

# AN ABSTRACT PATTERN FOR SOME DYNAMICAL MODELS

ILEANA TOMA

*Abstract.* Previously, several models from mechanics and physics were found that satisfy the same polynomial equation, whose coefficients are purely abstract. These models are: the nonlinear pendulum, the Bernoulli-Euler bar, P.P. Teodorescu's two-bar frame model and Troesch's plasma problem. In this paper, the analogy is completed with Euler-Poinsot's solid in a particular case, due to author's previous result concerning a connection between this last model and the nonlinear pendulum. All the above models can be solved by using author's linear equivalence method (LEM) by the same analytical formula. The normal LEM solution is then numerically compared in the case of Euler-Poinsot's solid with the corresponding Runge-Kutta solution.

*Key words:* linear equivalence method (LEM), Euler-Poinsot's solid, nonlinear pendulum, Bernoulli-Euler bar, P.P. Teodorescu's two-bar frame model, Troesch's plasma problem.

## 1. INTRODUCTION

In some previous papers [9, 10, 22], several mechanical and physical models were put into evidence, satisfying the same polynomial ODE with abstract coefficients; this is why it was called *intrinsic*. We firstly present four mechanical and physical phenomena, governed by the intrinsic equation: the nonlinear pendulum, the Bernoulli-Euler bar, the two-bar frame, a nonlinear model set up by P.P. Teodorescu [6, 7, 22] and Troesch's plasma problem [15, 22].

In this paper, we complete this analogy with another model: Euler-Poinsot's solid. In [24, 25] it was shown that the model of Euler-Poinsot's solid has as a mathematical core the nonlinear pendulum equation and its complementary; consequently, it can be introduced as a fifth term of the analogy.

The next task is to find analytical solutions for the intrinsic equation. To do this, we applied LEM – the linear equivalence method, an original method previously introduced by the author [14, 22]. The normal LEM solution of the intrinsic equation can be applied to all the above considered models. A parallel with the Runge-Kutta method shows that the analytic LEM formula holds on large intervals; this recommends its application to a qualitative study of the solutions in

---

Technical University of Civil Engineering of Bucharest, Dept. of Mathematics and Computer Science

all considered cases. Let us mention that other authors successfully applied LEM to their models (see e.g. [2, 4, 5]).

## 2. THE INTRINSIC EQUATION

This equation reads

$$z_j^{IV} z_j' - z_j''' z_j'' = (-1)^j z_j'' z_j'^3, \quad j = 1, 2. \quad (1)$$

It is a polynomial fourth order ODE that can be immediately reduced to the second order ODE

$$u'' + (-1)^m K^2 u = \frac{(-1)^j}{2} u^3, \quad u = z'. \quad (2)$$

In the above equation,  $K$  is a constant taking different values, according to the model and its corresponding initial conditions.

## 3. THE MODELS

### I. The nonlinear pendulum

We consider a rod of length  $l$ , fixed up at a point, carrying a bob of mass  $m$  and free to oscillate in a vertical plane. Neglecting the frictions, dissipations and the tension in the rod, the equation of motion of the pendulum is

$$\ddot{y} + \omega^2 \sin y = 0, \quad \omega^2 = \frac{g}{l}. \quad (3)$$

The unknown function  $y$  is the angle of deviation from the vertical (equilibrium) position.

Let us note that the physical significance of  $\omega^2$  may vary, according to the model.

### II. The Bernoulli-Euler (B.-E.) bar

Consider a cantilever bar of length  $l$ , acted upon by the constant axial force  $P$ . The mathematical model of the problem may be expressed in the general nonlinear form

$$\begin{aligned} \frac{dy}{ds} &= \sin \theta, \\ \frac{d\theta}{ds} &= \alpha^2 (f - y), \quad \alpha^2 = \frac{P}{EI}, \end{aligned} \quad (4)$$

where  $ds = \sqrt{dx^2 + dy^2}$ ,  $Ox$  – the direction along the bar axis,  $O$  corresponds to the bar left end,  $Oy$  is the transverse axis,  $\theta$  – the rotation of the bar cross section and  $EI$  is the constant bending rigidity of the bar ( $E$  is the modulus of the longitudinal elasticity and  $I$  – the moment of inertia of the cross section with respect to the neutral axis).

