AN INTRODUCTION TO DSmT. FIRST PART

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Abstract. The management and combination of uncertain, imprecise, fuzzy and even paradoxical or highly conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. In this introduction, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT), developed for dealing with imprecise, uncertain and conflicting sources of information. We focus our presentation on the foundations of DSmT and on its most important rules of combination, rather than on browsing specific applications of DSmT available in literature. Several simple examples are given throughout this presentation to show the efficiency and the generality of this new theory.

Key words: Dezert-Smarandache Theory, DSmT, quantitative and qualitative reasoning, information fusion.

1. INTRODUCTION

The management and combination of uncertain, imprecise, fuzzy and even paradoxical or highly conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. The combination (fusion) of information arises in many fields of applications nowadays (especially in defense, medicine, finance, geo-science, economy, etc). When several sensors, observers or experts have to be combined together to solve a problem, or if one wants to update our current estimation of solutions for a given problem with some new information available, we need powerful and solid mathematical tools for the fusion, specially when the information one has to deal with is imprecise and uncertain. In this chapter, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT) in the literature, developed for dealing with imprecise, uncertain and conflicting sources of information. Recent publications have shown the interest and the ability of DSmT to solve problems where other approaches fail, especially when conflict between sources becomes high. We focus this presentation rather on the foundations of DSmT, and on the main important rules of combination, than on browsing specific

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Rev. Roum. Sci. Techn. – Méc. Appl., Tome 56, Nº 2, P. 167–198, Bucarest, 2011

applications of DSmT available in literature. Successful applications of DSmT in target tracking, satellite surveillance, situation analysis, robotics, medicine, biometrics, etc, can be found in Parts II of [29, 33, 35] and on the world wide web [36]. Several simple examples are given in this paper to show the efficiency and the generality of DSmT.

2. FOUNDATIONS of DSmT

The development of DSmT (Dezert-Smarandache Theory of plausible and paradoxical reasoning [6, 29]) arises from the necessity to overcome the inherent limitations of DST (Dempster-Shafer Theory [22]) which are closely related with the acceptance of Shafer's model for the fusion problem under consideration (i.e. the frame of *discernment* Θ is implicitly defined as a finite set of *exhaustive* and *exclusive* hypotheses θ_i, $i = 1,\ldots,n$ since the masses of belief are defined only on the power set of Θ – see section 0 for details), the third middle excluded principle (i.e. the existence of the complement for any elements/propositions belonging to the power set of Θ), and the acceptance of Dempster's rule of combination (involving normalization) as the framework for the combination of independent sources of evidence. Discussions on limitations of DST and presentation of some alternative rules to Dempster's rule of combination can be found in [9, 13, 15–17, 19, 21, 29, 38, 46, 49, 50, 53–56] and therefore they will be not reported in details in this introduction. We argue that these three fundamental conditions of DST can be removed and another new mathematical approach for combination of evidence is possible. This is the purpose of DSmT.

The basis of DSmT is the refutation of the principle of the third excluded middle and Shafer's model, since for a wide class of fusion problems the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated. Many problems involving fuzzy continuous and relative concepts described in natural language and having no absolute interpretation like tallness/smallness, pleasure/pain, cold/hot, Sorites paradoxes, etc, enter in this category. DSmT starts with the notion of *free DSm model*, denoted $M^f(\Theta)$, and considers Θ only as a frame of exhaustive elements θ_i, $i = 1,\ldots,n$ which can potentially overlap. This model is *free* because no other assumption is done on the hypotheses, but the weak exhaustivity constraint which can always be satisfied according the closure principle explained in [29]. No other constraint is involved in the free DSm model. When the free DSm model holds, the commutative and associative classical DSm rule of combination, denoted DSmC, corresponding to the conjunctive consensus defined on the free Dedekind's lattice is performed.

Depending on the intrinsic nature of the elements of the fusion problem under consideration, it can however happen that the free model does not fit the reality because some subsets of Θ can contain elements known to be truly exclusive but also truly non existing at all at a given time (specially when working on dynamic fusion problem where the frame Θ varies with time with the revision of the knowledge available). These integrity constraints are then explicitly and formally introduced into the free DSm model $M^f(\Theta)$ in order to adapt it properly to fit as close as possible with the reality and permit to construct a *hybrid DSm model* $M(\Theta)$ on which the combination will be efficiently performed. Shafer's model, denoted $M^0(\Theta)$, corresponds to a very specific hybrid DSm model including all possible exclusivity constraints. DST has been developed for working only with $M^0(\Theta)$ while DSmT has been developed for working with any kind of hybrid model (including Shafer's model and the free DSm model), to manage as efficiently and precisely as possible imprecise, uncertain and potentially highly conflicting sources of evidence while keeping in mind the possible dynamicity of the information fusion problematic. The foundations of DSmT are therefore totally different from those of all existing approaches managing uncertainties, imprecisions and conflicts. DSmT provides a new interesting way to attack the information fusion problematic with a general framework in order to cover a wide variety of problems.

DSmT refutes also the idea that sources of evidence provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliability of sources, but also because of possible different and relative interpretation of Θ, *e.g.* what is considered as good for somebody can be considered as bad for somebody else. There is some unavoidable subjectivity in the belief assignments provided by the sources of evidence, otherwise it would mean that all bodies of evidence have a same objective and universal interpretation (or measure) of the phenomena under consideration, which unfortunately rarely occurs in reality, but when basic belief assignments (bba's) are based on some *objective probabilities* transformations. But in this last case, probability theory can handle properly and efficiently the information, and DST, as well as DSmT, becomes useless. If we now get out of the probabilistic background argumentation for the construction of bba, we claim that in most of cases, the sources of evidence provide their beliefs about elements of the frame of the fusion problem only based on their own limited knowledge and experience without reference to the (inaccessible) absolute truth of the space of possibilities.

2.1. THE POWER SET, HYPER-POWER SET AND SUPER-POWER SET

In DSmT, we take very care about the model associated with the set Θ of hypotheses where the solution of the problem is assumed to belong to. In particular, the three main sets (power set, hyper-power set and super-power set) can be used depending on their ability to fit adequately with the nature of hypotheses. In the following, we assume that $\Theta = \{\theta_1, \ldots, \theta_n\}$ is a finite set (called frame) of n exhaustive elements[3]. If $\Theta = \{\theta_1, \ldots, \theta_n\}$ is a priori not closed (Θ is said to be an open world/frame), one can always include in it a closure element, say θ_{n+1} in such away that we can work with a new closed world/frame $\{\theta_1, \ldots, \theta_n, \theta_{n+1}\}$. So without loss of generality, we will always assume that we work in a closed world by considering the frame Θ as a finite set of exhaustive elements. Before introducing the power set, the hyper-power set and the super-power set it is necessary to recall that subsets are regarded as propositions in Dempster-Shafer Theory (see Chapter 2 of [22]) and we adopt the same approach in DSmT.

- **Subsets as propositions**: Glenn Shafer in pages 35–37 of [22] considers the subsets as propositions in the case we are concerned with the true value of some quantity θ taking its possible values in Θ. Then the propositions $P_\theta(A)$ of interest are those of the form[4]:

  $$P_\theta(A) \triangleq \textit{The true value of } \theta \textit{ is in a subset A of } \Theta.$$

  Any proposition $P_\theta(A)$ is thus in one-to-one correspondence with the subset A of Θ. Such correspondence is very useful since it translates the logical notions of conjunction $\wedge$, disjunction $\vee$, implication $\Rightarrow$ and negation $\neg$ into the set-theoretic notions of intersection $\cap$, union $\cup$, inclusion $\subset$ and complementation $c(.)$. Indeed, if $P_\theta(A)$ and $P_\theta(B)$ are two propositions corresponding to subsets A and B of Θ, then the conjunction $P_\theta(A) \wedge P_\theta(B)$ corresponds to the intersection $A \cap B$ and the disjunction $P_\theta(A) \vee P_\theta(B)$ corresponds to the union $A \cup B$. A is a subset of B if and only if $P_\theta(A) \Rightarrow P_\theta(B)$ and A is the set-theoretic complement of B with respect to Θ (written $A = c_\Theta(B)$) if and only if $P_\theta(A) = \neg P_\theta(B)$. In other words, the following equivalences are then used between the operations on the subsets and on the propositions (Table 1).

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[3] We do not assume here that elements θ_i are necessary exclusive, unless specified. There is no restriction on θ_i but the exhaustivity.

[4] We use the symbol $\triangleq$ to mean *equals by definition*; the right-hand side of the equation is the definition of the left-hand side.

