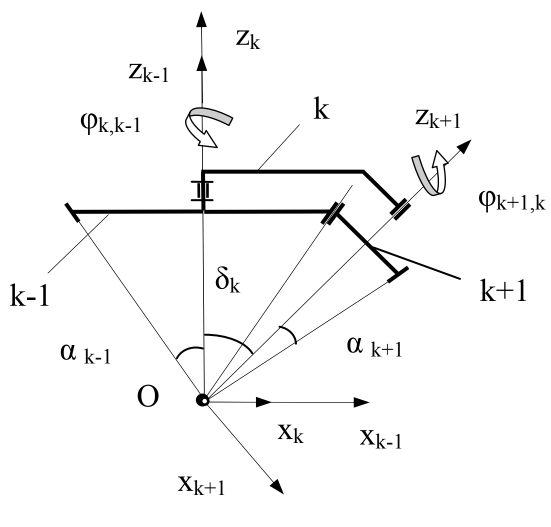

Fig. 2 – Gear fundamental circuit.

The variable angles $\varphi_{k,k-1}$ of rotation about the joint axis z_k are the parameters needed to bring the next link from a reference configuration to the next configuration. We call the matrix $a^{\varphi}_{k,k-1}$, for example, the orthogonal transformation 3×3 matrix of relative rotation with the angle $\varphi^{A}_{k,k-1}$ of link T^{A}_{k} around z^{A}_{k} axis. In the study of the kinematics on constrained systems, we are interested in deriving a matrix equation relating the location of an arbitrary T_k body to the joint variables. When the change of coordinates is successively considered, the corresponding matrices are multiplied. In what follows, we introduce a matrix approach that utilizes the *theory of fundamental circuits* aroused by Tsai [7]. There exists a *real* or *fictitious* carrier for every gear pair in a planetary gear train and a fundamental matrix equation for each loop can be written as

$$a_{k+1,k-1} = a^{\varphi}_{k+1,k}\theta^{\delta_k}_{y} a_{k,k-1}, \quad \varphi_{k,k-1} = n_{k+1,k-1}\varphi_{k+1,k}, \quad \delta_k = \alpha_{k-1} + \alpha_{k+1},$$

$$\theta^{\delta_k}_{y} = \begin{bmatrix} \cos\delta_k & 0 & -\sin\delta_k \\ 0 & 1 & 0 \\ \sin\delta_k & 0 & \cos\delta_k \end{bmatrix}, \tag{1}$$

where $\varphi_{k,k-1}$ and $\varphi_{k+1,k}$ denote the relative angles of rotation of the carrier T_k and the planet gear T_{k+1}, respectively, while α_{k-1}, α_{k+1} are the angles that characterize the geometry of the connected gears T_{k-1} and T_{k+1} (Fig. 2).

The ratio of a gear pair is defined as

$$n_{k+1,k-1} = r_{k+1}/r_{k-1} = z_{k+1}/z_{k-1}, \tag{2}$$

where r_{k-1}, r_{k+1} and z_{k-1}, z_{k+1} are the radius and the number of teeth of the two gears, respectively (Fig. 3).

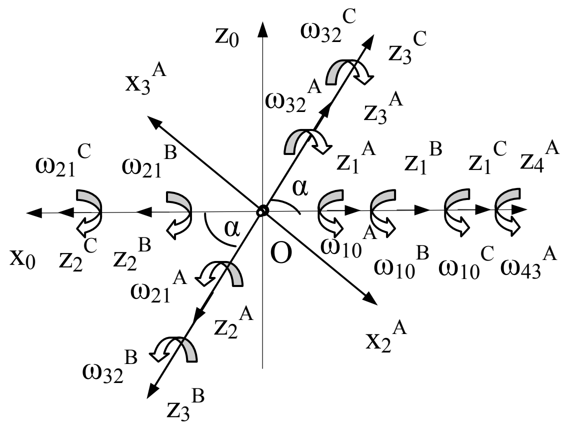

Fig. 3 – Moving frames sequence.

The motions of the six parts of the wrist mechanism are all concurrent rotations around the fixed point O. To simplify the graphical image of the kinematical scheme of the spherical mechanism, in what follows we will represent the intermediate reference systems by only two axes, so as is used in most of robotics papers. The z_k axis is represented, of course, for each component T_k. It is noted that the relative rotation with angle $\varphi_{k,k-1}$ of the body T_k must always be pointed about the direction of the z_k axis. Consequently, four appropriate frames for the first circuit A, three frames for the second kinematical chain B and three frames for the last circuit C are fixed in a same origin O (Fig. 3). Let us consider the rotations angles $\varphi_{10}^{A}, \varphi_{10}^{B}, \varphi_{10}^{C}$ of the three actuators A_1, B_1, C_1 as variables that give the instantaneous position of the mechanism.