### III. P.P. Teodorescu's nonlinear two-bar frame

This represents a fundamental problem of the statics of constructions. Its nonlinear model was established by P.P. Teodorescu [6,7], which turned out better than Koiter's linear model [1]. P.P. Teodorescu considered the frame composed of two bars: a horizontal and a vertical bar, of constant cross section and of bending rigidity  $EI$  in the frame plane. The two bars are supposed to be of equal length  $l$ ; this hypothesis, introduced only to the aim of simplifying calculus, is not essential and can be removed. The frame is acted upon by a vertical force  $P$ , which is responsible for the structure loss of stability (Fig. 1).

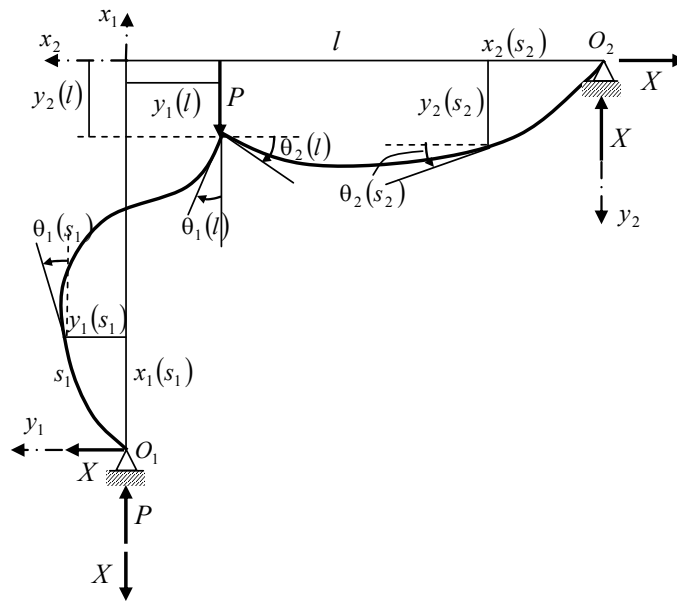


Fig. 1 – P.P. Teodorescu's two-bar frame.

The magnitudes corresponding to the vertical/horizontal bar will be indexed by 1 and 2 accordingly. Thus,  $y_j$  represents the bar displacement,  $\theta_j$  – the rotation of the cross section, each of them depending on the arc  $s_j$ ,  $j = 1, 2$ . The equations of the static equilibrium, established by P.P. Teodorescu, read

$$\begin{aligned}x_1' &= \cos \theta_1, \\y_1' &= \sin \theta_1, \\ \theta_1' &= \tilde{X}x_1 - (\alpha^2 - \omega^2)y_1,\end{aligned}\tag{5}$$

and

$$\begin{aligned}x_2' &= \cos \theta_2, \\y_2' &= \sin \theta_2, \\ \theta_2' &= -\omega^2x_2 + \tilde{X}y_2;\end{aligned}\tag{6}$$

they must be solved taking into account the conditions

$$\begin{aligned}x_j(0) &= 0, y_j(0) = 0, \quad j = 1, 2, \\y_1(l) &= f, l - x_1(l) = y_2(l), l - x_2(l) = f, \\ \theta_1(l) &= \theta_2(l) = -\theta_0, \quad \theta_0 > 0.\end{aligned}\tag{7}$$

#### ***IV. Troesch's plasma problem***

Troesch's model describes the confinement of plasma by radiation pressure and it requires the solution of the nonlinear ODE

$$y'' = \sinh y,\tag{8}$$

that must be solved with the two-point conditions

$$y(0) = 0, \quad y(n) = n;\tag{9}$$

the parameter  $n$  has the physical significance of plasma density.

