AN INTRODUCTION TO DSmT

Second part

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Abstract. The management and combination of uncertain, imprecise, fuzzy and even paradoxical or highly conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. In this introduction, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT), developed for dealing with imprecise, uncertain and conflicting sources of information. We focus our presentation on the foundations of DSmT and on its most important rules of combination, rather than on browsing specific applications of DSmT available in literature. Several simple examples are given throughout this presentation to show the efficiency and the generality of this new theory.

Key words: Dezert-Smarandache Theory, DSmT, quantitative and qualitative reasoning, information fusion.

1. INTRODUCTION

The management and combination of uncertain, imprecise, fuzzy and even paradoxical or highly conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. The combination (fusion) of information arises in many fields of applications nowadays (especially in defense, medicine, finance, geo-science, economy, etc). When several sensors, observers or experts have to be combined together to solve a problem, or if one wants to update our current estimation of solutions for a given problem with some new information available, we need powerful and solid mathematical tools for the fusion, specially when the information one has to deal with is imprecise and uncertain. In this chapter, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT) in the literature, developed for dealing with imprecise, uncertain and conflicting sources of information. Recent publications have shown the interest and the ability of DSmT to solve problems where other approaches fail, especially when conflict between sources becomes high. We focus this presentation rather on the foundations of

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DSmT, and on the main important rules of combination, than on browsing specific applications of DSmT available in literature. Successful applications of DSmT in target tracking, satellite surveillance, situation analysis, robotics, medicine, biometrics, etc, can be found in Parts II of [29, 33, 35] and on the world wide web [36]. Several simple examples are given in this paper to show the efficiency and the generality of DSmT.

2. UNIFORM AND PARTIALLY UNIFORM REDISTRIBUTION RULES

The principles of Uniform Redistribution Rule (URR) and Partially Uniform Redistribution Rule (PURR) have been proposed in 2006 with examples in [32].

The Uniform Redistribution Rule consists in redistributing the total conflicting mass k_{12} to all focal elements of G^Θ generated by the consensus operator. This way of redistributing mass is very simple and URR is different from Dempster's rule of combination, because Dempster's rule redistributes the total conflict proportionally with respect to the masses resulted from the conjunctive rule of non-empty sets. PCR5 rule presented previously does proportional redistributions of partial conflicting masses to the sets involved in the conflict. The URR formula for two sources is given by: $\forall A \neq \varnothing$

$$m_{12URR}(A) = m_{12}(A) + \frac{1}{n_{12}} \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = \varnothing}} m_1(X_1) m_2(X_2), \tag{1}$$

where $m_{12}(A)$ is the result of the conjunctive rule applied to belief assignments $m_1(.)$ and $m_2(.)$, and $n_{12} = \text{Card}\{Z \in G^\Theta, m_1(Z) \neq 0 \text{ or } m_2(Z) \neq 0\}$.

For $s \geq 2$ sources to combine: $\forall A \neq \varnothing$, one has

$$m_{12\ldots sURR}(A) = m_{12\ldots s}(A) + \frac{1}{n_{12\ldots s}} \sum_{\substack{X_1, X_2, \ldots, X_s \in G^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = \varnothing}} \prod_{i=1}^{s} m_1(X_i), \tag{2}$$

where $m_{12\ldots s}(A)$ is the result of the conjunctive rule applied to $m_i(.)$, for all $i \in \{1,2,\ldots,s\}$ and

$$n_{12\ldots s} = \text{Card}\{Z \in G^\Theta, m_1(Z) \neq 0 \text{ or } m_2(Z) \neq 0 \text{ or } \ldots \text{ or } m_s(Z) \neq 0\}.$$

As alternative (modified version of URR), we can also consider the cardinal of the ensemble of sets whose masses resulted from the conjunctive rule are non-null, *i.e.* the cardinality of the core of conjunctive consensus:

$$n_{12\ldots s}^c = \text{Card}\{Z \in G^\Theta, m_{12\ldots s}(Z) \neq 0\}.$$

It is also possible to do a uniformly partial redistribution, *i.e.* to uniformly redistribute the conflicting mass only to the sets involved in the conflict. For example, if $m_{12}(A \cap B) = 0.08$ and $A \cap B = \varnothing$, then 0.08 is equally redistributed to A and B only, supposing A and B are both non-empty, so 0.04 assigned to A and 0.04 to B.

The Partially Uniform Redistribution Rule (PURR) for two sources is defined as follows: $\forall A \neq \varnothing$

$$m_{12PURR}(A) = m_{12}(A) + \frac{1}{2} \sum_{\substack{X_1, X_2 \in G^{\Theta} \\ X_1 \cap X_2 = \varnothing \\ \hat{E} X_1 = A \text{ or } X_2 = A}} m_1(X_1) m_2(X_2), \tag{3}$$

where $m_{12}(A)$ is the result of the conjunctive rule applied to belief assignments $m_1(.)$ and $m_2(.)$.

For $s \geq 2$ sources to combine: $\forall A \neq \varnothing$, one has

$$m_{12\ldots sPURR}(A) = m_{12\ldots s}(A) +$$

$$+ \frac{1}{s} \sum_{\substack{X_1, X_2, \ldots, X_s \in G^{\Theta} \\ X_1 \cap X_2 \cap \ldots \cap X_s = \varnothing \\ \text{atleatone } X_j = A, j \in \{1, \ldots, s\}}} \mathrm{Card}_A(\{X_1, \ldots, X_s\}) \prod_{i=1}^{s} m_1(X_i), \tag{4}$$

where $\mathrm{Card}_A(\{X_1, \ldots, X_s\})$ is the number of A's occurring in $\{X_1, X_2, \ldots, X_s\}$.

If $A = \varnothing$, $m_{12PURR}(A) = 0$ and $m_{12\ldots sPURR}(A) = 0$.

These rules have a low computation cost with respect to Proportional Conflict Redistribution (PCR) rules developed in the DSmT framework and they preserve the neutrality of the vacuous belief assignment (VBA) since any bba $m_1(.)$ combined with VBA defined on any frame $\Theta = \{\theta_1, \ldots, \theta_n\}$ by $m_{VBA}(\theta_1 \cup \ldots \cup \theta_n) = 1$, using the conjunctive rule, gives $m_1(.)$, so no conflicting mass is needed to transfer. Of course these rules are very easy to implement but from a theoretical point of view they remain less precise in their transfer of conflicting beliefs since they do not take into account the proportional redistribution with respect to the mass of each set involved in the conflict. Reasonably, URR or PURR cannot outperform PCR5 but they may hopefully could appear as good enough in some specific fusion problems when the level of total conflict is not important. PURR does a more refined redistribution that URR and MURR but it requires a little more calculation.

3. RSC FUSION RULES

In this section, we briefly recall a new class of fusion rules based on the belief redistribution to subsets or complements and denoted CRSC (standing for Class of Redistribution rules to Subsets or Complements) for short. This class is presented in details in [35] with several examples.

Let $m_1(.)$ and $m_2(.)$ be two normalized basic belief assignments (bba's) defined[3] from S^{Θ} to $[0,1]$. We use the conjunctive rule to first combine $m_1(.)$ with $m_2(.)$ to get $m_{\cap}(.)$ and then the mass of conflict say $m_{\cap}(X \cap Y) = 0$, when $X \cap Y = \varnothing$ or even when $X \cap Y$ is different from the empty set is redistributed to subsets or complements in many ways (see [35] for details). The new class of fusion rule (denoted $CRSC_c$) for transferring the conflicting masses only is defined for $A \in S^{\Theta} \setminus \{\varnothing, I_t\}$ by:

$$m_{CRSC_c}(A) = m_{\cap}(A) + [\alpha \cdot m_{\cap}(A) + \beta \cdot \mathrm{Card}(A) + \gamma \cdot f(A)] \cdot$$

$$\cdot \sum_{\substack{X,Y \in S^{\Theta} \\ X \cap Y = \varnothing \\ A \subseteq M}} \frac{m_1(X) m_2(Y)}{\sum_{Z \in S^{\Theta}, Z \subseteq M} [\alpha \cdot m_{\cap}(Z) + \beta \cdot \mathrm{Card}(Z) + \gamma \cdot f(Z)]}, \tag{5}$$

where $I_t = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ represents the total ignorance when $\Theta = \{\theta_1, \ldots, \theta_n\}$. M can be $c(X \cup Y)$ (the complement of $X \cup Y$), or a subset of $c(X \cup Y)$, or $X \cup Y$, or a subset of $X \cup Y$; $\alpha, \beta, \gamma \in \{0,1\}$ but $\alpha + \beta + \gamma \neq 0$; in a weighted way we can take $\alpha, \beta, \gamma \in [0,1]$ also with $\alpha + \beta + \gamma \neq 0$; $f(X)$ is a function of X, i.e. another parameter that the mass of X is directly proportionally with respect to; $\mathrm{Card}(X)$ is the cardinal of X.