Starting from the reference origin O and pursuing the independent serial circuits A, B, C: $0-1a-2a-3a-4a$, $0-1b-2b$ and $0-1c-2c$, we obtain the following successive matrices of transformation [10, 11]:

$$
\begin{aligned}
&a_{10} = a_{10}^{\varphi}\theta_1,\ a_{21} = a_{21}^{\varphi}a_{\alpha}\theta_2,\ a_{32} = a_{32}^{\varphi}\theta_2,\ a_{43} = a_{43}^{\varphi}a_{\alpha}^{T}\\
&b_{10} = b_{10}^{\varphi}\theta_1, b_{21} = b_{21}^{\varphi}\theta_2, b_{32} = b_{32}^{\varphi}a_{\alpha}\\
&c_{10} = c_{10}^{\varphi}\theta_1,\ c_{21} = c_{21}^{\varphi}\theta_2,\ c_{32} = c_{32}^{\varphi}a_{\alpha}\theta_2,
\end{aligned}
\tag{3}
$$

where are denoted

$$
\theta_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \theta_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ a_{\alpha} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix},
$$

$$
p_{k,k-1}^{\varphi} = \begin{bmatrix} \cos\varphi_{k,k-1}^{i} & \sin\varphi_{k,k-1}^{i} & 0 \\ -\sin\varphi_{k,k-1}^{i} & \cos\varphi_{k,k-1}^{i} & 0 \\ 0 & 0 & 1 \end{bmatrix} \ (p = a,b,c)\ (i = A,B,C), \tag{4}
$$

$$
p_{k0} = \prod_{s=1}^{k} p_{k-s+1,k-s}\ ,\ (k = 1, 2, 3, 4).
$$

In what follows we consider that the end-effector is initially located at a *central configuration*, where this is not rotated with respect to the fixed base. In the inverse geometric problem however, a complete description of the orientation of the end-effector in the fixed frame is known by intermediate of three Euler angles ϕ_1, ϕ_2, ϕ_3 associated with three successive rotations, which can be expressed by the analytical functions

$$
\phi_l = \phi_l^{*}[1-\cos(\frac{\pi}{6}\ t)]\ \ (l = 1,\ 2,\ 3), \tag{5}
$$

where $2\phi_l^*$ represents the maximum of the angle of rotation ϕ_l.

Since all rotations take place successively about the moving coordinate axes, the general rotation matrix $R = d_{32}d_{21}d_{10}$ of the end-effector from $Ox_0y_0z_0$ to $Ox_4^Ay_4^Az_4^A$ reference system is obtained by multiplying three transformation matrices

$$d_{10} = a_1\theta_1,\ d_{21} = a_2a_\alpha,\ d_{32} = a_3a_\alpha^T, \tag{6}$$

where

$$a_l = \begin{bmatrix} \cos\phi_l & \sin\phi_l & 0 \\ -\sin\phi_l & \cos\phi_l & 0 \\ 0 & 0 & 1 \end{bmatrix} \ (l = 1, 2, 3). \tag{7}$$

Constraint geometric conditions for the rotation of the end-effector are given by the identities

$$a_{40} = R\,, a_{20} = b_{30}\,, a_{30} = c_{30}\,. \tag{8}$$

From these equations, we obtain the real-time evolution of all characteristic joint angles, as follows:

$$\begin{aligned}
&\varphi_{10}^A = \phi_1,\ \varphi_{21}^A = -\phi_2 + \frac{\phi_3}{n_2},\ \varphi_{32}^A = \frac{\phi_3}{n_2},\ \varphi_{43}^A = \phi_3, \\
&\varphi_{10}^B = \phi_1 - \frac{\phi_2}{n_1} + \frac{\phi_3}{n_1n_2},\ \varphi_{21}^B = -\frac{\phi_2}{n_1} + \frac{\phi_3}{n_1n_2},\ \varphi_{32}^B = -\phi_2 + \frac{\phi_3}{n_2}, \\
&\varphi_{10}^C = \phi_1 + \frac{\phi_2}{n_3}, \varphi_{21}^C = \frac{\phi_2}{n_3}, \varphi_{32}^C = \phi_2, \\
&n_1 = \frac{r_1}{r_2},\ n_2 = \frac{r_3}{r_4},\ n_3 = \frac{r_5}{r}.
\end{aligned} \tag{9}$$

3. KINEMATICS OF THE WRIST ROBOT

The analysis of the kinematics of bevel-gear wrist mechanisms of gyroscopic structure is very complex, due to the fact that the carriers and planet gears may possess simultaneous angular velocities about nonparallel axes. In the design of power transmission mechanisms, it is often necessary to analyze the velocity ratios between their input and output parts and angular velocities or angular accelerations of the intermediate parts.