Table 1

Correspondence between operations on subsets and on propositions

Operations	Subsets	Propositions
Intersection/conjunction	$A \cap B$	$P_\theta(A) \wedge P_\theta(B)$
Union/disjunction	$A \cup B$	$P_\theta(A) \vee P_\theta(B)$
Inclusion/implication	$A \subset B$	$P_\theta(A) \Rightarrow P_\theta(B)$
Complementation/negation	$A = c_\Theta(B)$	$P_\theta(A) = \neg P_\theta(B)$

- **Canonical form of a proposition**: In DSmT we consider all propositions/sets in a canonical form. We take the disjunctive normal form, which is a disjunction of conjunctions, and it is unique in Boolean algebra and simplest. For example, $X = A \cap B \cap (A \cup B \cup C)$ it is not in a canonical form, but we simplify the formula and $X = A \cap B$ is in a canonical form.
- **The power set**: $2^\Theta \triangleq (\Theta, \cup)$

Aside Dempster's rule of combination, the power set is one of the corner stones of Dempster-Shafer Theory (DST) since the basic belief assignments to combine are defined on the power set of the frame Θ. In mathematics, given a set Θ, the power set of Θ, written 2^Θ, is the set of all subsets of Θ. In ZermeloÐFraenkel set theory with the axiom of choice (ZFC), the existence of the power set of any set is postulated by the axiom of power set. In other words, Θ generates the power set 2^Θ with the $\cup$ (union) operator only.

More precisely, the power set 2^Θ is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$ operator such that:

1. $\varnothing, \theta_1, \ldots, \theta_n \in 2^\Theta$.
2. If $A, B \in 2^\Theta$, then $A \cup B \in 2^\Theta$.
3. No other elements belong to 2^Θ, except those obtained by using rules 1 and 2.

- **The hyper-power set**: $D^\Theta \triangleq (\Theta, \cup, \cap)$

One of the cornerstones of DSmT is the free Dedekind's lattice [3] denoted as *hyper-power set* in DSmT framework. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements. The hyper-power set D^Θ is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$ and $\cap$ operators such that:

1. $\varnothing, \theta_1, \ldots, \theta_n \in D^\Theta$.
2. If $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$.

3. No other elements belong to D^{Θ}, except those obtained by using rules 1 or 2.

Therefore by convention, we write $D^{\Theta}=(\Theta,\cup,\cap)$ which means that Θ generates D^{Θ} under operators $\cup$ and $\cap$. The dual (obtained by switching $\cup$ and $\cap$ in expressions) of D^{Θ} is itself. There are elements in D^{Θ} which are self-dual (dual to themselves), for example α_8 for the case when $n=3$ in the following example. The cardinality of D^{Θ} is majored by 2^{2^n} when the cardinality of Θ equals n, *i.e.* $|\Theta|=n$. The generation of hyper-power set D^{Θ} is closely related with the famous Dedekind's problem [2, 3] on enumerating the set of isotone Boolean functions. The generation of the hyper-power set is presented in [29]. Since for any given finite set Θ, $|D^{\Theta}|\geq|2^{\Theta}|$ we call D^{Θ} the *hyper-power set* of Θ.

The cardinality of hyper-power set D^{Θ} for $n\geq 1$ follows the sequence of Dedekind's numbers [24], *i.e.* 1, 2, 5, 19, 167, 7580, 7828353, ... and analytical expression of Dedekind's numbers has been obtained recently by Tombak in [45] (see [29] for details on generation and ordering of D^{Θ}). Interesting investigations on the programming of the generation of hyper-power sets for engineering applications have been done in Chapter 15 of [33] and in [35].

Shafer's model of a frame: More generally, when all the elements of a given frame Θ are known (or are assumed to be) truly exclusive, then the hyper-power set D^{Θ} reduces to the classical power set 2^{Θ}. Therefore, working on power set 2^{Θ} as Glenn Shafer has proposed in his Mathematical Theory of Evidence [22]) is equivalent to work on hyper-power set D^{Θ} with the assumption that all elements of the frame are exclusive. This is what we call *Shafer's model of the frame* Θ, written $M^0(\Theta)$, even if such model/assumption has not been clearly stated explicitly by Shafer himself in his milestone book.

- **The super-power set**: $S^{\Theta}\triangleq(\Theta,\cup,\cap,c(.))$

The notion of super-power set has been introduced by Smarandache in the Chapter 8 of [33]. It corresponds actually to the theoretical construction of the power set of the minimal[5] refined frame Θ^{ref} of Θ. Θ generates S^{Θ} under operators $\cup$, $\cap$ and complementation $c(.)$. $S^{\Theta}=(\Theta,\cup,\cap,c(.))$ is a Boolean algebra with respect to the union, intersection and complementation. Therefore working with the super-power set is equivalent to work with a minimal theoretical refined frame Θ^{ref} satisfying Shafer's model. More precisely, S^{Θ} is defined as the set of all composite propositions/subsets built from elements of Θ with $\cup$, $\cap$

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[5] The minimality refers here to the cardinality of the refined frames.

and $c(.)$ operators such that:

1. $\varnothing, \theta_1, \dots, \theta_n \in S^\Theta$.
2. If $A, B \in S^\Theta$, then $A \cap B \in S^\Theta$, $A \cup B \in S^\Theta$.
3. If $A \in S^\Theta$, then $c(A) \in S^\Theta$.
4. No other elements belong to S^Θ, except those obtained by using rules 1, 2 and 3.

As reported in [30], a similar generalization has been previously used in 1993 by Guan and Bell [12] for the Dempster-Shafer rule using propositions in sequential logic and reintroduced in 1994 by Paris in his book [18], page 4.

A one-to-one correspondence between the elements of S^Θ and $2^{\Theta^{ref}}$ can be defined for any cardinality $|\Theta| \geq 2$ of the frame Θ and thus one can consider S^Θ as the mathematical construction of the power set $2^{\Theta^{ref}}$ of the minimal refinement of the frame Θ. Of course, when Θ already satisfies Shafer's model, the hyper-power set and the super-power set coincide with the classical power set of Θ. It is worth to note that even if we have a mathematical tool to built the minimal refined frame satisfying Shafer's model, it doesn't mean necessary that one must work with this super-power set in general in real applications because most of the times the elements/granules of S^Θ have no clear physical meaning, not to mention the drastic increase of the complexity since one has $2^\Theta \subseteq D^\Theta \subseteq S^\Theta$ and

$$|2^\Theta| = 2^{|\Theta|} < |D^\Theta| < |S^\Theta| = 2^{|\Theta^{ref}|} = 2^{2^{|\Theta|}-1} \tag{1}$$

Typically, we have Table 2.

Table 2

Cardinalities of 2^Θ, D^Θ and S^Θ

$\lvert\Theta\rvert = n$	$\lvert 2^\Theta\rvert = 2^n$	$\lvert D^\Theta\rvert$	$\lvert S^\Theta\rvert = \lvert 2^{\Theta^{ref}}\rvert = 2^{2^n-1}$
2	4	5	$2^3 = 8$
3	8	19	$2^7 = 128$
4	16	167	$2^{15} = 32768$
5	32	7580	$2^{31} = 2\,147\,483\,648$

In summary, DSmT offers truly the possibility to build and to work on refined frames and to deal with the complement whenever necessary, but in most of applications either the frame Θ is already built/chosen to satisfy Shafer's model or the refined granules have no clear physical meaning which finally prevent to be considered/assessed individually so that working on the hyper-power set is usually sufficient for dealing with uncertain imprecise (quantitative or qualitative) and highly conflicting sources of evidences. Working with S^{Θ} is actually very similar to working with 2^{Θ} in the sense that in both cases we work with classical power sets; the only difference is that when working with S^{Θ} we have implicitly switched from the original frame Θ representation to a minimal refinement Θ^{ref} representation. Therefore, in the sequel we focus our discussions based mainly on hyper-power set rather than (super-) power set which has already been the basis for the development of DST. But as already mentioned, DSmT can easily deal with belief functions defined on 2^{Θ} or S^{Θ} similarly as those defined on D^{Θ}.

Generic notation: In the sequel, we use the generic notation G^{Θ} for denoting the sets (power set, hyper-power set and super-power set) on which the belief functions are defined.