#### ***V. Euler-Poinsot's solid***

The frictionless motion of a rigid solid around one of its points, considered fixed up with respect to a fixed inertial reference frame, is one of the basic problems of mechanics, both from the theoretical and practical point of view. This problem was firstly considered by d'Alembert in 1749, but the equations of motion received their final form due to Euler (1758). Later on, this problem was tackled by other scientists as J.L. Lagrange, L. Poinsot, S.D. Poinsot, C.G.C. Jacobi, Ch. Hermite and Sonia Kovalevskaia; it is a continuously attracting point for researchers. One of the cases of integrability is Euler-Poinsot's case, in which the moment of the external forces with respect to a fixe pole is null; for instance, the external forces have a null resultant or a resultant always passing through the fixed point. We consider here a particular case: that in which the solid is fixed up in its mass center, being acted upon by its own weight. The equations of motion become

$$\begin{aligned}
 I_1 \dot{\omega}_1 &= (I_2 - I_3) \omega_2 \omega_3, \\
 I_2 \dot{\omega}_2 &= (I_3 - I_1) \omega_3 \omega_1, \\
 I_3 \dot{\omega}_3 &= (I_1 - I_2) \omega_1 \omega_2,
 \end{aligned} \tag{10}$$

where  $\omega_1, \omega_2, \omega_3$  are the corresponding angular velocities and  $I_1, I_2, I_3$  are the principal moments of inertia with respect to the fixed pole, ordered by the relation  $I_1 > I_2 > I_3$ . We can write (10) in the form

$$\begin{aligned}
 \dot{\omega}_1 &= a^2 \omega_2 \omega_3, \\
 \dot{\omega}_2 &= -b^2 \omega_3 \omega_1, \quad a^2 = \frac{I_2 - I_3}{I_1}, \quad b^2 = \frac{I_1 - I_3}{I_2}, \quad c^2 = \frac{I_1 - I_2}{I_3}. \\
 \dot{\omega}_3 &= c^2 \omega_1 \omega_2,
 \end{aligned} \tag{11}$$

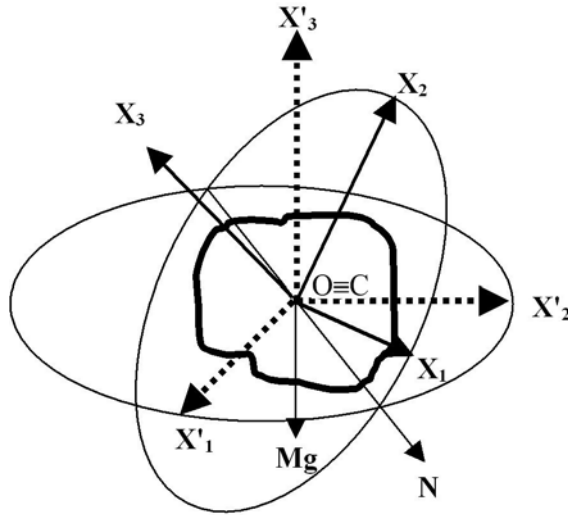


Fig. 2 – Euler-Poinsot's solid.

From this system it follows immediately that we can find  $\omega_1, \omega_2, \omega_3$  following one of the three different ways [24, 25]:

1. We take

$$\omega_1 = aC \cos \theta, \quad \omega_2 = bC \sin \theta, \quad \omega_3 = -\frac{1}{ab} \dot{\theta}, \tag{12}$$

where  $x = 2\theta$  satisfies the equation

$$\ddot{x} + a^2 b^2 c^2 C^2 \sin x = 0, \quad C = \text{const.}, \tag{13}$$

in which we recognize the *equation of the nonlinear rigid pendulum*, or

2. We take

$$\omega_1 = -\frac{1}{bc}\dot{\varphi}, \quad \omega_2 = bA \sin \varphi, \quad \omega_3 = cA \cos \varphi, \quad (14)$$

