

# TRAJECTORY PLANNING OF A PENDULUM-DRIVEN UNDERACTUATED CART

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*Abstract.* The control problem of the pendulum-driven cart has been well studied in several literatures. The challenge issue of control of such underactuated systems is to control the motions of the pendulum and the cart by applying only one control input. However, most literatures focused on addressing the dynamic modeling and the tracking control design. Few papers have considered the trajectory planning of the system which is still an open issue. This paper studies the optimization for the trajectory planning of the system by considering three questions: 1) the furthest cart displacement; 2) the fastest average cart speed; 3) the most efficient power consumption. The paper systematically studies the system dynamics, the system constraints, the tracking control method, some initializations for the optimization problems, and the solutions for the three optimal questions. Extensive simulation studies are given to identify the optimum trajectories of the system.

*Key words:* pendulum-driven cart, underactuated system, trajectory planning, tracking control.

## 1. INTRODUCTION

The pendulum-driven cart (PDC) [1, 2] is an underactuated system that has fewer independent control actuators than degrees of freedom to be controlled. It contains an inverted pendulum rotated by a motor mounted on the top of the cart shown in Fig. 1. The cart has four passive wheels which make it move horizontally on the ground. The control objective of the system is to control the cart movement by rotating the pendulum periodically. A similar system is the well-known classical cart-pole system [2]. Both systems are underactuated systems with two-degree-of-freedom and one control actuator. The difference between the classical cart-pole system and the PDC is that the former addresses a swing-up and balancing problem

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4, while the later addresses a trajectory planning and tracking problem 5. In this paper, the optimization of the trajectory planning for the PDC is considered.

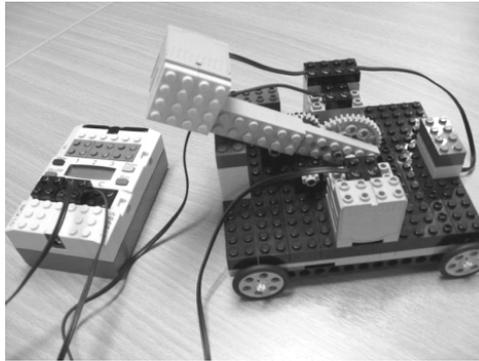


Fig. 1 – The PDC system.

Control of underactuated systems is currently considered among one of the most active fields of research due to their diverse applications [6, 7]. The difficulty of controlling such systems is that the reduced input space is less than the number of the system configuration to be controlled. Luca *et al.* 7 has identified the following three control problems for underactuated systems:

- 1) Trajectory planning: given an initial configuration  $q_0$  and a final desired configuration  $q_d$ , find a dynamically feasible trajectory that joins  $q_0$  and  $q_d$ ;
- 2) Trajectory tracking: given a dynamically feasible trajectory  $q_d(t)$ , design a feedback control law that asymptotically drives the tracking error  $e=q_d(t)-q(t)$  to zero;
- 3) Set-point regulation: given a desired configuration  $q_d$ , design a feedback control law to make the equilibrium state  $q=q_d, dq/dt=0$  asymptotically stable.

It is known that swing-up control of the classical cart-pole system 2, the Acrobot 8, and the Pendubot 10 belongs to the set-point regulation problem, while control of the mobile robot 11, the PDC, the capsule robot 12, and the double-pendulum driven cart 13 belongs to the trajectory planning and tracking problem.

A number of papers [1, 2, 14, 15, 16] address the trajectory planning and tracking problem for the PDC system. The first model of the PDC was proposed by Li *et al.* 14. In 14, the dynamic model and the trajectory planning problem of the system were initially studied. Later on, a six-step motion strategy and the tracking control problem were studied in detail by Yu *et al.* 1. In particular, the paper has been devoted to trajectory optimization, and finally, a group of optimum trajectory parameters for the six-step motion strategy was obtained. In 15, a number of tracking control methods was studied, and the tracking performance was compared in terms of the cart displacement. A further study of the control method was presented in 16, and its robustness to parameter uncertainty and disturbance was investigated thoroughly. From the literature review above, it is known that the

modeling and the trajectory tracking problems of the PDC have been extensively studied. Regarding to the issues of optimization, the following questions arise naturally:

- 1) What is the furthest cart displacement?
- 2) What is the fastest average cart speed?
- 3) What is the most efficient control consumption?

It is therefore that the focus of this paper is to study the optimization of the trajectory planning problem for the PDC system by addressing the questions above. The paper is organized as follows. In Section II, the system dynamics and the control problem are described, and are followed by some initializations. In Section III, the optimal problems are studied, and extensive optimal results are presented. Finally, conclusions are given in Section IV.