The main distinctions between DSmT and DST are summarized by the following points:

1. The refinement is not always (physically) possible, especially for elements from the frame of discernment whose frontiers are not clear, such as: colors, vague sets, unclear hypotheses, etc. in the frame of discernment; DST does not fit well for working in such cases, while DSmT does;
2. Even in the case when the frame of discernment can be refined (i.e. the *atomic* elements of the frame have all a distinct physical meaning), it is still easier to use DSmT than DST since in DSmT framework the refinement is done automatically by the mathematical construction of the super-power set;
3. DSmT offers better fusion rules, for example Proportional Conflict redistribution Rule # 5 (PCR5) – presented in the sequel – is better than Dempster's rule; hybrid DSm rule (DSmH) works for the dynamic fusion, while Dubois-Prade fusion rule does not (DSmH is an extension of Dubois-Prade rule);
4. DSmT offers the best qualitative operators (when working with labels) giving the most accurate and coherent results;
5. DSmT offers new interesting quantitative conditioning rules (BCRs) and qualitative conditioning rules (QBCRs), different from Shafer's conditioning rule (SCR). SCR can be seen simply as a combination of a prior mass of belief with the mass $m(A)=1$ whenever A is the conditioning event;
6. DSmT proposes a new approach for working with imprecise quantitative or qualitative information and not limited to interval-valued belief structures as proposed generally in the literature [4, 5, 47].

2.2. NOTION OF FREE AND HYBRID DSM MODELS

Free DSm model: The elements θ_i, $i=1,\ldots,n$ of Θ constitute the finite set of hypotheses/concepts characterizing the fusion problem under consideration. When there is no constraint on the elements of the frame, we call this model the *free DSm model*, written $M^f(\Theta)$. This free DSm model allows to deal directly with fuzzy concepts which depict a continuous and relative intrinsic nature and which cannot be precisely refined into finer disjoint information granules having an absolute interpretation because of the unreachable universal truth. In such case, the use of the hyper-power set D^Θ (without integrity constraints) is particularly well adapted for defining the belief functions one wants to combine.

Shafer's model: In some fusion problems involving discrete concepts, all the elements θ_i, $i=1,\ldots,n$ of Θ can be truly exclusive. In such case, all the exclusivity constraints on θ_i, $i=1,\ldots,n$ have to be included in the previous model to characterize properly the true nature of the fusion problem and to fit it with the reality. By doing this, the hyper-power set D^Θ as well as the super-power set S^Θ reduce naturally to the classical power set 2^Θ and this constitutes what we have called *Shafer's model*, denoted $M^0(\Theta)$. Shafer's model corresponds actually to the most restricted hybrid DSm model.

Hybrid DSm models: Between the class of fusion problems corresponding to the free DSm model $M^f(\Theta)$ and the class of fusion problems corresponding to Shafer's model $M^0(\Theta)$, there exists another wide class of hybrid fusion problems involving in Θ both fuzzy continuous concepts and discrete hypotheses. In such (hybrid) class, some exclusivity constraints and possibly some non-existential constraints (especially when working on dynamic[6] fusion) have to be taken into account. Each hybrid fusion problem of this class will then be characterized by a proper hybrid DSm model denoted $M(\Theta)$ with $M(\Theta) \neq M^f(\Theta)$ and $M(\Theta) \neq M^0(\Theta)$.

In any fusion problems, we consider as primordial at the very beginning and before combining information expressed as belief functions to define clearly the proper frame Θ of the given problem and to choose explicitly its corresponding model one wants to work with. Once this is done, the second important point is to select the proper set 2^Θ, D^Θ or S^Θ on which the belief functions will be defined. The third point concerns the choice of an efficient rule of combination of belief functions and finally the criteria adopted for decision-making.

In the sequel, we focus our presentation mainly on hyper-power set D^Θ (unless specified) since it is the most interesting new aspect of DSmT for readers already familiar with DST framework, but a fortiori we can work similarly on classical power set 2^Θ if Shafer's model holds, and even on $2^{\Theta^{ref}}$ (the power set of the minimal refined frame) whenever one wants to use it and if possible.

Examples of models for a frame Θ :

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[6] *i.e.* when the frame Θ and/or the model M is changing with time.

- Let's consider the 2D problem where $\Theta = \{\theta_1, \theta_2\}$ with $D^\Theta = \{\varnothing, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$ and assume now that θ_1 and θ_2 are truly exclusive (*i.e.* Shafer's model M^0 holds), then because $\theta_1 \cap \theta_2 \overset{M^0}{=} \varnothing$, one gets

$$D^\Theta = \{\varnothing, \theta_1 \cap \theta_2 \overset{M^0}{=} \varnothing, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = \{\varnothing, \theta_1, \theta_2, \theta_1 \cup \theta_2\} \equiv 2^\Theta .$$

- As another simple example of hybrid DSm model, let's consider the 3D case with the frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with the model $M \neq M^f$ in which we force all possible conjunctions to be empty, but $\theta_1 \cap \theta_2$. This hybrid DSm model is then represented with the Venn diagram on Fig. 1 (where boundaries of intersection of θ_1 and θ_2 are not precisely defined if θ_1 and θ_2 represent only fuzzy concepts like *smallness* and *tallness* by example).

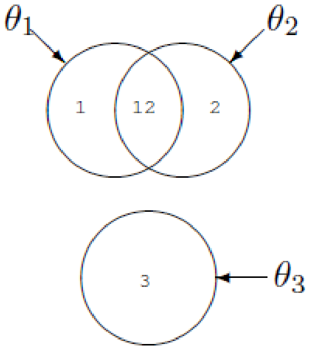

Fig. 1 – Venn diagram of a DSm hybrid model for a 3D frame.

2.3. GENERALIZED BELIEF FUNCTIONS

From a general frame Θ, we define a map $m(.): G^\Theta \to [0,1]$ associated to a given body of evidence B as

$$m(\varnothing) = 0 \qquad \text{and} \qquad \sum_{A \in G^\Theta} m(A) = 1 . \tag{2}$$

The quantity $m(A)$ is called the *generalized basic belief assignment/mass* (gbba) of A.

The *generalized belief and plausibility functions* are defined in almost the same manner as within DST, *i.e.*

$$\mathrm{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in G^\Theta}} m(B) \qquad \mathrm{Pl}(A) = \sum_{\substack{B \cap A \neq \varnothing \\ B \in G^\Theta}} m(B) . \tag{3}$$

We recall that G^{Θ} is the generic notation for the set on which the gbba is defined (G^{Θ} can be 2^{Θ}, D^{Θ} or even S^{Θ} depending on the model chosen for Θ). These definitions are compatible with the definitions of the classical belief functions in DST framework when $G^{\Theta} = 2^{\Theta}$ for fusion problems where Shafer's model $M^{0}(\Theta)$ holds. We still have $\forall A \in G^{\Theta}, \mathrm{Bel}(A) \leq \mathrm{Pl}(A)$. Note that when working with the free DSm model $M^{f}(\Theta)$, one has always $\mathrm{Pl}(A) = 1$ $\forall A \neq \varnothing \in (G^{\Theta} = D^{\Theta})$ which is normal.

2.4. THE CLASSIC DSM RULE OF COMBINATION

When the free DSm model $M^{f}(\Theta)$ holds for the fusion problem under consideration, the classic DSm rule of combination $m_{M^{f}(\Theta)} \equiv m(.) \underline{\underline{\Delta}} [m_1 \oplus m_2](.)$ of two independent[7] sources of evidences B_1 and B_2 over the same frame Θ with belief functions $\mathrm{Bel}_1(.)$ and $\mathrm{Bel}_2(.)$ associated with gbba $m_1(.)$ and $m_2(.)$ corresponds to the conjunctive consensus of the sources. It is given by [29]:

$$\forall C \in D^{\Theta}, \qquad m_{M^{f}(\Theta)}(C) \equiv m(C) = \sum_{\substack{A,B \in D^{\Theta} \\ A \cap B = C}} m_1(A) m_2(B). \tag{4}$$

Since D^{Θ} is closed under $\cup$ and $\cap$ set operators, this new rule of combination guarantees that $m(.)$ is a proper generalized belief assignment, i.e. $m(.): D^{\Theta} \to [0,1]$. This rule of combination is commutative and associative and can always be used for the fusion of sources involving fuzzy concepts when free DSm model holds for the problem under consideration. This rule has been extended for $s > 2$ sources in [29].