The mass of belief $m_{CRSC_c}(I_t)$ committed to the total ignorance is given by:

$$m_{CRSC_c}(I_t) = m_{\cap}(I_t) + \sum_{\substack{X,Y \in S^{\Theta} \\ \{X \cap Y = \varnothing \text{ and } M = \varnothing\} \\ \text{or} \{X \cap Y = \varnothing \text{ and } Den(Z) = 0\}}} m_1(X) m_2(Y), \tag{6}$$

where $\mathrm{Den}(Z) \triangleq \sum_{Z \in S^{\Theta}, Z \subseteq M} [\alpha \cdot m_{\cap}(Z) + \beta \cdot \mathrm{Card}(Z) + \gamma \cdot f(Z)]$.

A more general formula for the redistribution of conflict and non-conflict to subsets or complements class of rules for the fusion of masses of belief for two sources of evidence is defined $A \in (S^{\Theta} \setminus S_{\cap}^{non\varnothing}) \setminus \{\varnothing, \Theta\}$ by:

$$m_{CRSC}(A) = m_{\cap}(A) + \sum_{\substack{X,Y \in S^{\Theta} \\ \{X \cap Y = \varnothing, A \in T(X,Y)\} \\ or\ \{X \cap Y \in S_{\cap,r}^{non\varnothing}, A \in T'(X,Y)\}}} f(A) \frac{m_1(X) m_2(Y)}{\sum_{Z \in T(X,Y)} f(Z)} \tag{7}$$

[3] Since these rules use explicitely the complementation operator $c(.)$, they apply only with the super-power set S^{Θ} or on 2^{Θ} depending on the underlying model chosen for the frame Θ.

and for $A = I_t$:

$$m_{CRSC}(I_t) = m_{\cap}(I_t) + \sum_{\substack{X,Y \in S^{\Theta} \\ X \cap Y = \varnothing, \\ \{T(X,Y)=\varnothing \text{ or } \sum_{Z \in T(X,Y)} f(Z)=0\}}} m_1(X) m_2(Y) \tag{8}$$

where $S_{\cap} = \{X \in S^{\Theta} \mid X = Y \cap Z, \text{ where } Y, Z \in S^{\Theta} \setminus \{\varnothing\}\}$, all propositions are expressed in their canonical form and where X contains at least an $\cap$ symbol in its expression; $S_{\cap}^{\varnothing}$ be the set of all empty intersections from $S_{\cap}$ (i.e. the set of exclusivity constraints), and $S_{\cap}^{non\varnothing}$ the set of all non-empty intersections from $S_{\cap}$. $S_{\cap,r}^{non\varnothing}$ is the set of all non-empty intersections from $S_{\cap}^{non\varnothing}$ whose masses are redistributed to other sets/propositions. The set $S_{\cap,r}^{non\varnothing}$ highly depends on the model for the frame of the application under consideration. $f(.)$ is a mapping from S^{Θ} to R^{+}. For example, we can choose $f(X) = m_{\cap}(X)$, $f(X) = |X|$, $f^{T}(X) = \dfrac{|X|}{|T(X,Y)|}$, or $f(x) = m_{\cap}(X) + |X|$, etc. The function T specifies a subset of S^{Θ}, for example $T(X,Y) = \{c(X \cup Y)\}$, or $T(X,Y) = \{X \cup Y\}$ or can specify a set of subsets of S^{Θ}. For example, $T(X,Y) = \{A \subset c(X \cup Y)\}$, or $T(X,Y) = \{A \subset X \cup Y\}$. The function T' is a subset of S^{Θ}, for example $T'(X,Y) = \{X \cup Y\}$, or T' is a subset of $X \cup Y$, etc.

It is important to highlight that in formulas (5–6) one transfers only the conflicting masses, whereas the formulas (7–8) are more general since one transfers the conflicting masses or the non-conflicting masses as well depending on the preferences of the fusion system designer. The previous formulas have been directly extended for any $s \geq 2$ sources of evidence in [35]. All denominators in these CRSC formulas are naturally supposed different from zero. It is worth to note also that the extensions of these rules for including the reliabilities of the sources are also presented in [35].

4. THE GENERALIZED PIGNISTIC TRANSFORMATION (GPT)

4.1. THE CLASSICAL PIGNISTIC TRANSFORMATION

We follow here Philippe Smets' vision which considers the management of information as a two 2-levels process: credal (for combination of evidences) and pignistic[4] (for decision-making) , *i.e.* "*when someone must take a decision, he/she must then construct a probability function derived from the belief function that describes his/her credal state. This probability function is then used to make decisions*" [38] (p. 284). One obvious way to build this probability function corresponds to the so-called Classical Pignistic Transformation (CPT) defined in DST framework (*i.e.* based on the Shafer's model assumption) as [40]:

$$\mathrm{BetP}\{A\} = \sum_{X\in 2^{\Theta}} \frac{|X\cap A|}{|X|} m(X), \tag{9}$$

where $|A|$ denotes the cardinality of A (with convention $|\varnothing|/|\varnothing|=1$, to define $\mathrm{Bet}P\{\varnothing\}$). Decisions are achieved by computing the expected utilities of the acts using the subjective/pignistic $\mathrm{Bet}P\{.\}$ as the probability function needed to compute expectations. Usually, one uses the maximum of the pignistic probability as decision criterion. The maximum of $\mathrm{Bet}P\{.\}$ is often considered as a prudent betting decision criterion between the two other alternatives (max of plausibility or max. of credibility which appears to be respectively too optimistic or too pessimistic). It is easy to show that $\mathrm{Bet}P\{.\}$ is indeed a probability function (see [39]).

4.2. NOTION OF DSm CARDINALITY

One important notion involved in the definition of the Generalized Pignistic Transformation (GPT) is the *DSm cardinality*. The *DSm cardinality* of any element A of hyper-power set D^{Θ}, denoted $C_M(A)$, corresponds to the number of parts of A in the corresponding fuzzy/vague Venn diagram of the problem (model M) taking into account the set of integrity constraints (if any), *i.e.* all the possible intersections due to the nature of the elements θ_i. This *intrinsic cardinality* depends on the model M (free, hybrid or Shafer's model). M is the model that contains A, which depends both on the dimension $n=|\Theta|$ and on the number of non-empty intersections present in its associated Venn diagram (see [29] for details). The DSm cardinality depends on the cardinal of $\Theta=\{\theta_1,\theta_2,\ldots,\theta_n\}$ and on the model of D^{Θ} (*i.e.*, the number of intersections and between what elements of Θ – in a word the structure) at the same time; it is not necessarily that every singleton, say θ_i, has the same DSm cardinal, because each singleton has a different structure; if its structure is the simplest (no intersection of this elements with other elements) then $C_M(\theta_i)=1$, if the structure is more complicated (many

[4] Pignistic terminology has been coined by Philippe Smets and comes from *pignus*, a bet in Latin.

intersections) then $C_M(\theta_i) > 1$; let's consider a singleton θ_i: if it has 1 intersection only then $C_M C_M(\theta_i) = 2$, for 2 intersections only $C_M(\theta_i)$ is 3 or 4 depending on the model M, for m intersections it is between $m+1$ and 2^m depending on the model; the maximum DSm cardinality is 2^{n-1} and occurs for $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ in the free model M^f; similarly for any set from D^Θ: the more complicated structure it has, the bigger is the DSm cardinal; thus the DSm cardinality measures the complexity of an element from D^Θ, which is a nice characterization in our opinion; we may say that for the singleton θ_i not even $|\Theta|$ counts, but only its structure (= how many other singletons intersect θ_i). Simple illustrative examples are given in Chapter 3 and 7 of [29]. One has $1 \le C_M(A) \le 2^n - 1$. $C_M(A)$ must not be confused with the classical cardinality $|A|$ of a given set A (*i.e.* the number of its distinct elements) – that's why a new notation is necessary here. $C_M(A)$ is very easy to compute by programming from the algorithm of generation of D^Θ given explicated in [29].