The conventional tabular or analytical method, which concentrates on planar epicyclical gear trains, is no longer applicable. To overcome this difficulty, Freudenstein, Longman and Chen [12] applied the dual relative velocity and dual matrix of transformation for the analysis of epicyclical bevel-gear trains. Tsai, Chen and Lin [13], Chang and Tsai [14] and Hedman [15] showed that the kinematical analysis of geared robotic mechanisms can be accomplished by applying the theory of fundamental circuits.

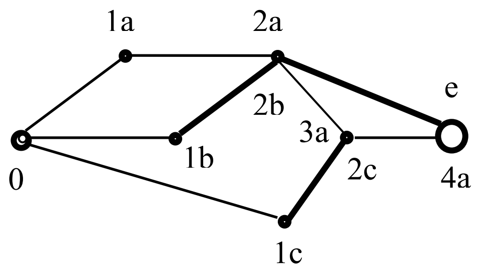

Fig. 4 – Associated graph of the mechanism.

Since a kinematical chain is an assemblage of links and joints, these can be symbolized in a more abstract form known as equivalent graph representation (Fig. 4). For the reason that will be clear later we use the *associated graph* to represent the topology of the mechanism. In the kinematical graph representation we denote the links by vertices and the joints by edges (Yan and Hsieh [16, 17]. Two small concentric circles label the vertex denoting the fixed forearm 0.

To distinguish the difference between the pair types, in the graph representation of the Cincinnati-Milacron wrist the three gear pairs $1b-2b$, $1c-2c$, $2a-4a$ are drawn in heavy edges and the revolute joints $0-1a$, $1a-2a$, $2a-3a$, $3a-4a$, $0-1b$ and $0-1c$ are sketched in thin edges. There are three significant independent loops and we identify three serial kinematical chains.

The kinematics of an element for each circuit is characterized by skew-symmetric matrices given by the recursive relations [18]:

$$\tilde{\omega}_{k0}^{i} = p_{k,k-1}\tilde{\omega}_{k-1,0}^{i} p_{k,k-1}^{T} + \tilde{\omega}_{k,k-1}^{i}, \tilde{\omega}_{k,k-1}^{i} = \varphi_{k,k-1}^{i}\tilde{u}_3,$$
$$p = a,b,c;\ i = A,\ B,C, \tag{10}$$

where $\tilde{u}_3$ is a skew-symmetric matrix associated with the unit vector $\vec{u}_3 - [0 \quad 0 \quad 1]^T$. These matrices are *associated* to the angular velocities

$$\vec{\omega}_{k0}^{i} = p_{k,k-1}\vec{\omega}_{k-1,0}^{i} + \vec{\omega}_{k,k-1}^{i}, \vec{\omega}_{k,k-1}^{i} = \dot{\varphi}_{k,k-1}^{i}\vec{u}_3. \tag{11}$$

Knowing the general rotation motion of the end-effector attached at the planet gear $4a$ by the relations (5), one develops the inverse kinematical problem and determines the absolute angular velocity and acceleration $\vec{\omega}_{k0}^{i}, \vec{\varepsilon}_{k0}^{i}$ of each moving link.

Based on the important remark

$$\omega_{k,k-1} = n_{k+1,k-1}\omega_{k+1,k}\,, \tag{12}$$

the derivatives with respect to time of the relations (9) lead to the relative angular velocities of all links as function of the angular velocities $\omega_1 = \dot{\phi}_1$, $\omega_2 = \dot{\phi}_2$, $\omega_3 = \dot{\phi}_3$ of the end-effector