According to Table 2, this classic DSm rule of combination looks very expensive in terms of computations and memory size due to the huge number of elements in D^{Θ} when the cardinality of Θ increases. This remark is however valid only if the cores (the set of focal elements of gbba) $K_1(m_1)$ and $K_2(m_2)$ coincide with D^{Θ}, *i.e.* when $m_1(A) > 0$ and $m_2(A) > 0$ for all $A \neq \varnothing \in D^{\Theta}$. Fortunately, it is important to note here that in most of the practical applications the sizes of $K_1(m_1)$ and $K_2(m_2)$ are much smaller than $|D^{\Theta}|$ because bodies of evidence generally allocate their basic belief assignments only over a subset of the

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[7] While independence is a difficult concept to define in all theories managing epistemic uncertainty, we follow here the interpretation of Smets in [37] and [38], p. 285 and consider that two sources of evidence are independent (*i.e.* distinct and noninteracting) if each leaves one totally ignorant about the particular value the other will take.

hyper-power set. This makes things easier for the implementation of the classic DSm rule (4). The DSm rule is actually very easy to implement. It suffices for each focal element of $K_1(m_1)$ to multiply it with the focal elements of $K_2(m_2)$ and then to pool all combinations which are equivalent under the algebra of sets. While very costly in term on memory storage in the worst case (i.e. when all $m(A) > 0$, $A \in D^{\Theta}$ or $A \in 2^{\Theta^{ref}}$), the DSm rule however requires much smaller memory storage than when working with S^{Θ}, *i.e.* working with a minimal refined frame satisfying Shafer's model.

In most fusion applications only a small subset of elements of D^{Θ} have a non null basic belief mass because all the commitments are just usually impossible to obtain precisely when the dimension of the problem increases. Thus, it is not necessary to generate and keep in memory all elements of D^{Θ} (or eventually S^{Θ}) but only those which have a positive belief mass. However there is a real technical challenge on how to manage efficiently all elements of the hyper-power set. This problem is obviously much more difficult when trying to work on a refined frame of discernment Θ^{ref} if one really prefers to use Dempster-Shafer theory and apply Dempster's rule of combination. It is important to keep in mind that the ultimate and minimal refined frame consisting in exhaustive and exclusive finite set of refined exclusive hypotheses is just impossible to justify and to define precisely for all problems dealing with fuzzy and ill-defined continuous concepts. A discussion on refinement with an example has be included in [29].

2.5. THE HYBRID DSM RULE OF COMBINATION

When the free DSm model $M^f(\Theta)$ does not hold due to the true nature of the fusion problem under consideration which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $M(\Theta) \neq M^f(\Theta)$. In such case, the hybrid DSm rule (DSmH) of combination based on the chosen hybrid DSm model $M(\Theta)$ for $k \geq 2$ independent sources of information is defined for all $A \in D^{\Theta}$ as [29]:

$$m_{DSmH}(A) = m_{M(\Theta)}(A) \underline{\underline{\Delta}} \phi(A)[S_1(A) + S_2(A) + S_3(A)] \qquad (5)$$

where all sets involved in formulas are in the canonical form and $\phi(A)$ is the *characteristic non-emptiness function* of a set A, *i.e.* $\phi(A) = 1$ if $A \notin \emptyset$ and $\phi(A) = 0$ otherwise, where $\emptyset \underline{\underline{\Delta}} \{\emptyset_M, \emptyset\}$. $\emptyset_{[}$ is the set of all elements of D^{Θ} which have been forced to be empty through the constraints of the model ⌞ and $\emptyset$ is the classical/universal empty set. $S_1(A) \equiv m_{M^f(\theta)}(A)$, $S_2(A)$, $S_3(A)$ are defined by

$$S_1(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \prod_{i=1}^{k} m_i(X_i), \tag{6}$$

$$S_2(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in \varnothing \\ \vee[(U \in \varnothing) \wedge (A = I_t)]}} \prod_{i=1}^{k} m_i(X_i), \tag{7}$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cup X_2 \cup \ldots \cup X_k = A \\ X_1 \cap X_2 \cap \ldots \cap X_k \in \varnothing}} \prod_{i=1}^{k} m_i(X_i), \tag{8}$$

with $U \triangleq u(X_1) \cup u(X_2) \cup \ldots \cup u(X_k)$ where $u(X)$ is the union of all θ_i that compose X, $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ is the total ignorance. $S_1(A)$ corresponds to the classic DSm rule for k independent sources based on the free DSm model $M^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $S_3(A)$ transfers the sum of relatively empty sets directly onto the canonical disjunctive form of non-empty sets.

The hybrid DSm rule of combination generalizes the classic DSm rule of combination and is not equivalent to Dempter's rule. It works for any models (the free DSm model, Shafer's model or any other hybrid models) when manipulating *precise* generalized (or eventually classical) basic belief functions. An extension of this rule for the combination of *imprecise* generalized (or eventually classical) basic belief functions is presented in next section. As already stated, in DSmT framework it is also possible to deal directly with complements if necessary depending on the problem under consideration and the information provided by the sources of evidence themselves.

The first and simplest way is to work with S^Θ on Shafer's model when a minimal refinement is possible and makes sense. The second way is to deal with partially known frame and introduce directly the complementary hypotheses into the frame itself. By example, if one knows only two hypotheses θ_1, θ_2 and their complements $\bar{\theta}_1$, $\bar{\theta}_2$, then we can choose to switch from original frame

$\Theta = \{\theta_1, \theta_2\}$ to the new frame $\Theta = \{\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2\}$. In such case, we don't necessarily assume that $\bar{\theta}_1 = \theta_2$ and $\bar{\theta}_2 = \theta_1$ because $\bar{\theta}_1$ and $\bar{\theta}_2$ may include other unknown hypotheses we have no information about (case of partial known frame). More generally, in DSmT framework, it is not necessary that the frame is built on pure/simple (possibly vague) hypotheses θ_i as usually done in all theories managing uncertainty. The frame Θ can also contain directly as elements conjunctions and/or disjunctions (or mixed propositions) and negations/ complements of pure hypotheses as well. The DSm rules also work in such non-classic frames because DSmT works on any distributive lattice built from Θ anywhere Θ is defined.

2.6. EXAMPLES OF COMBINATION RULES

Here are some numerical examples on results obtained by DSm rules of combination. More examples can be found in [29].

2.6.1. Example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, two independent experts, and the two following bbas

$$m_1(\theta_1) = 0.6 \quad m_1(\theta_3) = 0.4 \quad m_2(\theta_2) = 0.2 \quad m_2(\theta_4) = 0.8$$

represented in terms of mass matrix

$$\mathbf{M} = \begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}.$$

Dempster's rule cannot be applied because: $\forall 1 \le j \le 4$, one gets $m(\theta_j) = 0/0$ (undefined!).

But the classic DSm rule works because one obtains: $m(\theta_1) = m(\theta_2) = m(\theta_3) = m(\theta_4) = 0$, and $m(\theta_1 \cap \theta_2) = 0.12$, $m(\theta_1 \cap \theta_4) = 0.48$, $m(\theta_2 \cap \theta_3) = 0.08$, $m(\theta_3 \cap \theta_4) = 0.32$ (partial paradoxes/ conflicts).

Suppose now one finds out that all intersections are empty (Shafer's model), then one applies the hybrid DSm rule and one gets (index h stands here for *hybrid* rule): $m_h(\theta_1 \cup \theta_2) = 0.12$, $m_h(\theta_1 \cup \theta_4) = 0.48$, $m_h(\theta_2 \cup \theta_3) = 0.08$ and $m_h(\theta_3 \cup \theta_4) = 0.32$.

2.6.2. Generalization of Zadeh's example with $\Theta = \{\theta_1, \theta_2, \theta_3\}$

Let's consider $0 < \varepsilon_1, \varepsilon_2 < 1$ be two very tiny positive numbers (close to zero), the frame of discernment be $\Theta = \{\theta_1, \theta_2, \theta_3\}$, have two experts (independent sources of evidence s_1 and s_2) giving the belief masses

$$m_1(\theta_1) = 1 - \varepsilon_1 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = \varepsilon_1$$

$$m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1 - \varepsilon_2 \quad m_2(\theta_3) = \varepsilon_2 .$$

From now on, we prefer to use matrices to describe the masses, *i.e.*

$$\begin{bmatrix} 1-\varepsilon_1 & 0 & \varepsilon_1 \\ 0 & 1-\varepsilon_2 & \varepsilon_2 \end{bmatrix}.$$

Using Dempster's rule of combination, one gets

$$m(\theta_3) = \frac{(\varepsilon_1 \varepsilon_2)}{(1-\varepsilon_1)\cdot 0 + 0 \cdot (1-\varepsilon_2) + \varepsilon_1 \varepsilon_2} = 1,$$

which is absurd (or at least counter-intuitive). Note that whatever positive values for ε_1, ε_2 are, Dempster's rule of combination provides always the same result (one) which is abnormal. The only acceptable and correct result obtained by Dempster's rule is really obtained only in the trivial case when $\varepsilon_1 = \varepsilon_2 = 1$, *i.e.* when both sources agree in θ_3 with certainty which is obvious.