4.3. THE GENERALIZED PIGNISTIC TRANSFORMATION

To take a rational decision within DSmT framework, it is necessary to generalize the Classical Pignistic Transformation in order to construct a pignistic probability function from any generalized basic belief assignment $m(.)$ drawn from the DSm rules of combination. Here is the simplest and direct extension of the CPT to define the Generalized Pignistic Transformation:

$$\forall A \in D^\Theta, \qquad \mathrm{Bet}P\{A\} = \sum_{X \in D^\Theta} \frac{C_M(X \cap A)}{C_M(X)} m(X), \tag{10}$$

where $C_M(X)$ denotes the DSm cardinal of proposition X for the DSm model M of the problem under consideration.

The decision about the solution of the problem is usually taken by the maximum of pignistic probability function $\mathrm{Bet}P\{.\}$. Let's remark the close ressemblance of the two pignistic transformations (9) and (10). It can be shown that (10) reduces to (9) when the hyper-power set D^Θ reduces to classical power set 2^Θ if we adopt Shafer's model. But (10) is a generalization of (9) since it can be used for computing pignistic probabilities for any models (including Shafer's model). It has been proved in [29, Chap. 7], that $\mathrm{BetP}\{.\}$ defined in (10) is indeed a probability distribution. In the following section, we introduce a new alternative to BetP which is presented in details in [35].

5. THE DSmP TRANSFORMATION

In the theories of belief functions, the mapping from the belief to the probability domain is a controversial issue. The original purpose of such mappings was to make (hard) decision, but contrariwise to erroneous widespread idea/claim, this is not the only interest for using such mappings nowadays. Actually the probabilistic transformations of belief mass assignments (as the pignistic transformation mentioned previously) are for example very useful in modern multitarget multisensor tracking systems (or in any other systems) where one deals with soft decisions *(i.e.* where all possible solutions are kept for state estimation with their likelihoods). For example, in a Multiple Hypotheses Tracker using both kinematical and attribute data, one needs to compute all probabilities values for deriving the likelihoods of data association hypotheses and then mixing them altogether to estimate states of targets. Therefore, it is very relevant to use a mapping which provides a highly probabilistic information content (PIC) for expecting better performances.

In this section, we briefly recall a new probabilistic transformation, denoted DSmP and introduced in [8] which is explained in details in [35]. DSmP is straight and different from other transformations. The basic idea of DSmP consists in a new way of proportionalizations of the mass of each partial ignorance such as $A_1 \cup A_2$ or $A_1 \cup (A_2 \cap A_3)$ or $(A_1 \cap A_2) \cup (A_3 \cap A_4)$, etc. and the mass of the total ignorance $A_1 \cup A_2 \cup \ldots \cup A_n$, to the elements involved in the ignorances. This new transformation takes into account both the values of the masses and the cardinality of elements in the proportional redistribution process. We first remind what PIC criteria is and then shortly present the general formula for DSmP transformation with few numerical examples. More examples and comparisons with respect to other transformations are given in [35].

5.1. THE PROBABILISTIC INFORMATION CONTENT (PIC)

Following Sudano's approach [41, 42, 44], we adopt the Probabilistic Information Content (PIC) criterion as a metric depicting the strength of a critical decision by a specific probability distribution. It is an essential measure in any threshold-driven automated decision system. The PIC is the dual of the normalized Shannon entropy. A PIC value of one indicates the total knowledge to make a correct decision (one hypothesis has a probability value of one and the rest of zero). A PIC value of zero indicates that the knowledge to make a correct decision does not exist (all the hypotheses have an equal probability value), *i.e.* one has the maximal entropy. The PIC is used in our analysis to sort the performances of the different pignistic transformations through several numerical examples. We first recall what Shannon entropy and PIC measure are and their tight relationship.

• **Shannon entropy**

Shannon entropy, usually expressed in bits (binary digits), of a probability measure $P\{.\}$ over a discrete finite set $\Theta = \{\theta_1,\ldots,\theta_n\}$ is defined by[5] [23]:

$$H(P) \overset{\Delta}{=} -\sum_{i=1}^{n} P\{\theta_i\}\log_2(P\{\theta_i\}). \tag{11}$$

$H(P)$ is maximal for the uniform probability distribution over Θ, *i.e.* when $P\{\theta_i\} = 1/n$ for $i = 1,2,\ldots,n$. In that case, one gets $H(P) = H_{\max} = -\sum_{i=1}^{n} \frac{1}{n}\log_2\frac{1}{n} = \log_2 n$. $H(P)$ is minimal for a totally *deterministic* probability, *i.e.* for any $P\{.\}$ such that $P\{\theta_i\} = 1$ for some $i \in \{1,2,\ldots,n\}$ and $P\{\theta_j\} = 0$ for $j \neq i$. $H(P)$ measures the randomness carried by any discrete probability $P\{.\}$.

• **The PIC metric**

The Probabilistic Information Content (PIC) of a probability measure $P\{.\}$ associated with a probabilistic source over a discrete finite set $\Theta = \{\theta_1,\ldots,\theta_n\}$ is defined by [42]:

$$\mathrm{PIC}(P) = 1 + \frac{1}{H_{\max}} \cdot \sum_{i=1}^{n} P\{\theta_i\}\log_2(P\{\theta_i\}). \tag{12}$$

The PIC is nothing but the dual of the normalized Shannon entropy and thus is actually unit less. $\mathrm{PIC}(P)$ takes its values in $[0,1]$. $\mathrm{PIC}(P)$ is maximum, *i.e.* $\mathrm{PIC}_{\max} = 1$ with any *deterministic* probability and it is minimum, *i.e.* $\mathrm{PIC}_{\min} = 0$, with the uniform probability over the frame Θ. The simple relationships between $H(P)$ and $\mathrm{PIC}(P)$ are $\mathrm{PIC}(P) = 1 - (H(P)/H_{\max})$ and $H(P) = H_{\max} \cdot (1 - \mathrm{PIC}(P))$.

5.2. THE DSmP FORMULA

Let's consider a discrete frame Θ with a given model (free DSm model, hybrid DSm model or Shafer's model), the DSmP mapping is defined by $\mathrm{DSmP}_\varepsilon(\varnothing) = 0$ and $\forall X \in G^\Theta \setminus \{\varnothing\}$ by

$$\mathrm{DSmP}_\varepsilon(X) = \sum_{Y \in G^\Theta} \frac{\sum_{\substack{Z \subseteq X \cap Y \\ C(Z)=1}} m(Z) + \varepsilon \cdot C(X \cap Y)}{\sum_{\substack{Z \subseteq Y \\ C(Z)=1}} m(Z) + \varepsilon \cdot C(Y)} m(Y), \tag{13}$$

[5] With common convention $0\log_2 0 = 0$.

where $\varepsilon \geq 0$ is a tuning parameter and G^Θ corresponds to the generic set (2^Θ, S^Θ or D^Θ including eventually all the integrity constraints (if any) of the model M); $C(X \cap Y)$ and $C(Y)$ denote the DSm cardinals[6] of the sets $X \cap Y$ and Y respectively. ε allows to reach the maximum PIC value of the approximation of $m(.)$ into a subjective probability measure. The smaller ε, the better/bigger PIC value. In some particular degenerate cases however, the $\mathrm{DSm}P_{\varepsilon=0}$ values cannot be derived, but the $\mathrm{DSm}P_{\varepsilon>0}$ values can however always be derived by choosing ε as a very small positive number, say $\varepsilon = 1/1000$ for example in order to be as close as we want to the maximum of the PIC. When $\varepsilon = 1$ and when the masses of all elements Z having $\mathcal{C}(Z) = 1$ are zero, (13) reduces to (10), *i.e.* $\mathrm{DSm}P_{\varepsilon=1} = \mathrm{Bet}P$. The passage from a free DSm model to a Shafer's model involves the passage from a structure to another one, and the cardinals change as well in the formula (13).