where  $y = 2\varphi$  satisfies one and the same *pendulum equation*, with the only difference that the constant  $C$  is replaced by another constant  $A$  :

$$\ddot{y} + a^2 b^2 c^2 A^2 \sin y = 0, \quad A = \text{const}. \quad (15)$$

3. We can also take (changing, if necessary, the roles of  $\omega_1, \omega_2$ , depending on the sign of  $\omega_{10}^2/a^2 - \omega_{30}^2/c^2$ )

$$\omega_1 = aB \cosh \psi, \quad \omega_2 = \frac{1}{ac}\dot{\psi}, \quad \omega_3 = cB \sinh \psi. \quad (16)$$

This time,  $z = 2\psi$  satisfies the equation complementary to that of the pendulum

$$\ddot{z} + a^2 b^2 c^2 B^2 \sinh z = 0, \quad B = \text{const}. \quad (17)$$

#### 4. THE NORMAL LEM REPRESENTATIONS

While LEM can be applied to more general ODSs, as the involved equation (2) is polynomial and with constant coefficients, we will restrict to this case. Consider therefore the polynomial ODS

$$\mathcal{P} \mathbf{y} \equiv \frac{d\mathbf{y}}{dt} - \mathbf{P}(\mathbf{y}) = \mathbf{0}, \quad \mathbf{P} = [P_j(\mathbf{y})]_{j=\overline{1,n}}, \quad P_j(\mathbf{y}) \equiv \sum_{|\eta| \leq p_j} a_{j\eta} \mathbf{y}^\eta, \quad (18)$$

$$a_{j\mu} \in \mathfrak{R}, \quad j = \overline{1,n}, \quad |\mu| \leq p_j, \quad j = \overline{1,n}.$$

As it was mentioned – firstly in [14] – the LEM mapping is

$$v(t, \xi) = e^{\langle \xi, \mathbf{y} \rangle}, \quad \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathfrak{R}^n, \quad (19)$$

where  $\xi$  are newly introduced parameters. This mapping associates to the nonlinear ODS two linear equivalents [22]:

- a linear PDE, always of first order with respect to  $t$

$$\mathcal{L} v(t, \xi) \equiv \frac{\partial v}{\partial t} - \langle \xi, \mathbf{P}(D) \rangle v = 0, \quad (20)$$

and

- a linear, while infinite, first order ODS, that may be also written in matrix form

$$\begin{aligned} \mathcal{S} \mathbf{V} &\equiv \frac{d\mathbf{V}}{dt} - \mathbf{A} \mathbf{V} = \mathbf{0}, \\ \mathbf{V} &= (\mathbf{V}_j)_{j \in \mathcal{N}}, \quad \mathbf{V}_j = (v_\gamma)_{\|\gamma\|=j}. \end{aligned} \quad (21)$$

The second LEM equivalent, the system (21) is obtained from the first one, by searching the unknown function  $v$  in the class of analytic in  $\xi$  functions

$$v(t, \xi) = 1 + \sum_{|\gamma|=1}^{\infty} v_\gamma(t) \frac{\xi^\gamma}{j!}. \quad (22)$$

The LEM matrix  $\mathbf{A}$  is row and column-finite, as the differential operator is polynomial. It has a cell-diagonal structure. The involved cells  $\mathbf{A}_{k, k+s}$  are generated by those  $f_{j\mu}$  with  $|\mu| = s + 1$  only; for instance,  $\mathbf{A}_{11}$  is the linear part of the operator. This special form of  $\mathbf{A}$  allows the calculus by block partitioning. Let us associate to (18) the initial conditions

$$\mathbf{y}(t_0) = \mathbf{y}_0, \quad t_0 \in I. \quad (23)$$

By LEM, they are transferred to

$$v(t_0, \xi) = e^{\langle \xi, \mathbf{y}_0 \rangle}, \quad \xi \in \mathfrak{R}^n, \quad (24)$$

a condition that must be associated to (20), and

$$\mathbf{V}(t_0) = (\mathbf{y}_0^\gamma)_{|\gamma| \in \mathcal{N}}, \quad (25)$$

indicating an initial condition for the second LEM equivalent (21).