Using the DSm rule of combination based on free-DSm model, one gets $m(\theta_3) = \varepsilon_1 \varepsilon_2$, $m(\theta_1 \cap \theta_2) = (1-\varepsilon_1)(1-\varepsilon_2)$, $m(\theta_1 \cap \theta_3) = (1-\varepsilon_1)\varepsilon_2$, $m(\theta_2 \cap \theta_3) = (1-\varepsilon_2)\varepsilon_1$ and the others are zero which appears more reliable/trustable.

Going back to Shafer's model and using the hybrid DSm rule of combination, one gets $m(\theta_3) = \varepsilon_1 \varepsilon_2$, $m(\theta_1 \cup \theta_2) = (1-\varepsilon_1)(1-\varepsilon_2)$, $m(\theta_1 \cup \theta_3) = (1-\varepsilon_1)\varepsilon_2$, $m(\theta_2 \cup \theta_3) = (1-\varepsilon_2)\varepsilon_1$ and the others are zero.

Note that in the special case when $\varepsilon_1 = \varepsilon_2 = 1/2$, one has

$$m_1(\theta_1) = 1/2 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = 1/2$$

$$m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1/2 \quad m_2(\theta_3) = 1/2 .$$

Dempster's rule of combinations still yields $m(\theta_3) = 1$ while the hybrid DSm rule based on the same Shafer's model yields now $m(\theta_3) = 1/4$, $m(\theta_1 \cup \theta_2) = 1/4$, $m(\theta_1 \cup \theta_3) = 1/4$, $m(\theta_2 \cup \theta_3) = 1/4$ which is normal.

2.6.3. Comparison with Smets, Yager and Dubois & Prade rules

We compare the results provided by DSmT rules and the main common rules of combination on the following very simple numerical example where only 2 independent sources (a priori assumed equally reliable) are involved and providing their belief initially on the 3D frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$. It is assumed in this example that Shafer's model holds and thus the belief assignments $m_1(.)$ and $m_2(.)$ do not commit belief to internal conflicting information. $m_1(.)$ and $m_2(.)$ are chosen as follows:

$$m_1(\theta_1) = 0.1 \qquad m_1(\theta_2) = 0.4 \qquad m_1(\theta_3) = 0.2 \qquad m_1(\theta_1 \cup \theta_2) = 0.3$$

$$m_2(\theta_1) = 0.5 \qquad m_2(\theta_2) = 0.1 \qquad m_2(\theta_3) = 0.3 \qquad m_2(\theta_1 \cup \theta_2) = 0.1.$$

These belief masses are usually represented in the form of a belief mass matrix $\mathbf{M}$ given by

$$\mathbf{M} = \begin{bmatrix} 0.1 & 0.4 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.3 & 0.1 \end{bmatrix}, \tag{9}$$

where index i for the rows corresponds to the index of the source no. i and the indexes j for columns of $\mathbf{M}$ correspond to a given choice for enumerating the focal elements of all sources. In this particular example, index $j = 1$ corresponds to θ_1, $j = 2$ corresponds to θ_2, $j = 3$ corresponds to θ_3 and $j = 4$ corresponds to $\theta_1 \cup \theta_2$.

Now let's imagine that one finds out that θ_3 is actually truly empty because some extra and certain knowledge on θ_3 is received by the fusion center. As example, θ_1, θ_2 and θ_3 may correspond to three suspects (potential murders) in a police investigation, $m_1(.)$ and $m_2(.)$ corresponds to two reports of independent witnesses, but it turns out that finally θ_3 has provided a strong alibi to the criminal police investigator once arrested by the policemen. This situation corresponds to set up a hybrid model M with the constraint $\theta_3 \overset{M}{=} \varnothing$.

Let's examine the result of the fusion in such situation obtained by the Smets', Yager's, Dubois & Prade's and hybrid DSm rules of combinations. First note that, based on the free DSm model, one would get by applying the classic DSm rule (denoted here by index $DSmC$) the following fusion result

$$m_{DSmC}(\theta_1) = 0.21 \qquad m_{DSmC}(\theta_2) = 0.11$$

$$m_{DSmC}(\theta_3) = 0.06 \qquad m_{DSmC}(\theta_1 \cup \theta_2) = 0.03$$

$$m_{DSmC}(\theta_1 \cap \theta_2) = 0.21 \qquad m_{DSmC}(\theta_1 \cap \theta_3) = 0.13$$

$$m_{DSmC}(\theta_2 \cap \theta_3) = 0.14 \qquad m_{DSmC}(\theta_3 \cap (\theta_1 \cup \theta_2)) = 0.11\,.$$

But because of the exclusivity constraints (imposed here by the use of Shafer's model and by the non-existential constraint $\theta_3 \overset{M}{=} \varnothing$), the total conflicting mass is actually given by $k_{12} = 0.06 + 0.21 + 0.13 + 0.14 + 0.11 = 0.65$.

- If one applies **Dempster's rule** [22] (denoted here by index DS), one gets:

$$m_{DS}(\varnothing) = 0$$

$$m_{DS}(\theta_1) = 0.21/[1-k_{12}] = 0.21/[1-0.65] = 0.21/0.35 = 0.600000$$

$$m_{DS}(\theta_2) = 0.11/[1-k_{12}] = 0.11/[1-0.65] = 0.11/0.35 = 0.314286$$

$$m_{DS}(\theta_1 \cup \theta_2) = 0.03/[1-k_{12}] = 0.03/[1-0.65] = 0.03/0.35 = 0.085714\,.$$

- If one applies **Smets' rule** [39, 40] (*i.e.* the non normalized version of Dempster's rule with the conflicting mass transferred onto the empty set), one gets:

$$m_S(\varnothing) = m(\varnothing) = 0.65 \qquad \text{(conflicting mass)}$$

$$m_S(\theta_1) = 0.21$$

$$m_S(\theta_2) = 0.11$$

$$m_S(\theta_1 \cup \theta_2) = 0.03\,.$$

- If one applies **Yager's rule** [48-50], one gets:

$$m_Y(\varnothing) = 0$$

$$m_Y(\theta_1) = 0.21$$

$$m_Y(\theta_2) = 0.11$$

$$m_Y(\theta_1 \cup \theta_2) = 0.03 + k_{12} = 0.03 + 0.65 = 0.68\,.$$

- If one applies **Dubois & Prade's rule** [10], one gets because $\theta_3 \overset{M}{=} \varnothing$:

$$m_{DP}(\varnothing) = 0 \qquad \text{(by definition of Dubois \& Prade's rule)}$$

$$\begin{aligned} m_{DP}(\theta_1) &= [m_1(\theta_1)m_2(\theta_1) + m_1(\theta_1)m_2(\theta_1 \cup \theta_2) \\ &\quad + m_2(\theta_1)m_1(\theta_1 \cup \theta_2)] \\ &\quad + [m_1(\theta_1)m_2(\theta_3) + m_2(\theta_1)m_1(\theta_3)] \\ &= [0.1 \cdot 0.5 + 0.1 \cdot 0.1 + 0.5 \cdot 0.3] + [0.1 \cdot 0.3 + 0.5 \cdot 0.2] \\ &= 0.21 + 0.13 = 0.34 \end{aligned}$$

$$\begin{aligned} m_{DP}(\theta_2) &= [0.4 \cdot 0.1 + 0.4 \cdot 0.1 + 0.1 \cdot 0.3] + [0.4 \cdot 0.3 + 0.1 \cdot 0.2] \\ &= 0.11 + 0.14 = 0.25 \end{aligned}$$

$$\begin{aligned} m_{DP}(\theta_1 \cup \theta_2) &= [m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)] \\ &\quad + [m_1(\theta_1 \cup \theta_2)m_2(\theta_3) + m_2(\theta_1 \cup \theta_2)m_1(\theta_3)] \\ &\quad + [m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] \\ &= [0.30.1] + [0.3 \cdot 0.3 + 0.1 \cdot 0.2] + [0.1 \cdot 0.1 + 0.5 \cdot 0.4] \\ &= [0.03] + [0.09 + 0.02] + [0.01 + 0.20] \\ &= 0.03 + 0.11 + 0.21 = 0.35\,. \end{aligned}$$

Now if one adds up the masses, one gets $0 + 0.34 + 0.25 + 0.35 = 0.94$ which is less than 1. Therefore Dubois & Prade's rule of combination does not work when a singleton, or an union of singletons, becomes empty (in a dynamic fusion problem). The products of such empty-element columns of the mass matrix $\mathbf{M}$ are lost; this problem is fixed in DSmT by the sum $S_2(.)$ in (5) which transfers these products to the total or partial ignorances.