DSm*P* works for all models (free, hybrid and Shafer's). In order to apply classical transformation (Pignistic, Cuzzolin's one, Sudano's ones, etc – see [35]), we need at first to refine the frame (on the cases when it is possible!) in order to work with Shafer's model, and then apply their formulas. In the case where refinement makes sense, then one can apply the other subjective probabilities on the refined frame. DSm*P* works on the refined frame as well and gives the same result as it does on the non-refined frame. Thus DSm*P* with $\varepsilon > 0$ works on any models and so is very general and appealing. DSmP does a redistribution of the ignorance mass with respect to both the singleton masses and the singletons' cardinals in the same time. Now, if all masses of singletons involved in all ignorances are different from zero, then we can take $\varepsilon = 0$, and DSmP gives the best result, i.e. the best PIC value. In summary, DSm*P* does an 'improvement' over previous known probabilistic transformations in the sense that DSm*P* mathematically makes a more accurate redistribution of the ignorance masses to the singletons involved in ignorances. DSm*P* and Bet*P* work in both theories: DST (= Shafer's model) and DSmT (= free or hybrid models) as well.

6. FUSION OF QUALITATIVE BELIEFS

We recall here the notion of qualitative belief assignment to model beliefs of human experts expressed in natural language (with linguistic labels). We show how qualitative beliefs can be efficiently combined using an extension of DSmT to qualitative reasoning. A more detailed presentation can be found in [33, 35]. The derivations are based on a new arithmetic on linguistic labels which allows a direct extension of all quantitative rules of combination and conditioning. The qualitative version of PCR5 rule and DSmP is also presented in the sequel.

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[6] We have omitted the index of the model M for the notation convenience.

6.1. QUALITATIVE OPERATORS

Computing with words (CW) and qualitative information is more vague, less precise than computing with numbers, but it offers the advantage of robustness if done correctly. Here is a general arithmetic we propose for computing with words (i.e. with linguistic labels). Let's consider a finite frame $\Theta = \{\theta_1, \ldots, \theta_n\}$ of n (exhaustive) elements θ_i, $i = 1, 2, \ldots, n$, with an associated model $M(\Theta)$ on Θ (either Shafer's model $M^0(\Theta)$, free-DSm model $M^f(\Theta)$, or more general any Hybrid-DSm model [29]). A model $M(\Theta)$ is defined by the set of integrity constraints on elements of Θ (if any); Shafer's model $M^0(\Theta)$ assumes all elements of Θ truly exclusive, while free-DSm model $M^f(\Theta)$ assumes no exclusivity constraints between elements of the frame Θ. Let's define a finite set of linguistic labels $\widetilde{L} = \{L_1, L_2, \ldots, L_m\}$ where $m \geq 2$ is an integer. $\widetilde{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \ldots \prec L_m$. To work on a close linguistic set under linguistic addition and multiplication operators, we extends $\widetilde{L}$ with two extreme values L_0 and L_{m+1} where L_0 corresponds to the minimal qualitative value and L_{m+1} corresponds to the maximal qualitative value, in such a way that

$$L_0 \prec L_1 \prec L_2 \prec \ldots \prec L_m \prec L_{m+1},$$

where $\prec$ means inferior to, or less (in quality) than, or smaller (in quality) than, etc. hence a relation of order from a qualitative point of view. But if we make a correspondence between qualitative labels and quantitative values on the scale $[0,1]$, then $L_{\min} = L_0$ would correspond to the numerical value 0, while $L_{\max} = L_{m+1}$ would correspond to the numerical value 1, and each L_i would belong to $[0,1]$, *i.e.*

$$L_{\min} = L_0 < L_1 < L_2 < \ldots < L_m < L_{m+1} = L_{\max}.$$

From now on, we work on extended ordered set L of qualitative values

$$L = \{L_0, \widetilde{L}, L_{m+1}\} = \{L_0, L_1, L_2, \ldots, L_m, L_{m+1}\}.$$

In our previous works, we did propose approximate qualitative operators, but in [35] we propose to use better and accurate operators for qualitative labels. Since these new operators are defined in details in the chapter of [35] devoted on the DSm Field and Linear Algebra of Refined Labels (FLARL), we just briefly introduce here only the the main ones (i.e. the accurate label addition, multiplication and division). In FLARL, we can replace the "qualitative quasi-normalization" of qualitative operators we used in our previous papers by "qualitative normalization" since in FLARL we have exact qualitative calculations and exact normalization.

- Label addition :

$$L_a + L_b = L_{a+b}, \tag{14}$$

since $\dfrac{a}{m+1} + \dfrac{b}{m+1} = \dfrac{a+b}{m+1}$.

- Label multiplication :

$$L_a \times L_b = L_{(ab)/(m+1)}, \tag{15}$$

since $\dfrac{a}{m+1} \cdot \dfrac{b}{m+1} = \dfrac{(ab)/(m+1)}{m+1}$.

- Label division (when $L_b \neq L_0$):

$$L_a \div L_b = L_{(a/b)(m+1)}, \tag{16}$$

since $\dfrac{a}{m+1} \div \dfrac{b}{m+1} = \dfrac{a}{b} = \dfrac{(a/b)(m+1)}{m+1}$.

More accurate qualitative operations (substraction, scalar multiplication, scalar root, scalar power, etc) can be found in [35]. Of course, if one really needs to stay within the original set of labels, an approximation will be necessary at the very end of the calculations.

6.2. QUALITATIVE BELIEF ASSIGNMENT

A qualitative belief assignment[7] (qba) is a mapping function $qm(.): G^{\Theta} \to L$ where G^{Θ} corresponds either to 2^{Θ}, to D^{Θ} or even to S^{Θ} depending on the model of the frame Θ we choose to work with. In the case when the labels are equidistant, *i.e.* the qualitative distance between any two consecutive labels is the same, we get an exact qualitative result, and a qualitative basic belief assignment (bba) is considered normalized if the sum of all its qualitative masses is equal to $L_{\max} = L_{m+1}$. If the labels are not equidistant, we still can use all qualitative operators defined in the FLARL, but the qualitative result is approximate, and a qualitative bba is considered quasi-normalized if the sum of all its masses is equal to $L_{\max}$. Using the qualitative operator of FLARL, we can easily extend all the combination and conditioning rules from quantitative to qualitative. In the sequel we will consider $s \geq 2$ qualitative belief assignments $qm_1(.),\ldots,qm_s(.)$ defined over the same space G^{Θ} and provided by s independent sources $S_1,\ldots,S_s$ of evidence.

Note. The addition and multiplication operators used in all qualitative fusion formulas in next sections correspond to *qualitative addition* and *qualitative*

[7] We call it also qualitative belief mass or *q*-mass for short.

multiplication operators and must not be confused with classical addition and multiplication operators for numbers.

6.3. QUALITATIVE CONJUNCTIVE RULE

The qualitative Conjunctive Rule (qCR) of $s \geq 2$ sources is defined similarly to the quantitative conjunctive consensus rule, *i.e.*

$$qm_{qCR}(X) = \sum_{\substack{X_1,\ldots,X_s \in G^{\Theta} \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i). \tag{17}$$

The total qualitative conflicting mass is given by

$$K_{1\ldots s} = \sum_{\substack{X_1,\ldots,X_s \in G^{\Theta} \\ X_1 \cap \ldots \cap X_s = \varnothing}} \prod_{i=1}^{s} qm_i(X_i).$$

6.4. QUALITATIVE DSm CLASSIC RULE

The qualitative DSm Classic rule (q-DSmC) for $s \geq 2$ is defined similarly to DSm Classic fusion rule (DSmC) as follows : $qm_{qDSmC}(\varnothing) = L_0$ and for all $X \in D^{\Theta} \setminus \{\varnothing\}$,

$$qm_{qDSmC}(X) = \sum_{\substack{X_1,\ldots,X_s \in D^{\Theta} \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i). \tag{18}$$

6.5. QUALITATIVE HYBRID DSm RULE

The qualitative hybrid DSm rule (q-DSmH) is defined similarly to quantitative hybrid DSm rule [29] as follows:

$$qm_{qDSmH}(\varnothing) = L_0, \tag{19}$$

and for all $X \in G^{\Theta} \setminus \{\varnothing\}$

$$qm_{qDSmH}(X) \stackrel{\Delta}{=} \phi(X) \cdot [qS_1(X) + qS_2(X) + qS_3(X)], \tag{20}$$

where all sets involved in formulas are in the canonical form and $\phi(X)$ is the *characteristic non-emptiness function* of a set X, *i.e.* $\phi(X) = L_{m+1}$ if $X \notin \varnothing$ and $\phi(X) = L_0$ otherwise, where $\varnothing \underline{\underline{\Delta}} \{\varnothing_M, \varnothing\}$. $\varnothing_M$ is the set of all elements of D^{Θ} which have been forced to be empty through the constraints of the model M and $\varnothing$ is the classical/universal empty set. $qS_1(X) \equiv qm_{qDSmC}(X)$, $qS_2(X)$,