$$\begin{aligned}
&\omega_{10}^{A} = \dot{\phi}_1\,,\ \omega_{21}^{A} = -\dot{\phi}_2 + \frac{\dot{\phi}_3}{n_2}\,,\ \omega_{32}^{A} = \frac{\dot{\phi}_3}{n_2}\,,\ \omega_{43}^{A} = \dot{\phi}_3\,,\\
&\omega_{10}^{B} = \dot{\phi}_1 + \frac{1}{n_1}(-\dot{\phi}_2 + \frac{\dot{\phi}_3}{n_2}),\ \omega_{21}^{B} = \frac{1}{n_1}(-\dot{\phi}_2 + \frac{\dot{\phi}_3}{n_2}),\ \omega_{32}^{B} = -\dot{\phi}_2 + \frac{\dot{\phi}_3}{n_2}\,,\\
&\omega_{10}^{C} = \dot{\phi}_1 + \frac{\dot{\phi}_2}{n_3}\,,\ \omega_{21}^{C} = \frac{\dot{\phi}_2}{n_3}\,,\ \omega_{32}^{C} = \dot{\phi}_2\,.
\end{aligned} \tag{13}$$

Starting from the relationship $\vec{\omega}_{10} = J\vec{\omega}$, the expression of the Jacobian matrix is easily written in an invariant form

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/n_1 & 1/(n_1 n_2) \\ 0 & 1/n_3 & 0 \end{bmatrix},\quad J^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -n_3 & 0 & n_3 \\ -n_2(n_1+n_3) & n_1 n_2 & n_2 n_3 \end{bmatrix}, \tag{14}$$

where $\vec{\omega}_{10} = [\omega_{10}^{A} \quad \omega_{10}^{B} \quad \omega_{10}^{C}]^T$ denotes the input vector of three angular velocities and $\vec{\omega} = [\dot{\phi}_1 \quad \dot{\phi}_2 \quad \dot{\phi}_3]^T$ is the vector of joint angular velocities of the end-effector. This square invertible matrix is an essential element for the analysis of the loci into robot workspace.

Let us assume now that the robot has successively three independent virtual motions. Characteristic virtual velocities expressed as function of mechanism's position are given by the relations (13). First, we consider the following input angular velocities $\omega_{10a}^{Av} = 1, \omega_{10a}^{Bv} = 0, \omega_{10a}^{Cv} = 0$ and we obtain a set of virtual velocities:

$$\omega_{21a}^{Av} = -n_1\,,\ \omega_{32a}^{Av} = -(n_1 + n_3)\,,\ \omega_{43a}^{Av} = -n_2(n_1 + n_3)\,. \tag{15}$$

A second virtual motion is defined by the input velocities $\omega_{10b}^{Bv} = 1, \omega_{10b}^{Cv} = 0$, $\omega_{10b}^{Av} = 0$ and the following results:

$$\omega_{21b}^{Av} = n_1\,,\ \omega_{32b}^{Av} = n_1\,,\ \omega_{43b}^{Av} = n_1 n_2\,. \tag{16}$$

Finally, from the third virtual motion $\omega_{10c}^{Cv} = 1$, $\omega_{10c}^{Av} = 0$, $\omega_{10c}^{Bv} = 0$ we obtain

$$\omega_{21c}^{Av} = 0\,,\ \omega_{32c}^{Av} = n_3\,,\ \omega_{43c}^{Av} = n_2 n_3\,. \tag{17}$$

Concerning the relative angular accelerations of the compounding elements of the mechanism, these are immediately given by deriving the relations of the velocities (13): $\varepsilon^i_{k,k-1} = \dot{\omega}^i_{k,k-1}$.

The angular accelerations $\vec{\varepsilon}^{\,i}_{k0}$ and the useful square matrices $\tilde{\omega}^i_{k0}\tilde{\omega}^i_{k0} + \tilde{\varepsilon}^i_{k0}$ are calculated with the following formulae [19]:

$$\vec{\varepsilon}^{\,i}_{k0} = p_{k,k-1}\vec{\varepsilon}^{\,i}_{k-1,0} + \varepsilon^i_{k,k-1}\vec{u}_3 + \omega^i_{k,k-1}p_{k,k-1}\tilde{\omega}^i_{k-1,0}p^T_{k,k-1}\vec{u}_3\,, \tag{18}$$

$$\tilde{\omega}^i_{k0}\tilde{\omega}^i_{k0} + \tilde{\varepsilon}^i_{k0} = p_{k,k-1}\left(\tilde{\omega}^i_{k-1,0}\tilde{\omega}^i_{k-1,0} + \tilde{\varepsilon}^i_{k-1,0}\right)p^T_{k,k-1} +$$
$$+\omega^i_{k,k-1}\omega^i_{k,k-1}\tilde{u}_3\tilde{u}_3 + \varepsilon^i_{k,k-1}\tilde{u}_3 + 2\omega^i_{k,k-1}p_{k,k-1}\tilde{\omega}^i_{k-1,0}p^T_{k,k-1}\tilde{u}_3.$$