- Finally, if one applies **DSmH rule**, one gets because $\theta_3 \overset{M}{=} \varnothing$:

$$m_{DSmH}(\varnothing) = 0 \qquad \text{(by definition of DSmH)}$$

$$m_{DSmH}(\theta_1) = 0.34 \qquad (\text{same as } \mathrm{m}_{\mathrm{DP}}(\theta_1))$$

$$m_{DSmH}(\theta_2) = 0.25 \qquad (\text{same as } \mathrm{m}_{\mathrm{DP}}(\theta_2))$$

$$\begin{aligned} m_{DSmH}(\theta_1 \cup \theta_2) &= [m_1(\theta_1 \cup \theta_2) m_2(\theta_1 \cup \theta_2)] \\ &+ [m_1(\theta_1 \cup \theta_2) m_2(\theta_3) + m_2(\theta_1 \cup \theta_2) m_1(\theta_3)] \\ &+ [m_1(\theta_1) m_2(\theta_2) + m_2(\theta_1) m_1(\theta_2)] + [m_1(\theta_3) m_2(\theta_3)] \\ &= 0.03 + 0.11 + 0.21 + 0.06 = 0.35 + 0.06 = 0.41 \\ &\neq m_{DP}(\theta_1 \cup \theta_2). \end{aligned}$$

We can easily verify that $m_{DSmH}(\theta_1) + m_{DSmH}(\theta_2) + m_{DSmH}(\theta_1 \cup \theta_2) = 1$. In this example, using the hybrid DSm rule, one transfers the product of the empty-element θ_3 column, $m_1(\theta_3) m_2(\theta_3) = 0.2 \cdot 0.3 = 0.06$, to $m_{DSmH}(\theta_1 \cup \theta_2)$, which becomes equal to $0.35 + 0.06 = 0.41$. Clearly, DSmH rule doesn't provide the same result as Dubois and Prade's rule, but only when working on static frames of discernment (restricted cases).

2.7. FUSION OF IMPRECISE BELIEFS

In many fusion problems, it seems very difficult (if not impossible) to have precise sources of evidence generating precise basic belief assignments (especially when belief functions are provided by human experts), and a more flexible plausible and paradoxical theory supporting imprecise information becomes necessary. In the previous sections, we presented the fusion of *precise* uncertain and conflicting/paradoxical generalized basic belief assignments (gbba) in DSmT framework. We mean here by precise gbba, basic belief functions/masses $m(.)$ defined precisely on the hyper-power set D^Θ where each mass $m(X)$, where X belongs to D^Θ, is represented by only one real number belonging to $[0,1]$ such that $\sum_{X \in D^\Theta} m(X) = 1$. In this section, we present the DSm fusion rule for dealing with *admissible imprecise generalized basic belief assignments* $m^I(.)$ defined as real subunitary intervals of $[0,1]$, or even more general as real subunitary sets [i.e. sets, not necessarily intervals].

An imprecise belief assignment $m^I(.)$ over D^Θ is said *admissible* if and only if there exists for every $X \in D^\Theta$ at least one real number $m(X) \in m^I(X)$

such that $\sum_{X\in D^{\Theta}} m(X)=1$. The idea to work with imprecise belief structures represented by real subset intervals of $[0,1]$ is not new and has been investigated in [4, 5, 14] and references therein. The proposed works available in the literature, upon our knowledge were limited only to sub-unitary interval combination in the framework of Transferable Belief Model (TBM) developed by Smets [39, 40]. We extend the approach of Lamata & Moral and Denœux based on subunitary interval-valued masses to subunitary set-valued masses; therefore the closed intervals used by Denœux to denote imprecise masses are generalized to any sets included in [0,1], *i.e.* in our case these sets can be unions of (closed, open, or half-open/half-closed) intervals and/or scalars all in $[0,1]$. Here, the proposed extension is done in the context of DSmT framework, although it can also apply directly to fusion of imprecise belief structures within TBM as well if the user prefers to adopt TBM rather than DSmT.

Before presenting the general formula for the combination of generalized imprecise belief structures, we remind the following set operators involved in the DSm fusion formulas. Several numerical examples are given in the chapter 6 of [29].

- **Addition of sets**

$$S_1+S_2=S_2+S_1\triangleq\{x \mid x=s_1+s_2, s_1\in S_1, s_2\in S_2\}$$

- **Subtraction of sets**

$$S_1 - S_2\triangleq\{x \mid x=s_1-s_2, s_1\in S_1, s_2\in S_2\}$$

- **Multiplication of sets**

$$S_1\cdot S_2\triangleq\{x \mid x=s_1\cdot s_2, s_1\in S_1, s_2\in S_2\}$$

- **Division of sets**: If 0 doesn't belong to S_2,

$$S_1 / S_2\triangleq\{x \mid x=s_1/s_2, s_1\in S_1, s_2\in S_2\}$$

2.7.1. DSm rule of combination for imprecise beliefs

We present the generalization of the DSm rules to combine any type of imprecise belief assignment which may be represented by the union of several sub-unitary (half-) open intervals, (half-)closed intervals and/or sets of points belonging to [0,1]. Several numerical examples are also given. In the sequel, one uses the notation (a,b) for an open interval, $[a,b]$ for a closed interval, and $(a,b]$ or $[a,b)$ for a half open and half closed interval. From the previous operators on sets, one can generalize the DSm rules (classic and hybrid) from scalars to sets in the following way [29] (chap. 6): $\forall A\neq\varnothing\in D^{\Theta}$,

$$m^I(A) = \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1 \cap X_2 \cap \ldots \cap X_k) = A}} \boxed{\sum} \prod_{i=1,\ldots,k} \boxed{\prod} m_i^I(X_i), \tag{10}$$

where $\boxed{\sum}$ and $\boxed{\prod}$ represent the summation, and respectively product, of sets.

Similarly, one can generalize the hybrid DSm rule from scalars to sets in the following way:

$$m^I_{DSmH}(A) = m^I_{(\square\Theta)}(A) \square \phi(A) \square[S_1^I(A) \square S_2^I(A) \square S_3^I(A)], \tag{11}$$

where all sets involved in formulas are in the canonical form and $\phi(A)$ is the *characteristic non emptiness function* of the set A and $S_1^I(A)$, $S_2^I(A)$ and $S_3^I(A)$ are defined by

$$S_1^I(A) \underline{\underline{\Delta}} \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \boxed{\sum} \prod_{i=1,\ldots,k} \boxed{\prod} m_i^I(X_i) \tag{12}$$

$$S_2^I(A) \underline{\underline{\Delta}} \sum_{\substack{X_1,X_2,\ldots,X_k \in \varnothing \\ [\mathrm{U}=A] \vee [(\mathrm{U} \in \varnothing) \wedge (A = I_t)]}} \boxed{\sum} \prod_{i=1,\ldots,k} \boxed{\prod} m_i^I(X_i) \tag{13}$$

$$S_3^I(A) \underline{\underline{\Delta}} \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cup X_2 \cup \ldots \cup X_k = A \\ X_1 \cap X_2 \cap \ldots \cap X_k \in \varnothing}} \boxed{\sum} \prod_{i=1,\ldots,k} \boxed{\prod} m_i^I(X_i) \tag{14}$$

In the case when all sets are reduced to points (numbers), the set operations become normal operations with numbers; the sets operations are generalizations of numerical operations. When imprecise belief structures reduce to precise belief structure, DSm rules (10) and (11) reduce to their precise version (4) and (5) respectively.

3. PROPORTIONAL CONFLICT REDISTRIBUTION RULE

Instead of applying a direct transfer of partial conflicts onto partial uncertainties as with DSmH, the idea behind the Proportional Conflict Redistribution (PCR) rule [31, 33] is to transfer (total or partial) conflicting masses to non-empty sets involved in the conflicts proportionally with respect to the masses assigned to them by sources as follows:

1. calculation the conjunctive rule of the belief masses of sources;
2. calculation the total or partial conflicting masses;
3. redistribution of the (total or partial) conflicting masses to the non-empty sets involved in the conflicts proportionally with respect to their masses assigned by the sources.