$qS_3(X)$ are defined by

$$qS_1(X) \triangleq \sum_{\substack{X_1, X_2, \dots, X_s \in D^\Theta \\ X_1 \cap X_2 \cap \dots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i), \tag{21}$$

$$qS_2(X) \triangleq \sum_{\substack{X_1, X_2, \dots, X_s \in \varnothing \\ [U = X] \vee [(U \in \varnothing) \wedge (X = I_t)]}} \prod_{i=1}^{s} qm_i(X_i), \tag{22}$$

$$qS_3(X) \triangleq \sum_{\substack{X_1, X_2, \dots, X_k \in D^\Theta \\ X_1 \cup X_2 \cup \dots \cup X_s = X \\ X_1 \cap X_2 \cap \dots \cap X_s \in \varnothing}} \prod_{i=1}^{s} qm_i(X_i), \tag{23}$$

with $U \stackrel{\Delta}{=} u(X_1) \cup \dots \cup u(X_s)$ where $u(X)$ is the union of all θ_i that compose X, $I_t \triangleq \theta_1 \cup \dots \cup \theta_n$ is the total ignorance. $qS_1(X)$ is nothing but the qDSmC rule for s independent sources based on $M^f(\Theta)$; $qS_2(X)$ is the qualitative mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $qS_3(X)$ transfers the sum of relatively empty sets directly onto the canonical disjunctive form of non-empty sets. qDSmH generalizes qDSmC works for any models (free DSm model, Shafer's model or any hybrid models) when manipulating qualitative belief assignments.

6.6. QUALITATIVE PCR5 RULE (qPCR5)

In classical (*i.e.* quantitative) DSmT framework, the Proportional Conflict Redistribution rule no. 5 (PCR5) defined in [33] has been proven to provide very good and coherent results for combining (quantitative) belief masses, see [7, 31]. When dealing with qualitative beliefs within the DSm Field and Linear Algebra of Refined Labels [35] we get an exact qualitative result no matter what fusion rule is used (DSm fusion rules, Dempster's rule, Smets's rule, Dubois-Prade's rule, etc.). The exact qualitative result will be a refined label (but the user can round it up or down to the closest integer index label).

6.7. A SIMPLE EXAMPLE OF QUALITATIVE FUSION OF qba'S

Let's consider the following set of ordered linguistic labels

$$L = \{L_0, L_1, L_2, L_3, L_4, L_5\}$$

(for example, L_1, L_2, L_3 and L_4 may represent the values: $L_1 \triangleq \text{very poor}$, $L_2 \triangleq \text{poor}$, $L_3 \triangleq \text{good}$ and $L_4 \triangleq \text{very good}$, where $\triangleq$ symbol means *by definition*).

Let's consider now a simple two-source case with a 2D frame $\Theta = \{\theta_1, \theta_2\}$, Shafer's model for Θ, and qba's expressed as follows:

$$qm_1(\theta_1) = L_1, \quad qm_1(\theta_2) = L_3, \quad qm_1(\theta_1 \cup \theta_2) = L_1,$$

$$qm_2(\theta_1) = L_2, \quad qm_2(\theta_2) = L_1, \quad qm_2(\theta_1 \cup \theta_2) = L_2.$$

The two qualitative masses $qm_1(.)$ and $qm_2(.)$ are normalized since:

$$qm_1(\theta_1) + qm_1(\theta_2) + qm_1(\theta_1 \cup \theta_2) = L_1 + L_3 + L_1 = L_{1+3+1} = L_5$$

and

$$qm_2(\theta_1) + qm_2(\theta_2) + qm_2(\theta_1 \cup \theta_2) = L_2 + L_1 + L_2 = L_{2+1+2} = L_5.$$

We first derive the result of the conjunctive consensus. This yields:

$$\begin{aligned} qm_{12}(\theta_1) &= qm_1(\theta_1)qm_2(\theta_1) + qm_1(\theta_1)qm_2(\theta_1 \cup \theta_2) + qm_1(\theta_1 \cup \theta_2)qm_2(\theta_1) \\ &= L_1 \times L_2 + L_1 \times L_2 + L_1 \times L_2 = \\ &= L_{\frac{1\cdot 2}{5}} + L_{\frac{1\cdot 2}{5}} + L_{\frac{1\cdot 2}{5}} = L_{\frac{2}{5}+\frac{2}{5}+\frac{2}{5}} = L_{\frac{6}{5}} = L_{1.2}, \end{aligned}$$

$$\begin{aligned} qm_{12}(\theta_2) &= qm_1(\theta_2)qm_2(\theta_2) + qm_1(\theta_2)qm_2(\theta_1 \cup \theta_2) + qm_1(\theta_1 \cup \theta_2)qm_2(\theta_2) = \\ &= L_3 \times L_1 + L_3 \times L_2 + L_1 \times L_1 = \\ &= L_{\frac{3\cdot 1}{5}} + L_{\frac{3\cdot 2}{5}} + L_{\frac{1\cdot 1}{5}} = L_{\frac{3}{5}+\frac{6}{5}+\frac{1}{5}} = L_{\frac{10}{5}} = L_2, \end{aligned}$$

$$qm_{12}(\theta_1 \cup \theta_2) = qm_1(\theta_1 \cup \theta_2)qm_2(\theta_1 \cup \theta_2) = L_1 \times L_2 = L_{\frac{1\cdot 2}{5}} = L_{\frac{2}{5}} = L_{0.4}$$

$$\begin{aligned} qm_{12}(\theta_1 \cap \theta_2) &= qm_1(\theta_1)qm_2(\theta_2) + qm_1(\theta_2)qm_2(\theta_1) = \\ &= L_1 \times L_1 + L_2 \times L_3 = L_{\frac{1\cdot 1}{5}} + L_{\frac{2\cdot 3}{5}} = \\ &= L_{\frac{1}{5}+\frac{6}{5}} = L_{\frac{7}{5}} = L_{1.4}. \end{aligned}$$

Therefore we get:

- for the fusion with qDSmC, when assuming $\theta_1 \cap \theta_2 \neq \emptyset$,

$$qm_{qDSmC}(\theta_1) = L_{1.2}, \qquad qm_{qDSmC}(\theta_2) = L_2,$$

$$qm_{qDSmC}(\theta_1 \cup \theta_2) = L_{0.4}, \qquad qm_{qDSmC}(\theta_1 \cap \theta_2) = L_{1.4};$$

- for the fusion with qDSmH, when assuming $\theta_1 \cap \theta_2 = \varnothing$. The mass of $\theta_1 \cap \theta_2$ is transferred to $\theta_1 \cup \theta_2$. Hence:

$$qm_{qDSmH}(\theta_1) = L_{1.2}, \qquad qm_{qDSmH}(\theta_2) = L_2,$$

$$qm_{qDSmH}(\theta_1 \cap \theta_2) = L_0, \quad qm_{qDSmH}(\theta_1 \cup \theta_2) = L_{0.4} + L_{1.4} = L_{1.8};$$

- for the fusion with qPCR5, when assuming $\theta_1 \cap \theta_2 = \varnothing$. The mass $qm_{12}(\theta_1 \cap \theta_2) = L_{1.4}$ is transferred to θ_1 and to θ_2 in the following way:

$$qm_{12}(\theta_1 \cap \theta_2) = qm_1(\theta_1)qm_2(\theta_2) + qm_2(\theta_1)qm_1(\theta_2).$$

Then, $qm_1(\theta_1)qm_2(\theta_2) = L_1 \times L_1 = L_{\frac{1\cdot 1}{5}} = L_{\frac{1}{5}} = L_{0.2}$ is redistributed to θ_1 and θ_2 proportionally with respect to their qualitative masses put in the conflict L_1 and respectively L_1:

$$\frac{x_{\theta_1}}{L_1} = \frac{y_{\theta_2}}{L_1} = \frac{L_{0.2}}{L_1 + L_1} = \frac{L_{0.2}}{L_{1+1}} = \frac{L_{0.2}}{L_2} = L_{\frac{0.2}{2}\cdot 5} = L_{\frac{1}{2}} = L_{0.5},$$

whence $x_{\theta_1} = y_{\theta_2} = L_1 \times L_{0.5} = L_{\frac{1\cdot 0.5}{5}} = L_{\frac{0.5}{5}} = L_{0.1}$.