The velocity $\vec{v}^{\,Ci}_k$ and the acceleration $\vec{\gamma}^{\,Ci}_k$ of mass centre of T^i_k rigid body are calculated from two known matrix relations

$$\vec{v}^{\,Ci}_k = \tilde{\omega}^i_{k0}\vec{r}^{\,Ci}_k\,,\ \vec{\gamma}^{\,Ci}_k = \{\tilde{\omega}^i_{k0}\tilde{\omega}^i_{k0} + \tilde{\varepsilon}^i_{k0}\}\vec{r}^{\,Ci}_k\,, \tag{19}$$

where

$$z^{CA}_1 = 0\,,\ z^{CA}_2 = 0.5(l_1 - l_2)\,,\ z^{CA}_3 = r/3\,,\ z^{CA}_4 = l_2 + 0.8l_3$$
$$z^{CB}_1 = -l_1\,, z^{CC}_1 = -l_4\,, \vec{r}^{\,Ci}_k = [0 \quad 0 \quad z^{Ci}_k]^T\,, (i = A, B, C)\,. \tag{20}$$

4. DYNAMICS EQUATIONS

4.1. PRINCIPLE OF VIRTUAL WORK

Three torques of moment $\vec{m}^A_{10} = m^A_{10}\vec{u}_3\,, \vec{m}^B_{10} = m^B_{10}\vec{u}_3\,, \vec{m}^C_{10} = m^C_{10}\vec{u}_3$ can control by intermediate of electric motors the motion of the wrist mechanism. The derivation of a dynamic model has a very important effect in the determination of the actuator torques (Tsai [20]; Muller, Mannhardt and Glover [21]; Castillo [22]).

In the inverse dynamic problem, in the present paper, one applies the principle of virtual work in order to establish some recursive matrix relations for the torques and the powers of the active systems. The parallel mechanism can artificially be transformed in a set of three open serial chains C_i $(i = A, B, C)$ subjected to the constraints. This is possible by cutting open at three bevel gear pairs $2a - 4a$, $2a - 1b$ and $3a - 1c$ and taking their effects into account by introducing the corresponding constraint conditions.

Considering that the end-effector motion is given, the position, angular velocity, angular acceleration as well as the velocity and acceleration of the centre of mass are known of each element. The force of inertia and the resulting moment of the forces of inertia of an arbitrary rigid body T^A_k, for example

$$\vec{f}^{\,inA}_{k0} = -m^A_k(\tilde{\omega}^A_{k0}\tilde{\omega}^A_{k0} + \tilde{\varepsilon}^A_{k0})\vec{r}^{\,CA}_k\,,\ \vec{m}^{inA}_{k0} = -\hat{J}^A_k\vec{\varepsilon}^{\,A}_{k0} - \tilde{\omega}^A_{k0}\hat{J}^A_k\vec{\omega}^A_{k0} \tag{21}$$

are determined with respect to the common centre of rotation O. On the other hand, the wrench of two vectors $\vec{f}_k^{*A}$ and $\vec{m}_k^{*A}$ evaluates the influence of the action of the external and internal forces applied to the same element T_k^A or of its weight $m_k^A \vec{g}$, for example:

$$\vec{f}_k^{*A} = m_k^A g a_{k0} \vec{u}_3 ,\ \vec{m}_k^{*A} = m_k^A g \tilde{r}_k^{CA} a_{k0} \vec{u}_3 \quad (k = 1,2,...,6). \tag{22}$$

Finally, two recursive relations generate the vectors

$$\begin{aligned} \vec{F}_k^A &= \vec{F}_{k0}^A + a_{k+1,k}^T \vec{F}_{k+1}^A , \\ \vec{M}_k^A &= \vec{M}_{k0}^A + a_{k+1,k}^T \vec{M}_{k+1}^A + \tilde{r}_{k+1,k}^A a_{k+1,k}^T \vec{F}_{k+1}^A , \end{aligned} \tag{23}$$

where one denoted

$$\vec{F}_{k0}^A = -\vec{f}_{k0}^{inA} - \vec{f}_k^{*A} ,\ \vec{M}_{k0}^A = -\vec{m}_{k0}^{inA} - \vec{m}_k^{*A} . \tag{24}$$

In the context of the real-time control, neglecting the frictional forces and considering the gravitational effect and the action of a resistant torque M_r through the vectors $\vec{f}_{2r}^{*E} = \vec{0}$, $\vec{m}_{2r}^{*E} = -M_r \vec{u}_3$, the relevant objective of a dynamic model is to determine the input torques, which must be exerted by the actuators in order to produce a given trajectory of the end-effector.