The way the conflicting mass is redistributed yields actually several versions of PCR rules. These PCR fusion rules work for any degree of conflict, for any DSm models (Shafer's model, free DSm model or any hybrid DSm model) and both in DST and DSmT frameworks for static or dynamical fusion situations. We present below only the most sophisticated proportional conflict redistribution rule denoted PCR5 in [31, 33]. PCR5 rule is what we feel the most efficient PCR fusion rule developed so far. This rule redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the conjunctive normal form of the partial conflict. PCR5 is what we think the most mathematically exact redistribution of conflicting mass to non-empty sets following the logic of the conjunctive rule. It does a better redistribution of the conflicting mass than Dempster's rule since PCR5 goes backwards on the tracks of the conjunctive rule and redistributes the conflicting mass only to the sets involved in the conflict and proportionally to their masses put in the conflict. PCR5 rule is quasi-associative and preserves the neutral impact of the vacuous belief assignment because in any partial conflict, as well in the total conflict (which is a sum of all partial conflicts), the conjunctive normal form of each partial conflict does not include Θ since Θ is a neutral element for intersection (conflict), therefore Θ gets no mass after the redistribution of the conflicting mass. We have proved in [33] the continuity property of the fusion result with continuous variations of bba's to combine.

3.1. PCR FORMULAS

The PCR5 formula for the combination of two sources ($s = 2$) is given by: $m_{PCR5}(\varnothing) = 0$ and $\forall X \in G^{\Theta} \setminus \{\varnothing\}$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G^{\Theta} \setminus \{X\} \\ X \cap Y = \varnothing}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}\right] \tag{15}$$

where all sets involved in formulas are in canonical form and where G^{Θ} corresponds to classical power set 2^{Θ} if Shafer's model is used, or to a constrained hyper-power set D^{Θ} if any other hybrid DSm model is used instead, or to the super-power set S^{Θ} if the minimal refinement Θ^{ref} of Θ is used; $m_{12}(X) \equiv m_{\cap}(X)$ corresponds to the conjunctive consensus on X between the

$s=2$ sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded.

A general formula of PCR5 for the fusion of $s>2$ sources has been proposed in [33], but a more intuitive PCR formula (denoted PCR6) which provides good results in practice has been proposed by Martin and Osswald in [33] (pages 69-88) and is given by: $m_{PCR6}(\varnothing)=0$ and $\forall X \in G^{\Theta} \setminus \{\varnothing\}$

$$m_{PCR6}(X) = m_{12\ldots s}(X) + \sum_{i=1}^{s} m_i(X)^2 \sum_{\substack{\overset{s-1}{\underset{k=1}{\cap}} Y_{\sigma_i(k)} \cap X \equiv \varnothing \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(s-1)}) \in (G^{\Theta})^{s-1}}} \left(\frac{\prod_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right), \quad (16)$$

where σ_i counts from 1 to s avoiding i:

$$\begin{cases} \sigma_i(j) = j & f\ j < i, \\ \sigma_i(j) = j+1 & f\ j \geq i. \end{cases} \quad (17)$$

Since Y_i is a focal element of expert/source i, $m_i(X) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$; the belief mass assignment $m_{12\text{K}s}(X) \equiv m_{\cap}(X)$ corresponds to the conjunctive consensus on X between the $s>2$ sources. For two sources ($s=2$), PCR5 and PCR6 formulas coincide.

3.2. EXAMPLES

- **Example 1**: Let's take $\Theta=\{A,B\}$ of exclusive elements (Shafer's model), and the following bba:

	A	B	$A \cup B$
$m_1(.)$	0.6	0	0.4
$m_2(.)$	0	0.3	0.7
$m_{\cap}(.)$	0.42	0.12	0.28

The conflicting mass is $k_{12} = m_{\cap}(A \cap B)$ and equals $m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18$. Therefore A and B are the only focal elements involved in the conflict. Hence according to the PCR5 hypothesis only A and B deserve a part of the conflicting mass and $A \cup B$ do not deserve. With PCR5, one redistributes the conflicting mass $k_{12}=0.18$ to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B

respectively.

Here are the results obtained from Dempster's rule, DSmH and PCR5:

	A	B	$A \cup B$
m_{DS}	0.512	0.146	0.342
m_{DSmH}	0.420	0.120	0.460
m_{PCR5}	0.540	0.180	0.280

- **Example 2**: Let's modify example 1 and consider

	A	B	$A \cup B$
$m_1(.)$	0.6	0	0.4
$m_2(.)$	0.2	0.3	0.5
$m_\cap(.)$	0.50	0.12	0.20

The conflicting mass $k_{12} = m_\cap(A \cap B)$ as well as the distribution coefficients for the PCR5 remains the same as in the previous example but one gets now

	A	B	$A \cup B$
m_{DS}	0.609	0.146	0.231
m_{DSmH}	0.500	0.120	0.380
m_{PCR5}	0.620	0.180	0.200

- **Example 3**: Let's modify example 2 and consider

	A	B	$A \cup B$
$m_1(.)$	0.6	0.3	0.1
$m_2(.)$	0.2	0.3	0.5
$m_\cap(.)$	0.44	0.27	0.05

The conflicting mass $k_{12} = 0.24 = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.24$ is now different from previous examples, which means that $m_2(A) = 0.2$ and $m_1(B) = 0.3$ did make an impact on the conflict. Therefore A and B are the only focal elements involved in the conflict and thus only A and B deserve a part of the conflicting mass. PCR5 redistributes the partial conflicting mass 0.18 to A and

B proportionally with the masses $m_1(A)$ and $m_2(B)$ and also the partial conflicting mass 0.06 to A and B proportionally with the masses $m_2(A)$ and $m_1(B)$. After all derivations (see [11] for details), one finally gets:

	A	B	$A \cup B$
m_{DS}	0.579	0.355	0.066
m_{DSmH}	0.440	0.270	0.290
m_{PCR5}	0.584	0.366	0.050

One clearly sees that $m_{DS}(A \cup B)$ gets some mass from the conflicting mass although $A \cup B$ does not deserve any part of the conflicting mass (according to PCR5 hypothesis) since $A \cup B$ is not involved in the conflict (only A and B are involved in the conflicting mass). Dempster's rule appears to us less exact than PCR5 and Inagaki's rules [13]. It can be showed [11] that Inagaki's fusion rule (with an optimal choice of tuning parameters) can become in some cases very close to PCR5 but upon our opinion PCR5 result is more exact (at least less ad-hoc than Inagaki's one).

• **Example 4 (A more concrete example)**: Three people, John (J), George (G), and David (D) are suspects to a murder. So the frame of discernment is $\Theta \triangleq \{J, G, D\}$. Two sources $m_1(.)$ and $m_2(.)$ (witnesses) provide the following information:

	J	G	D
m_1	0.9	0	0.1
m_2	0	0.8	0.2

We know that John and George are friends, but John and David hate each other, and similarly George and David.

a) Free model, i. e. all intersections are nonempty: $J \cap G \neq \varnothing$, $J \cap D \neq \varnothing$, $G \cap D \neq \varnothing$, $J \cap G \cap D \neq \varnothing$. Using the DSm classic rule one gets:

	J	G	D	$J \cap G$	$J \cap D$	$G \cap D$	$J \cap G \cap D$
m_{DSmC}	0	0	0.02	0.72	0.18	0.08	0

So we can see that John and George together ($J \cap G$) are most likely to have committed the crime, since the mass $m_{DSmC}(J \cap G) = 0.72$ is the biggest resulting mass after the fusion of the two sources. In Shafer's model, only one suspect could commit the crime, but the free and hybrid models allow two or more people to have committed the same crime – which happens in reality.

b) Let's consider the hybrid model, *i.e.* some intersections are empty, and others are not. According to the above statement about the relationships between the three suspects, we can deduce that $J \cap G \neq \varnothing$, while $J \cap D = G \cap D = J \cap G \cap D = \varnothing$. Then we first apply the DSm Classic rule, and then the transfer of the conflicting masses is done with PCR5:

	J	G	D	$J \cap G$	$J \cap D$	$G \cap D$	$J \cap G \cap D$
m_1	0.9	0	0.1				
m_2	0	0.8	0.2				
m_{DSmC}	0	0	0.02	0.72	0.18	0.08	0

Using PCR5 now we transfer $m(J \cap D) = 0.18$, since $J \cap D = \varnothing$, to J and D proportionally with 0.9 and 0.2 respectively, so J gets 0.15 and D gets 0.03 since:

$$xJ/0.9 = z1D/0.2 = 0.18/(0.9+0.2) = 0.18/1.1$$

whence $xJ = 0.9(0.18/1.1) = 0.15$ and $z1D = 0.2(0.18/1.1) = 0.03$.