Actually, we could easier see that $qm_1(\theta_1)qm_2(\theta_2) = L_{0.2}$ had in this case to be equally split between θ_1 and θ_2 since the mass put in the conflict by θ_1 and θ_2 was the same for each of them: L_1. Therefore $\frac{L_{0.2}}{2} = L_{\frac{0.2}{2}} = L_{0.1}$.

Similarly, $qm_2(\theta_1)qm_1(\theta_2) = L_2 \times L_3 = L_{\frac{2\cdot 3}{5}} = L_{\frac{6}{5}} = L_{1.2}$ has to be redistributed to θ_1 and θ_2 proportionally with L_2 and L_3 respectively:

$$\frac{x'_{\theta_1}}{L_2} = \frac{y'_{\theta_2}}{L_3} = \frac{L_{1.2}}{L_2 + L_3} = \frac{L_{1.2}}{L_{2+3}} = \frac{L_{1.2}}{L_5} = L_{\frac{1.2}{5}\cdot 5} = L_{1.2},$$

whence $\begin{cases} x'_{\theta_1} = L_2 \times L_{1.2} = L_{\frac{2\cdot 1.2}{5}} = L_{\frac{2.4}{5}} = L_{0.48} \\ y'_{\theta_2} = L_3 \times L_{1.2} = L_{\frac{3\cdot 1.2}{5}} = L_{\frac{3.6}{5}} = L_{0.72} \end{cases}$ Now, add all these to the qualitative masses of θ_1 and θ_2 respectively:

$$qm_{qPCR5}(\theta_1) = qm_{12}(\theta_1) + x_{\theta_1} + x'_{\theta_1} = L_{1.2} + L_{0.1} + L_{0.48} = L_{1.2+0.1+0.48} = L_{1.78},$$

$$qm_{qPCR5}(\theta_2) = qm_{12}(\theta_2) + y_{\theta_2} + y'_{\theta_2} = L_2 + L_{0.1} + L_{0.72} = L_{2+0.1+0.72} = L_{2.82},$$

$$qm_{qPCR5}(\theta_1 \cup \theta_2) = qm_{12}(\theta_1 \cup \theta_2) = L_{0.4},$$

$$qm_{qPCR5}(\theta_1 \cap \theta_2) = L_0.$$

The qualitative mass results using all fusion rules (qDSmC,qDSmH,qPCR5) remain normalized in FLARL.

Naturally, if one prefers to express the final results with qualitative labels belonging in the original discrete set of labels $L = \{L_0, L_1, L_2, L_3, L_4, L_5\}$, some approximations will be necessary to round continuous indexed labels to their closest integer/discrete index value; by example, $qm_{qPCR5}(\theta_1) = L_{1.78} \approx L_2$, $qm_{qPCR5}(\theta_2) = L_{2.82} \approx L_3$ and $qm_{qPCR5}(\theta_1 \cup \theta_2) = L_{0.4} \approx L_0$.

6.8. A SIMPLE EXAMPLE FOR THE qDSmP TRANSFORMATION

We first recall that the qualitative extension of (13), denoted $qDSmP_\varepsilon(.)$ is given by $q\mathrm{DSm}P_\varepsilon(\emptyset) = 0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$ by

$$q\mathrm{DSm}P_\varepsilon(X) = \sum_{Y \in G^\Theta} \frac{\sum_{\substack{Z \subseteq X \cap Y \\ C(Z)=1}} qm(Z) + \varepsilon \cdot C(X \cap Y)}{\sum_{\substack{Z \subseteq Y \\ C(Z)=1}} qm(Z) + \varepsilon \cdot C(Y)} qm(Y), \quad (24)$$

where all operations in (24) are referred to labels, that is q-operators on linguistic labels and not classical operators on numbers.

Let's consider the simple frame $\Theta = \{\theta_1, \theta_2\}$ (here $n = |\Theta| = 2$) with Shafer's model (i.e. $\theta_1 \cap \theta_2 = \emptyset$) and the following set of linguistic labels $L = \{L_0, L_1, L_2, L_3, L_4, L_5\}$, with $L_0 = L_{\min}$ and $L_5 = L_{\max} = L_{m+1}$ (here $m = 4$) and the following qualitative belief assignment: $qm(\theta_1) = L_1$, $qm(\theta_2) = L_3$ and $qm(\theta_1 \cup \theta_2) = L_1$. $qm(.)$ is quasi-normalized since $\sum_{X \in 2^\Theta} qm(X) = L_5 = L_{\max}$. In this example and with DSmP transformation, $qm(\theta_1 \cup \theta_2) = L_1$ is redistributed to θ_1 and θ_2 proportionally with respect to their qualitative masses L_1 and L_3 respectively. Since both L_1 and L_3 are different from L_0, we can take the tuning parameter $\varepsilon = 0$ for the best transfer. ε is taken different from zero when a mass of a set involved in a partial or total ignorance is zero (for qualitative masses, it means L_0).

Therefore using (16), one has

$$\frac{x_{\theta_1}}{L_1} = \frac{x_{\theta_2}}{L_3} = \frac{L_1}{L_1 + L_3} = \frac{L_1}{L_4} = L_{\frac{1}{4}\cdot 5} = L_{\frac{5}{4}} = L_{1.25}$$

and thus using (15), one gets

$$x_{\theta_1} = L_1 \times L_{1.25} = L_{\frac{1\cdot(1.25)}{5}} = L_{\frac{1.25}{5}} = L_{0.25},$$

$$x_{\theta_2} = L_3 \times L_{1.25} = L_{\frac{3\cdot(1.25)}{5}} = L_{\frac{3.75}{5}} = L_{0.75}.$$

Therefore,

$$q\text{DSm}P_{\varepsilon=0}(\theta_1 \cap \theta_2) = q\text{DSm}P_{\varepsilon=0}(\varnothing) = L_0,$$

$$q\text{DSm}P_{\varepsilon=0}(\theta_1) = L_1 + x_{\theta_1} = L_1 + L_{0.25} = L_{1.25},$$

$$q\text{DSm}P_{\varepsilon=0}(\theta_2) = L_3 + x_{\theta_2} = L_3 + L_{0.75} = L_{3.75}.$$

Naturally in our example, one has also

$$q\text{DSm}P_{\varepsilon=0}(\theta_1 \cup \theta_2) = q\text{DSm}P_{\varepsilon=0}(\theta_1) + q\text{DSm}P_{\varepsilon=0}(\theta_2) - q\text{DSm}P_{\varepsilon=0}(\theta_1 \cap \theta_2) =$$
$$= L_{1.25} + L_{3.75} - L_0 = L_5 = L_{\max}.$$

Since $H_{\max} = \log_2 n = \log_2 2 = 1$, using the qualitative extension of PIC formula (12), one obtains the following qualitative PIC value:

$$\text{PIC} = 1 + \frac{1}{1} \cdot [q\text{DSm}P_{\varepsilon=0}(\theta_1) \log_2 (q\text{DSm}P_{\varepsilon=0}(\theta_1)) +$$
$$+ q\text{DSm}P_{\varepsilon=0}(\theta_2) \log_2 (q\text{DSm}P_{\varepsilon=0}(\theta_2))]$$
$$= 1 + L_{1.25} \log_2 (L_{1.25}) + L_{3.75} \log_2 (L_{3.75}) \approx L_{0.94},$$

since we considered the isomorphic transformation $L_i = i/(m+1)$, in our particular example $m = 4$ interior labels.

7. BELIEF CONDITIONING RULES

7.1. SHAFER'S CONDITIONING RULE (SCR)

Until very recently, the most commonly used conditioning rule for belief revision was the one proposed by Shafer [22] and referred here as Shafer's

Conditioning Rule (SCR). The SCR consists in combining the prior bba $m(.)$ with a specific bba focused on A with Dempster's rule of combination for transferring the conflicting mass to non-empty sets in order to provide the revised bba. In other words, the conditioning by a proposition A, is obtained by SCR as follows :

$$m_{SCR}(.|A) = [m \oplus m_S](.), \tag{25}$$

where $m(.)$ is the prior bba to update, A is the conditioning event, $m_S(.)$ is the bba focused on A defined by $m_S(A)=1$ and $m_S(X)=0$ for all $X \neq A$ and $\oplus$ denotes Dempster's rule of combination [22].