The fundamental principle of virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism. Applying *the fundamental equations of parallel robots dynamics* [23], following compact matrix relations results for the torques of three actuators

$$\begin{aligned} m_{10}^A &= \vec{u}_3^T \{\omega_{10a}^{Av} \vec{M}_1^A + \omega_{21a}^{Av} \vec{M}_2^A + \omega_{32a}^{Av} \vec{M}_3^A + \omega_{43a}^{Av} \vec{M}_4^A \} , \\ m_{10}^B &= \vec{u}_3^T \{\omega_{10b}^{Bv} \vec{M}_1^B + \omega_{21a}^{Av} \vec{M}_2^A + \omega_{32a}^{Av} \vec{M}_3^A + \omega_{43b}^{Av} \vec{M}_4^A \} , \\ m_{10}^C &= \vec{u}_3^T \{\omega_{10c}^{Cv} \vec{M}_1^C + \omega_{21c}^{Av} \vec{M}_2^A + \omega_{32c}^{Av} \vec{M}_3^A + \omega_{43c}^{Av} \vec{M}_4^A \} . \end{aligned} \tag{25}$$

The relations (23) and (25) represent the *inverse dynamic model* of the Cincinnati-Milacron wrist mechanism. The procedure leads to very good estimates of the actuators torques for given displacement of end-effector, provided that the inertial properties of the gears are known with sufficient accuracy and that friction is not significant. This new dynamic approach developed here can be extended to any gyroscopic bevel-gear train with revolute actuators.

4.2. EQUATIONS OF LAGRANGE

A solution of the dynamics problem of the Cincinnati-Milacron mechanism can be developed based on the Lagrange equations of second kind. The generalized coordinates of the robot are represented by the rotation angles of the three actuators: $q_1 = \varphi_{10}^A$, $q_2 = \varphi_{10}^B$, $q_3 = \varphi_{10}^C$.

The Lagrange's equations will be expressed by three differential relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\frac{\partial L}{\partial \dot{q}_j}\} - \frac{\partial L}{\partial q_j} = Q_j \quad (j = 1,\ 2,\ 3)\,, \tag{26}$$

that contain following generalized forces

$$Q_1 = m_{10}^A + M_r n_2 (n_1 + n_3)\,, Q_2 = m_{10}^B - M_r n_1 n_2\,, Q_3 = m_{10}^C - M_r n_2 n_3\,. \tag{27}$$

The components of the general expression $L = \sum_{k=1}^{4} L_k^A + L_1^B + L_1^C$ of the *Lagrange function* are expressed as analytical functions of the generalized coordinates and their first derivatives with respect to time:

$$L_k^A = \frac{1}{2}\vec{\omega}_{k0}^{AT}\hat{J}_k^A\vec{\omega}_{k0}^A - m_k^A g \vec{u}_3^T a_{k0}^T \vec{r}_k^{CA}\,,$$

$$L_1^B = \frac{1}{2}\vec{\omega}_{10}^{BT}\hat{J}_1^B\vec{\omega}_{10}^B\,, L_1^C = \frac{1}{2}\vec{\omega}_{10}^{CT}\hat{J}_1^C\vec{\omega}_{10}^C\,, \tag{28}$$

where the angular velocities have the expressions:

$$\vec{\omega}_{10}^A = \dot{q}_1\vec{u}_3\,, \vec{\omega}_{21}^A = -n_1(\dot{q}_1 - \dot{q}_2)\vec{u}_3\,,\ \vec{\omega}_{32}^A = -(n_1 + n_3)(\dot{q}_1 - \dot{q}_3)\vec{u}_3\,,$$

$$\vec{\omega}_{43}^A = -n_2[(n_1 + n_3)\dot{q}_1 - n_1\dot{q}_2 - n_3\dot{q}_3]\vec{u}_3\ \vec{\omega}_{10}^B = \dot{q}_2\vec{u}_3\,, \vec{\omega}_{10}^C = \dot{q}_3\vec{u}_3\,. \tag{29}$$