Again using PCR5, we transfer $m(G \cap D) = 0.08$, since $G \cap D = \varnothing$, to G and D proportionally with 0.8 and 0.1 respectively, so G gets 0.07 and D gets 0.01 since:

$$yG/0.8 = z2D/0.1 = 0.08/(0.8+0.1) = 0.08/0.9$$

whence $yG = 0.8(0.08/0.9) = 0.07$ and $zD = 0.1(0.08/0.9) = 0.01$. Adding we get finally:

	J	G	D	$J \cap G$	$J \cap D$	$G \cap D$	$J \cap G \cap D$
m_{PCR5}	0.15	0.07	0.06	0.72	0	0	0

So one has a high belief that the criminals are John and George (both of them committed the crime) since $m(J \cap D) = 0.72$ and it is by far the greatest fusion mass.

In Shafer’s model, if we try to refine we get the disjoint parts: D, $J \cap G$, $J \setminus (J \cap G)$, and $G \setminus (J \cap G)$, but the last two are ridiculous (what is the real/physical nature of $J \setminus (J \cap G)$ or $G \setminus (J \cap G)$? Half of a person(!) ?), so the refining does not work here in reality. That’s why the hybrid and free models are needed.

• **Example 5 (Imprecise PCR5)**: The PCR5 formula can naturally work also for the combination of imprecise bba's. This has been already presented in section 1.11.8 page 49 of [33] with a numerical example to show how to apply it. This example will therefore not be reincluded here.

3.3. ZADEH'S EXAMPLE

We compare here the solutions for well-known Zadeh's example [53, 56] provided by several fusion rules. A detailed presentation with more comparisons can be found in [29, 33]. Let's consider $\Theta = \{M, C, T\}$ as the frame of three potential origins about possible diseases of a patient (M standing for *meningitis*, C for *concussion* and T for *tumor*), the Shafer's model and the two following belief assignments provided by two independent doctors after examination of the same patient.

$$m_1(M) = 0.9 \quad m_1(C) = 0 \quad m_1(T) = 0.1$$

$$m_2(M) = 0 \quad m_2(C) = 0.9 \quad m_2(T) = 0.1$$

The total conflicting mass is high since it is

$$m_1(M)m_2(C) + m_1(M)m_2(T) + m_2(C)m_1(T) = 0.99$$

- with Dempster's rule and Shafer's model (DS), one gets the counter-intuitive result (see justifications in [9, 29, 46, 50, 53]): $m_{DS}(T) = 1$
- with Yager's rule [50] and Shafer's model: $m_Y(M \cup C \cup T) = 0.99$ and $m_Y(T) = 0.01$
- with DSmH and Shafer's model:

$$m_{DSmH}(M \cup C) = 0.81 \qquad m_{DSmH}(T) = 0.01$$

$$m_{DSmH}(M \cup T) = m_{DSmH}(C \cup T) = 0.09$$

- The Dubois & Prade's rule (DP) [9] based on Shafer's model provides in Zadeh's example the same result as DSmH, because DP and DSmH coincide in all static fusion problems[8].
- with PCR5 and Shafer's model: $m_{PCR5}(M) = m_{PCR5}(C) = 0.486$ and $m_{PCR5}(T) = 0.028$.

One sees that when the total conflict between sources becomes high, DSmT is able (upon authors opinion) to manage more adequately through DSmH or PCR5

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[8] Indeed DP rule has been developed for static fusion only while DSmH has been developed to take into account the possible dynamicity of the frame itself and also its associated model.

rules the combination of information than Dempster's rule, even when working with Shafer's model, which is only a specific hybrid model. DSmH rule is in agreement with DP rule for the static fusion, but DSmH and DP rules differ in general (for non degenerate cases) for dynamic fusion while PCR5 rule is the most exact proportional conflict redistribution rule. Besides this particular example, we showed in [29] that there exist several infinite classes of counter-examples to Dempster's rule which can be solved by DSmT.

In summary, DST based on Dempster's rule provides counter-intuitive results in Zadeh's example, or in non-Bayesian examples similar to Zadeh's and no result when the conflict is 1. Only ad-hoc discounting techniques allow to circumvent troubles of Dempster's rule or we need to switch to another model of representation/frame; in the later case the solution obtained doesn't fit with the Shafer's model one originally wanted to work with. We want also to emphasize that in dynamic fusion when the conflict becomes high, both DST [22] and Smets' Transferable Belief Model (TBM) [39] approaches fail to respond to new information provided by new sources. This can be easily showed by the very simple following example.

Example (where TBM doesn't respond to new information): Let $\Theta = \{A, B, C\}$ with the (precise) bba's $m_1(A) = 0.4$, $m_1(C) = 0.6$ and $m_2(A) = 0.7$, $m_2(B) = 0.3$. Then one gets[9] with Dempster's rule, Smets' TBM (i.e. the non-normalized version of Dempster's combination), DSmH and PCR5: $m_{DS}^{12}(A) = 1$, $m_{TBM}^{12}(A) = 0.28$, $m_{TBM}^{12}(\varnothing) = 0.72$,

$$\begin{cases} m_{DSmH}^{12}(A) = 0.28 \\ m_{DSmH}^{12}(A \cup B) = 0.12 \\ m_{DSmH}^{12}(A \cup C) = 0.42 \\ m_{DSmH}^{12}(B \cup C) = 0.18 \end{cases} \quad \text{and} \quad \begin{cases} m_{PCR5}^{12}(A) = 0.574725 \\ m_{PCR5}^{12}(B) = 0.111429 \\ m_{PCR5}^{12}(C) = 0.313846 \end{cases} \tag{1}$$

Now let's consider a temporal fusion problem and introduce a third source $m_3(.)$ with $m_3(B) = 0.8$ and $m_3(C) = 0.2$. Then one sequentially combines the results obtained by $m_{TBM}^{12}(.)$, $m_{DS}^{12}(.)$, $m_{DSmH}^{12}(.)$ and $m_{PCR}^{12}(.)$ with the new evidence $m_3(.)$ and one sees that $m_{DS}^{(12)3}$ becomes not defined (division by zero) and $m_{TBM}^{(12)3}(\varnothing) = 1$ while (DSmH) and (PCR5) provide

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[9] We introduce here explicitly the indexes of sources in the fusion result since more than two sources are considered in this example.

$$\begin{cases} m_{DSmH}^{(12)3}(B) = 0.240 \\ m_{DSmH}^{(12)3}(C) = 0.120 \\ m_{DSmH}^{(12)3}(A \cup B) = 0.224 \\ m_{DSmH}^{(12)3}(A \cup C) = 0.056 \\ m_{DSmH}^{(12)3}(A \cup B \cup C) = 0.360 \end{cases} \quad and \quad \begin{cases} m_{PCR5}^{(12)3}(A) = 0.277490 \\ m_{PCR5}^{(12)3}(B) = 0.545010 \\ m_{PCR5}^{(12)3}(C) = 0.177500. \end{cases} \qquad (1)$$

When the mass committed to empty set becomes one at a previous temporal fusion step, then both DST and TBM do not respond to new information[10]. Let's continue the example and consider a fourth source $m_4(.)$ with $m_4(A) = 0.5$, $m_4(B) = 0.3$ and $m_4(C) = 0.2$. Then it is easy to see that $m_{DS}^{((12)3)4}(.)$ is not defined since at previous step $m_{DS}^{(12)3}(.)$ was already not defined, and that $m_{TBM}^{((12)3)4}(\varnothing) = 1$ whatever $m_4(.)$ is because at the previous fusion step one had $m_{TBM}^{(12)3}(\varnothing) = 1$. Therefore for a number of sources $n \geq 2$, DST and TBM approaches do not respond to new information incoming in the fusion process while both (DSmH) and (PCR5) rules respond to new information. To make DST and/or TBM working properly in such cases, it is necessary to introduce ad-hoc temporal discounting techniques which are not necessary to introduce if DSmT is adopted. If there are good reasons to introduce temporal discounting, there is obviously no difficulty to apply the DSm fusion of these discounted sources. An analysis of this behavior for target type tracking is presented in [7, 33].

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[10] Actually Dempster's rule doesn't respond also to new compatible information/bba as soon as a total mass of belief is already committed by a source to only one focal element. For example, if one considers $\Theta = \{A, B\}$ with Shafer's model ($A \cap B = \varnothing$) and with $m_1(A) = 1$, $m_2(A) = 0.2$ and $m_2(B) = 0.8$, then Dempster's rule always provides $m_{DS}(A) = 1$ whatever are the values taken by $m_2(A) > 0$ and $m_2(B) > 0$.

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