The linear equivalents are consistent on Exp-type spaces [22].

The following result holds true:

**Theorem 1** [16, 22]. *The solution of the nonlinear initial problem (18), (23)*

i) *coincides with the first  $n$  components of the infinite vector*

$$\mathbf{V}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{V}(t_0), \quad (26)$$

*where the exponential matrix, defined as for finite matrices, can be computed by block partitioning, each step involving finite sums;*

ii) *coincides with the series*

$$y_j(t) = y_{j0} + \sum_{l=1}^{\infty} \sum_{|\gamma|=l} u_{j\gamma}(t) y_0^\gamma, \quad j = \overline{1, n}, \quad (27)$$

where  $u_{j\gamma}(t)$  satisfy the finite linear ODSs

$$\frac{d\mathbf{U}_k}{dt} = \mathbf{A}_{1k}^T \mathbf{U}_1 + \mathbf{A}_{2k}^T \mathbf{U}_2 + \dots + \mathbf{A}_{kk}^T \mathbf{U}_k, \quad k = \overline{1, l}, \quad \mathbf{U}_s(t) = [u_\gamma(t)]_{|\gamma|=s}, \quad (28)$$

and the Cauchy conditions

$$\mathbf{U}_1(t_0) = \mathbf{e}_j^n \equiv [\delta_i^j]_{i=1, n}, \quad \mathbf{U}_s(t_0) = \mathbf{0}, \quad s = \overline{2, l} \quad (29)$$

T standing for transpose matrix and  $\delta$  for Kronecker's delta.

The representation (27) was called *normal* by analogy with the linear case [16, 22]. The eigenvalues of the diagonal cells  $\mathbf{A}_{kk}$  are always known [16, 22]. It was used in many applications requiring the qualitative behavior of the solution and in stability problems, in general (see e.g. [8, 19–21, 23]).

Applying Theorem 1 to the equation (2), for arbitrary Cauchy data  $u(0) = \alpha, u'(0) = \beta$ , we find the following normal LEM solution, going as far as third order effects:

- for  $m = 2$ ,

$$u(x) \cong \frac{1}{K} (\alpha K \cos Kx + \beta \sin Kx) + \frac{(-1)^j}{2^6} [\alpha^3 \varphi_1(Kx) + \alpha^2 \beta \varphi_2(Kx) + \alpha \beta^2 \varphi_3(Kx) + \beta^3 \varphi_4(Kx)], \quad (30)$$

where:

$$\begin{aligned} K^2 \varphi_1(\tau) &= \cos 3\tau - \cos \tau - 12\tau \sin \tau, \\ K^3 \varphi_2(\tau) &= 3 \sin 3\tau - 21 \sin \tau + 12\tau \cos \tau, \\ K^4 \varphi_3(\tau) &= -3 \cos 3\tau + 3 \cos \tau - 12\tau \sin \tau, \\ K^5 \varphi_4(\tau) &= -\sin 3\tau - 9 \sin \tau + 12\tau \cos \tau; \end{aligned} \quad (31)$$

- for  $m = 1$

$$u(x) \cong \frac{1}{K} (\alpha K \cosh Kx + \beta \sinh Kx) + \frac{(-1)^j}{2^6} [\alpha^3 \psi_1(Kx) + \alpha^2 \beta \psi_2(Kx) + \alpha \beta^2 \psi_3(Kx) + \beta^3 \psi_4(Kx)], \quad (32)$$

where:



$$\begin{aligned}
K^2\psi_1(\tau) &= \cosh 3\tau - \cosh \tau - 12\tau \sinh \tau, \\
K^3\psi_2(\tau) &= 3(-\sinh 3\tau + 7\sinh \tau - 4\tau \cosh \tau), \\
K^4\psi_3(\tau) &= 3(\cosh 3\tau - \cosh \tau + 4\tau \sinh \tau), \\
K^5\psi_4(\tau) &= -\sinh 3\tau - 9\sinh \tau + 12\tau \cosh \tau.
\end{aligned} \tag{33}$$

### 5. CONCLUSIONS

The above LEM formulae (30) and (32) can be used to get solutions corresponding to each of the above five models. In Table 1 we specify how the above formulae (30), (32) must be used for every model.