The SCR approach based on Dempster's rule of combination of the prior bba with the bba focused on the conditioning event remains *subjective* since actually in such belief revision process both sources are subjective and in our opinions SCR doesn't manage satisfactorily the objective nature/absolute truth carried by the conditioning term. Indeed, when conditioning a prior mass $m(.)$, *knowing* (or assuming) that the truth is in A, means that we have in hands an absolute (not subjective) knowledge, *i.e.* the truth in *A* has occurred (or is assumed to have occurred), thus *A* is realized (or is assumed to be realized) and this is (or at least must be interpreted as) an absolute truth. The conditioning term "Given *A*" must therefore be considered as an absolute truth, while $m_S(A)=1$ introduced in SCR cannot refer to an absolute truth actually, but only to a *subjective certainty* on the possible occurrence of *A* from a *virtual* second source of evidence. The advantage of SCR remains undoubtedly in its simplicity and the main argument in its favor is its coherence with the conditional probability when manipulating Bayesian belief assignment. But in our opinion, SCR should better be interpreted as the fusion of $m(.)$ with a particular subjective bba $m_S(A)=1$ rather than an objective belief conditioning rule. This fundamental remark motivated us to develop a new family of BCR [33] based on hyper-power set decomposition (HPSD) explained briefly in the next section. It turns out that many BCR are possible because the redistribution of masses of elements outside of *A* (the conditioning event) to those inside *A* can be done in *n*-ways. This will be briefly presented right after the next section.

7.2. HYPER-POWER SET DECOMPOSITION (HPSD)

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, $n \geq 2$, a model $M(\Theta)$ associated for Θ (free DSm model, hybrid or Shafer's model) and its corresponding hyper-power set D^Θ. Let's consider a (quantitative) basic belief assignment (bba) $m(.): D^\Theta \mapsto [0,1]$ such that $\sum_{X \in D^\Theta} m(X) = 1$. Suppose one finds out that the truth is in the set $A \in D^\Theta \setminus \{\emptyset\}$. Let $\mathrm{P_D}(A) = 2^A \cap D^\Theta \setminus \{\emptyset\}$, *i.e.* all non-empty parts (subsets) of A which are included in D^Θ. Let's consider the normal cases when $A \neq \emptyset$ and

$\sum_{Y \in P_D(A)} m(Y) > 0$. For the degenerate case when the truth is in $A = \varnothing$, we consider Smets' open-world, which means that there are other hypotheses $\Theta' = \{\theta_{n+1}, \theta_{n+2}, \ldots \theta_{n+m}\}$, $m \geq 1$, and the truth is in $A \in D^{\Theta'} \setminus \{\varnothing\}$. If $A = \varnothing$ and we consider a close-world, then it means that the problem is impossible. For another degenerate case, when $\sum_{Y \in \mathrm{P_D}(A)} m(Y) = 0$, *i.e.* when the source gave us a totally (100%) wrong information $m(.)$, then, we define: $m(A \mid A) \overset{\Delta}{=} 1$ and, as a consequence, $m(X \mid A) = 0$ for any $X \neq A$. Let $s(A) = \{\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_p}\}$, $1 \leq p \leq n$, be the singletons/atoms that compose A (for example, if $A = \theta_1 \cup (\theta_3 \cap \theta_4)$ then $s(A) = \{\theta_1, \theta_3, \theta_4\}$). The Hyper-Power Set Decomposition (HPSD) of $D^{\Theta} \setminus \varnothing$ consists in its decomposition into the three following subsets generated by A:

- $D_1 = P_D(A)$, the parts of A which are included in the hyper-power set, except the empty set;
- $D_2 = \{(\Theta \setminus s(A)), \cup, \cap\} \setminus \{\varnothing\}$, *i.e.* the sub-hyper-power set generated by $\Theta \setminus s(A)$ under $\cup$ and $\cap$, without the empty set;
- $D_3 = (D^{\Theta} \setminus \{\varnothing\}) \setminus (D_1 \cup D_2)$; each set from D_3 has in its formula singletons from both $s(A)$ and $\Theta \setminus s(A)$ in the case when $\Theta \setminus s(A)$ is different from empty set.

D_1, D_2 and D_3 have no element in common two by two and their union is $D^{\Theta} \setminus \{\varnothing\}$.

Simple example of HPSD. Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with Shafer's model (*i.e.* all elements of Θ are exclusive) and let's assume that the truth is in $\theta_2 \cup \theta_3$, *i.e.* the conditioning term is $\theta_2 \cup \theta_3$. Then one has the following HPSD: $D_1 = \{\theta_2, \theta_3, \theta_2 \cup \theta_3\}$, $D_2 = \{\theta_1\}$ and $D_3 = \{\theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3\}$. More complex and detailed examples can be found in [33].

7.3. QUANTITATIVE BELIEF CONDITIONING RULES (BCR)

Since there exists actually many ways for redistributing the masses of elements outside of A (the conditioning event) to those inside A, several BCR's have been proposed in [33]. In this introduction, we will not browse all the possibilities for doing these redistributions and all BCR's formulas but only one, the BCR number 17 (*i.e.* BCR17) which does in our opinion the most refined redistribution since:

– the mass $m(W)$ of each element W in $D_2 \cup D_3$ is transferred to those

$X \in D_1$ elements which are included in W if any proportionally with respect to their non-empty masses;

– if no such X exists, the mass $m(W)$ is transferred in a pessimistic/prudent way to the k-largest element from D_1 which are included in W (in equal parts) if any;

– if neither this way is possible, then $m(W)$ is indiscriminately distributed to all $X \in D_1$ proportionally with respect to their nonzero masses.

BCR17 is defined by the following formula (see [33], Chap. 9 for detailed explanations and examples):

$$m_{BCR17}(X \mid A) = m(X) \cdot \left[S_{D_1} + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W) \neq 0}} \frac{m(W)}{S(W)} \right] + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W,\, X \text{ is } k\text{-largest} \\ S(W) = 0}} m(W)/k, \qquad (26)$$

where "X is *k-largest*" means that X is the k-largest (with respect to inclusion) set included in W and

$$S(W) \triangleq \sum_{Y \in D_1, Y \subset W} m(Y),$$

$$S_{D_1} \triangleq \frac{\displaystyle\sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \,|\, \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z)}{\displaystyle\sum_{Y \in D_1} m(Y)}.$$

Note. The authors mentioned in an Erratum to the printed version of the second volume of DSmT book series (http://fs.gallup.unm.edu//Erratum.pdf) and they also corrected the online version of the aforementioned book (see page 240 in http://fs.gallup.unm.edu//DSmT-book2.pdf that all denominators of the BCR's formulas are naturally supposed to be different from zero. Of course, Shafer's conditioning rule as stated in Theorem 3.6, page 67 of [22] does not work when the denominator is zero and that's why Shafer has introduced the condition $\text{Bel}(\overline{B}) < 1$ (or equivalently $Pl(B) > 0$) in his theorem when the conditioning term is B.

A simple example for BCR17. Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with Shafer's model (i.e. all elements of Θ are exclusive) and let's assume that the truth is in $\theta_2 \cup \theta_3$, *i.e.* the conditioning term is $A \overset{\Delta}{=} \theta_2 \cup \theta_3$. Then one has the following HPSD:

$$D_1 = \{\theta_2, \theta_3, \theta_2 \cup \theta_3\}, \qquad D_2 = \{\theta_1\},$$

$$D_3 = \{\theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3\}.$$

Let's consider the following prior bba: $m(\theta_1) = 0.2$, $m(\theta_2) = 0.1$, $m(\theta_3) = 0.2$, $m(\theta_1 \cup \theta_2) = 0.1$, $m(\theta_2 \cup \theta_3) = 0.1$ and $m(\theta_1 \cup \theta_2 \cup \theta_3) = 0.3$.