The absolute angular velocities and the first derivatives of orthogonal matrices $p_{k,k-1}$ are expressed as follows:

$$\vec{\omega}_{20}^A = a_{21}\vec{\omega}_{10}^A + \vec{\omega}_{21}^A\,,\ \vec{\omega}_{30}^A = a_{32}\vec{\omega}_{20}^A + \vec{\omega}_{32}^A\,,\ \vec{\omega}_{40}^A = a_{43}\vec{\omega}_{30}^A + \vec{\omega}_{43}^A\,,$$

$$\dot{p}_{k,k-1} = \dot{\varphi}_{k,k-1}^i \tilde{u}_3^T p_{k,k-1}\,, \dot{p}_{k,k-1}^T = \dot{\varphi}_{k,k-1}^i p_{k,k-1}^T \tilde{u}_3\,, \tag{30}$$

$$\frac{\partial p_{k,k-1}}{\partial \varphi_{k,k-1}^{i}} = \tilde{u}_3^T p_{k,k-1}\,,\ \frac{\partial p_{k,k-1}^T}{\partial \varphi_{k,k-1}^{i}} = p_{k,k-1}^T \tilde{u}_3 \ \ (p = a,\ b,\ c)\,.$$

In the inverse dynamics problem, a long calculus of the derivatives with respect to time $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$ $(j = 1,\ 2,\ 3)$ of above functions leads finally to the same expressions (25) for the input torques m_{10}^A, m_{10}^C, m_{10}^C required by actuators.

For simulation purposes let us consider a mechanism which has the following characteristics

$$l_0 = 0.075\,\mathrm{m},\ l_1 = 0.045\,\mathrm{m},\ l_2 = 0.035\,\mathrm{m},\ l_3 = 0.035\,\mathrm{m},\ l_4 = 0.055\,\mathrm{m},$$

$$r = 0.065\ \mathrm{m}, r_1 = 0.025\,\mathrm{m}, r_2 = 0.025\,\mathrm{m}\ , r_3 = 0.02\,\mathrm{m}, r_4 = 0.02\,\mathrm{m}\ , r_5 = 0.03\,\mathrm{m},$$

$$m_1^A = 0.75\,\mathrm{kg}, m_2^A = 0.6\,\mathrm{kg}, m_3^A = 0.6\,\mathrm{kg}, m_4^A = 1.05\,\mathrm{kg}, m_1^B = 0.35\,\mathrm{kg},\ m_1^C = 0.4\,\mathrm{kg},$$

$$\phi_1^* = \frac{\pi}{2}, \phi_2^* = \frac{\pi}{3}\pi, \phi_3^* = \pi\ ,\ \alpha = \frac{\pi}{3}\ ,\ M_r = 0.05\ \mathrm{Nm}\ ,\ \Delta t = 6\ \mathrm{s}.$$

Based on the algorithm derived from the above relations (23), (25), a computer program solves the inverse dynamics modelling of the mechanism, using the MATLAB software. Assuming that a resistant torque of constant moment $M_r = 0.05\,\mathrm{Nm}$ applied at the end-effector and the weights $m_k^i \vec{g}$ of compounding rigid bodies constitute the external forces acting on the mechanism during its evolution, a numerical computation in the dynamics is developed, based on the determination of the three input torques m_{10}^A, m_{10}^B, m_{10}^C (Fig. 5) and their active powers $P_{10}^A = \omega_{10}^A m_{10}^A$, $P_{10}^B = \omega_{10}^b m_{10}^B$ and $P_{10}^C = \omega_{10}^C m_{10}^C$ (Fig. 6).

5. CONCLUSIONS

Within the inverse kinematics analysis, some exact matrix relations giving the position, velocity and acceleration of each link for the Cincinnati-Milacron 3-DOF wrist mechanism have been established.

Most of dynamical models based on the Lagrange formalism neglect the weight of intermediate bodies and take into consideration the active forces or moments only and the wrench of applied forces on the end-effector. The number of relations given by this approach is equal to the total number of the position variables and Lagrange multipliers inclusive. Also, the analytical calculations involved in these equations present a risk of errors. The commonly known Newton-Euler method, which takes into account the free-body-diagrams of the mechanism, leads to a large number of equations with unknowns including also the connecting forces in the joints. Finally, the actuating torques could be obtained.