Table 1

The correspondence between the LEM solutions of the intrinsic equation and the five models

Model	ODS/ODE	Formula	$m$	$j$
Pendulum	(3)	(30)	2	1
B.-E. bar	(4)	(30)	2	1
Two-bar frame: vertical	(5)	(30)	2	1
Two-bar frame: horizontal	(6)	(32)	1	1
Troesch's model	(8)	(32)	2	2
E.-P. solid I	(13)	(30)	2	1
E.-P. solid II	(15)	(30)	2	1
E.-P. solid III	(17)	(32)	2	1

For Euler-Poinsot's solid, we set up Table 2, in which the Runge-Kutta solution of (11) is compared with the solution of the same system, by using the analytic LEM formula, applied to equation (13).

We took  $a^2 = 0.4$ ,  $b^2 = 1.3333$ ,  $c^2 = 2$  and made the comparison on the large enough interval  $[0, 10\pi]$ . This comparison resulted in relative errors around 0.05.

Let us mention that a complete numerical study should also include the other two above specified paths, having LEM representations as a kernel. It is possible that the Euler-Poinsot solid should present a "mathematical kernel" in the form of pendulum equation in other cases too; one could thus obtain analytical representations of the solution based on LEM. Finally, it should be very interesting to identify the physical significance of the magnitudes  $\theta$ ,  $\varphi$ ,  $\psi$  and, if possible, to

establish the role they play in the motion of the solid around a fixed point. These questions are among the perspectives of the present paper.

Table 2

The comparison between the LEM solution and the Runge-Kutta solution in the case of Euler-Poinsot's solid

$\omega_1(0)$	$\omega_2(0)$	$\omega_3(0)$	Relative error
1	0	0.2	0.0609
0.5	0.2	0.2	0.0703
0.5	0.1	0.1	0.0319
1.5	0.1	0.1	0.0673
1.5	-0.1	-0.1	0.1030
-1	-0.1	0.1	0.0250

Received on September 11, 2010

#### REFERENCES

1. KOITER, W.T., *Elastic stability and postbuckling behaviour*, Proc. Symp. Nonlinear Problems, ed. by R.E. Langer, Univ. Wisconsin Press, p. 257, 1963.
2. MUNTEANU, L., DONESCU, Ș., *Introduction to the soliton theory, Applications to mechanics*, Book Series: Fundamental Theories of Physics, Kluwer Academic Publishers, 2004.
3. SOARE, M.V., TEODORESCU, P.P., TOMA, I., *Ordinary differential equations with applications to mechanics*, Springer, Dordrecht, 2007.
4. TEODORESCU, P.P., *Mechanical systems. Classical models*, Springer, Dordrecht, 2007-2009.
5. TEODORESCU, P.P., NICOROVICI, N.-A., *Applications of the theory of groups in mechanics and physics*, Book Series: Fundam. Phys, Kluwer Acad. Publ., Boston, Dordrecht, London, 2004.
6. TEODORESCU, P.P., TOMA, I., *Third order effects in the buckling problem of a frame*, 2<sup>nd</sup> European Solid Mechanics Conference (EUROMECH), 12-16.09.1994, Genova, Italia, R7.
7. TEODORESCU, P.P., TOMA, I., *A new formulation in the stability problem of an elastic frame*, Bul. celei de a XVIII-a Conf. de Mecanica Solidelor, 9-11.06.1994, Constanța, I, pp. 7-64, 1994.
8. TEODORESCU, P.P., TOMA, I., *Asupra stabilității unor sisteme de bare în tratare modulară*, Lucrările celei de a XIX-a Conf. de Mecanica Solidelor, 2-3.06.1995, Târgoviște, pp. 15-18, 1995.
9. TEODORESCU, P.P., TOMA, I., *On some elastic structures governed by the same intrinsic equation*, Bull. For Appl. & Computer Math. (BAM) 1567-1591/98, LXXXVI-B, PC-122/98, pp. 8-14, 1998.
10. TEODORESCU, P.P., TOMA, I., *A class of elastic structures with the same mathematical core*, Honorary volume dedicated to professor emeritus Ioannis D. Mittas, Aristotle Univ. of Thessaloniki, Fac. of Eng., Dept. of Math, Phys.Sci., Division of Math., pp. 499-508, 2000.
11. TEODORESCU, P.P., TOMA, I., *Normal LEM representations for the nonlinear pendulum*, GAMM 2001 Annual Scientific Conference, 12-15.02.2001, ETH Zürich, p. 144, 2001.
12. TEODORESCU, P.P., TOMA, I., *New integral LEM formulae applied to the nonlinear bar*, Mech. Res. Comm., **31**, 1, pp. 161-168, 2004.