With BCR17, for D_2, $m(\theta_1) = 0.2$ is transferred proportionally to all elements of D_1, i.e. $\frac{x_{\theta_2}}{0.1} = \frac{y_{\theta_3}}{0.2} = \frac{z_{\theta_2 \cup \theta_3}}{0.1} = \frac{0.2}{0.4} = 0.5$ whence the parts of $m(\theta_1)$ redistributed to θ_2, θ_3 and $\theta_2 \cup \theta_3$ are respectively $x_{\theta_2} = 0.05$, $y_{\theta_3} = 0.10$, and $z_{\theta_2 \cup \theta_3} = 0.05$. For D_3, there is actually no need to transfer $m(\theta_1 \cup \theta_3)$ because $m(\theta_1 \cup \theta_3) = 0$ in this example; whereas $m(\theta_1 \cup \theta_2) = 0.1$ is transferred to θ_2 (no case of k-elements herein); $m(\theta_1 \cup \theta_2 \cup \theta_3) = 0.3$ is transferred to θ_2, θ_3 and $\theta_2 \cup \theta_3$ proportionally to their corresponding masses:

$$x_{\theta_2}/0.1 = y_{\theta_3}/0.2 = z_{\theta_2 \cup \theta_3}/0.1 = 0.3/0.4 = 0.75,$$

whence $x_{\theta_2} = 0.075$, $y_{\theta_3} = 0.15$, and $z_{\theta_2 \cup \theta_3} = 0.075$. Finally, one gets

$$m_{BCR17}(\theta_2 \mid \theta_2 \cup \theta_3) = 0.10 + 0.05 + 0.10 + 0.075 = 0.325,$$

$$m_{BCR17}(\theta_3 \mid \theta_2 \cup \theta_3) = 0.20 + 0.10 + 0.15 = 0.450,$$

$$m_{BCR17}(\theta_2 \cup \theta_3 \mid \theta_2 \cup \theta_3) = 0.10 + 0.05 + 0.075 = 0.225,$$

which is different from the result obtained with SCR, since one gets in this example:

$$m_{SCR}(\theta_2 \mid \theta_2 \cup \theta_3) = m_{SCR}(\theta_3 \mid \theta_2 \cup \theta_3) = 0.25,$$

$$m_{SCR}(\theta_2 \cup \theta_3 \mid \theta_2 \cup \theta_3) = 0.50.$$

More complex and detailed examples can be found in [33].

7.4. QUALITATIVE BELIEF CONDITIONING RULES

In this section we present only the qualitative belief conditioning rule no 17 which extends the principles of the previous quantitative rule BCR17 in the qualitative domain using the operators on linguistic labels defined previously. We consider from now on a general frame $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, a given model $M(\Theta)$ with its hyper-power set D^Θ and a given extended ordered set L of qualitative values $L=\{L_0,L_1,L_2,...,L_m,L_{m+1}\}$. The prior qualitative basic belief assignment (qbba) taking its values in L is denoted $qm(.)$. We assume in the sequel that the conditioning event is $A\neq\emptyset$, $A\in D^{\Theta}$, *i.e.* the absolute truth is in A. The approach we present here is a direct extension of BCR17 using FLARL operators. Such extension can be done with all quantitative BCR's rules proposed in [33], but only qBCR17 is presented here for the sake of space limitations.

7.4.1. Qualitative belief conditioning rule no 17 (qBCR17)

Similarly to BCR17, qBCR17 is defined by the following formula:

$$qm_{qBCR17}(X\mid A)=qm(X)\cdot[qS_{D_1}+\sum_{\substack{W\in D_2\cup D_3\\ X\subset W\\ qS(W)\neq 0}}\frac{qm(W)}{qS(W)}]+\sum_{\substack{W\in D_2\cup D_3\\ X\subset W,X\,\text{is}\,k-\text{largest}\\ qS(W)=0}}qm(W)/k, \tag{27}$$

where "*X* is *k*-largest" means that X is the k-largest (with respect to inclusion) set included in W and

$$qS(W)\stackrel{\Delta}{=}\sum_{Y\in D_1,Y\subset W}qm(Y),$$

$$S_{D_1}\stackrel{\Delta}{=}\frac{\displaystyle\sum_{\substack{Z\in D_1,\\ or\,Z\in D_2\,\exists Y\in D_1\,with\,Y\subset Z}}qm(Z)}{\displaystyle\sum_{Y\in D_1}qm(Y)}.$$

Naturally, all operators (summation, product, division, etc) involved in the formula (27) are the operators defined in FLARL working on linguistic labels. It is worth to note that the formula (27) requires also the division of the label $qm(W)$ by a scalar k. This division is defined as follows:

Let $r\in R, r\neq 0$. Then the label division by a scalar is defined by

$$\frac{L_a}{r}=L_{a/r} \tag{28}$$

Let's consider $L=\{L_0,L_1,L_2,L_3,L_4,L_5,L_6\}$ a set of ordered linguistic labels. For example, L_1, L_2, L_3, L_4 and L_5 may represent the values: $L_1\stackrel{\Delta}{=}\text{very poor}$, $L_2\stackrel{\Delta}{=}\text{poor}$, $L_3\stackrel{\Delta}{=}\text{medium}$, $L_4\stackrel{\Delta}{=}\text{good}$ and $L_5\stackrel{\Delta}{=}\text{very good}$. Let's consider also the frame $\Theta=\{A,B,C,D\}$ with the hybrid model corresponding to the Venn diagram (Fig. 1).

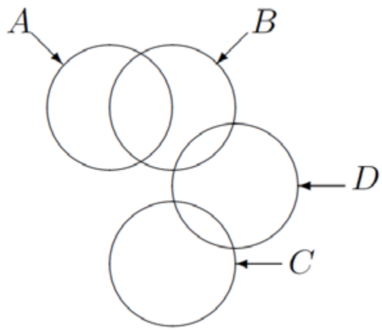

Fig. 1 – Venn diagram for the hibrid model for this example.

7.4.2. A simple example for qBCR17

We assume that the prior qualitative bba $qm(.)$ is given by:

$$qm(A) = L_1, \quad qm(C) = L_1, \quad qm(D) = L_4$$

and the qualitative masses of all other elements of G^{Θ} take the minimal/zero value L_0. This qualitative mass is normalized since $L_1 + L_1 + L_4 = L_{1+1+4} = L_6 = L_{\max}$.

If we assume that the conditioning event is the proposition $A \cup B$, i.e. the absolute truth is in $A \cup B$, the hyper-power set decomposition (HPSD) is obtained as follows: D_1 is formed by all parts of $A \cup B$, D_2 is the set generated by $\{(C,D),\cup,\cap\} \setminus \varnothing = \{C, D, C \cup D, C \cap D\}$, and $D_3 = \{A \cup C, A \cup D, B \cup C, B \cup D, A \cup B \cup C, A \cup (C \cap D), \ldots\}$. Because the truth is in $A \cup B$, $qm(D) = L_4$ is transferred in a prudent way to $(A \cup B) \cap D = B \cap D$ according to our hybrid model, because $B \cap D$ is the first-largest element from $A \cup B$ which is included in D. While $qm(C) = L_1$ is transferred to A only, since it is the only element in $A \cup B$ whose qualitative mass $qm(A)$ is different from L_0 (zero); hence:

$$qm_{qBCR17}(A \mid A \cup B) = qm(A) + qm(C) = L_1 + L_1 = L_{1+1} = L_2.$$

Therefore, one finally gets:

$$qm_{qBCR17}(A \mid A \cup B) = L_2, \qquad qm_{qBCR17}(C \mid A \cup B) = L_0,$$

$$qm_{qBCR17}(D \mid A \cup B) = L_0, \qquad qm_{qBCR17}(B \cap D \mid A \cup B) = L_4,$$

which is a normalized qualitative bba.

More complicated examples based on other qBCR's can be found in [34].

10. CONCLUSION

A general presentation of the foundations of DSmT has been proposed in this introduction. DSmT proposes new quantitative and qualitative rules of combination for uncertain, imprecise and highly conflicting sources of information. Several applications of DSmT have been proposed recently in the literature and show the potential and the efficiency of this new theory. DSmT offers the possibility to work in different fusion spaces depending on the nature of problem under consideration. Thus, one can work either in $2^{\Theta} = (\Theta,\cup)$ (*i.e.* in the classical power set as in DST framework), in $D^{\Theta} = (\Theta,\cup,\cap)$ (the hyper-power set Ń also known as Dedekind's lattice) or in the super-power set $S^{\Theta} = (\Theta,\cup,\cap,c(.))$, which includes 2^{Θ} and D^{Θ} and which represents the power set of the minimal refinement of the frame Θ when the refinement is possible (because for vague elements whose frontiers are not well known the refinement is not possible). We have enriched the DSmT with a subjective probability ($\mathrm{DSm}P_{\varepsilon}$) that gets the best Probabilistic Information Content (PIC) in comparison with other existing subjective probabilities. Also, we have defined and developed the DSm Field and Linear Algebra of Refined Labels that permit the transformation of any fusion rule to a corresponding qualitative fusion rule which gives an exact qualitative result (*i.e.* a refined label), so far the best in literature.

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