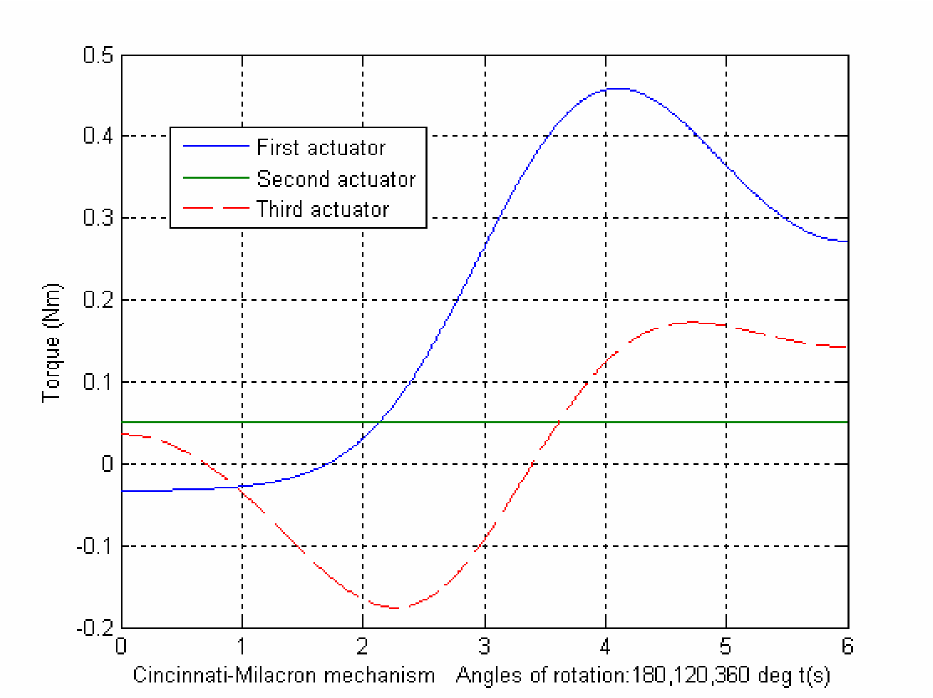

Fig. 5 – Input torques m_{10}^{A}, m_{10}^{B}, m_{10}^{C} of the three actuators.

The dynamics model developed in the present paper takes into consideration the mass, the tensor of inertia and the action of weight and inertia force introduced by all compounding elements of the mechanism. Based on the principle of virtual work, this approach is very efficient, can eliminate all forces of internal joints and establishes a direct determination of the time-history evolution of powers required by the actuators. The method described above is quit available in forward and inverse mechanics of all serial or parallel mechanisms, the platform of which behaves in translation, rotation evolution or general six-degrees-of-freedom motion. The matrix relations, given by this dynamic simulation, can be transformed in a model for automatic control of the spherical mechanism.

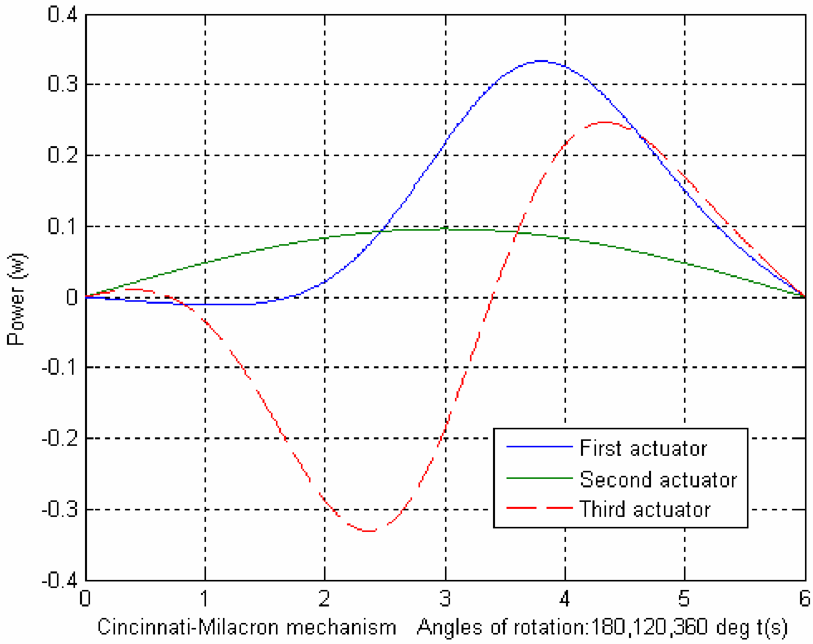

Fig. 6 – Input powers P_{10}^{A}, P_{10}^{B}, P_{10}^{C} of the three actuators.

Received January 9, 2010

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