13. TEODORESCU, P.P., TOMA, I., *Nonlinear elastic deformations treated by LEM*, in: *Topics in Applied Mechanics – Vol. 2* (eds. V. Chiroiu, T. Sireteanu), Romanian Academy Publishing House, Bucharest, pp. 391-442, 2004.
14. TOMA, I., *On polynomial differential equations*, Bull. Math. Soc. Sci. Math. de la Roumanie, **24(72)**, 4, pp. 417-424, 1980.
15. TOMA, I., *On Troesch's plasma problem*, Rev. Roum. des Sci. Techn. – Méc. Appl., **31**, 1, pp. 13-18, 1986.
16. TOMA, I., *Techniques of computation by linear equivalence*, Bull. Math. Soc. Sci. Mat. de la Roumanie, **33(81)**, 4, pp. 363-373, 1989.
17. TOMA, I., *Normal LEM representations for the non-linear forced pendulum*, Proceedings of the International Conference NNMAE II (Nonsmooth/Nonconvex Mechanics with Applications in Engineering), Thessaloniki, Greece, 7-8.07.2006, pp. 329-332, 2006.
18. TOMA, I., *Specific LEM techniques for some polynomial dynamical systems*, in: *Topics in Applied Mechanics – Vol. 3* (eds. V. Chiroiu, T. Sireteanu), Romanian Academy Publishing House, Bucharest, pp. 427-459, 2006.
19. TOMA, I., *The nonlinear pendulum from a LEM perspective*, in: *Research Trends in Mechanics – Vol. 1* (eds. D. Popa, V. Chiroiu, I. Toma), Romanian Academy Publishing House, Bucharest, 395-422, 2007.
20. TOMA, I., *Extensions of LEM to non-autonomous systems*, in: *Research Trends in Mechanics – Vol. 2* (eds. D. Popa, V. Chiroiu, I. Toma), Romanian Academy Publishing House, Bucharest, pp. 361-378, 2008.
21. TOMA, I., *LEM solutions in mechanics and engineering*, Proc. of ICTCAM, 20-23.06.2007, Bucharest, Romania, pp. 123-128, 2007.
22. TOMA, I., *Metoda echivalenței lineare și aplicațiile ei în mecanică*, Edit. Tehnică, Bucharest, 2008.
23. TOMA, I., *New periodic LEM solutions for the nonlinear pendulum*, Rev. Roum. Sci. Techn. – Méc. Appl., **53**, 2, pp. 135-146, 2008.
24. TOMA, I., *Soluții LEM pentru solidul Euler-Poinsot*, <http://instal.utcb.ro/conferinta/lucrari>.
25. TOMA, I., *Normal LEM solutions for Euler-Poinsot's solid*, Scientific Bull. of TUCEB, Series: Mathematical Modelling (to appear).