## 2. SYSTEM DYNAMICS

### A. Equations of Motion

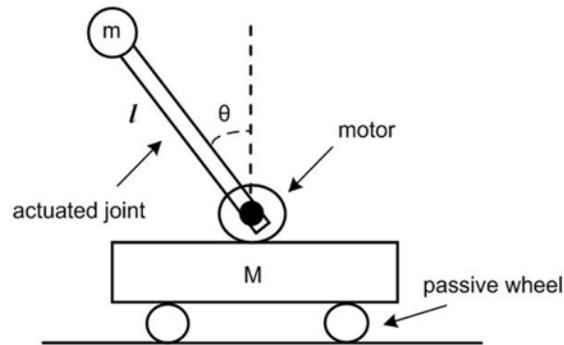


Fig. 2 – Physical model of the PDC.

Consider the PDC in Fig. 2 where the inverted pendulum rotated by a DC motor is mounted on the top of the cart. The cart has four passive wheels and moves horizontally on ground. Particularly, discontinuous friction effect exists in the motion of the cart. The friction governed by the cart velocity is various depending on the normal force applied on the cart. The parameters of the PDC are defined as follows.  $M$  is the cart mass,  $m$  the ball mass,  $l$  the length from the pivot to the mass centre of the ball,  $\mu$  the friction coefficient between the cart and ground,  $\theta$  the pendulum angle from vertical,  $x$  the cart displacement along horizontal, and  $\tau$  the torque applied to the pendulum by the DC motor. From the Newton's Second Law, the dynamic model of the PDC can be obtained as follows [1]

$$(M + m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta + \mu[(M + m)g - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta]\sigma(\dot{x}) = 0, \quad (1)$$

$$-ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mgl\sin\theta = \tau, \quad (2)$$

where  $\sigma(\dot{x})$  is a multivalued mapping governed by

$$\sigma(\dot{x}) = \begin{cases} 1 & \dot{x} > 0 \\ [-1, 1] & \dot{x} = 0 \\ -1 & \dot{x} < 0 \end{cases}$$

The control goal is to drive the cart to move in one direction only by applying the control input  $\tau$ .

### B. System Constraints

In order to keep the cart on ground, the normal force of the cart must be positive,

$$Mg + F_y = (M + m)g - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta > 0, \quad (3)$$

where  $F_y$  is the normal force of the pendulum applied on the cart. Based on the constraint (3), the following lemma is given.

**Lemma 1.** *If the following condition is satisfied,*

$$\ddot{\theta}^2 + \dot{\theta}^4 < \lambda^2 \quad (4)$$

where  $\lambda = (M + m)g/(ml)$ , the cart will be kept on ground.

*Proof.* Using the constraint (3), it yields

$$\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta < \lambda. \quad (5)$$

If  $\ddot{\theta}^2 + \dot{\theta}^4 = 0$ , the constraint (3) is held.

If  $\ddot{\theta}^2 + \dot{\theta}^4 \neq 0$ , letting  $\alpha = \arccos(\ddot{\theta}/\sqrt{\ddot{\theta}^2 + \dot{\theta}^4})$ , it yields

$$\sqrt{\ddot{\theta}^2 + \dot{\theta}^4}\sin(\theta + \alpha) < \lambda. \quad (6)$$

Enlarging the inequation above by using

$$\sqrt{\ddot{\theta}^2 + \dot{\theta}^4}\sin(\theta + \alpha) \leq \sqrt{\ddot{\theta}^2 + \dot{\theta}^4}, \quad (7)$$

which leads to

$$\sqrt{\ddot{\theta}^2 + \dot{\theta}^4} < \lambda. \quad (8)$$

This completes the proof.

Consider keeping the cart still when the pendulum is rotated, the internal force  $F_x$  along horizontal direction must be less than the maximal static friction,

$$|F_x| \leq \mu(Mg + F_y), \quad (9)$$

which gives the following lemma.

**Lemma 2.** *If the following condition is satisfied,*

$$\ddot{\theta}^2 + \dot{\theta}^4 \leq \left( \frac{\mu\lambda}{1+\mu} \right)^2 \quad (10)$$

*the cart will be kept still while pendulum is rotated.*

*Proof.* using the constraint (9) gives that

$$\begin{aligned} |m\ddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta| &\leq \\ &\leq \mu(Mg + mg - m\ddot{\theta} \sin \theta - ml\dot{\theta}^2 \cos \theta) \end{aligned} \quad (11)$$

First, consider one side of the above inequation,

$$\begin{aligned} ml(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) &\leq \\ &\leq \mu(M + m)g - \mu ml(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \end{aligned} \quad (12)$$

If  $\ddot{\theta}^2 + \dot{\theta}^4 = 0$ , the constraint (10) is hold.

If  $\ddot{\theta}^2 + \dot{\theta}^4 \neq 0$ , choosing a constant such that

$$\alpha = \arccos(\ddot{\theta} / \sqrt{\ddot{\theta}^2 + \dot{\theta}^4}),$$

which satisfies

$$\sqrt{\ddot{\theta}^2 + \dot{\theta}^4} \cos(\theta + \alpha) + \mu \sqrt{\ddot{\theta}^2 + \dot{\theta}^4} \sin(\theta + \alpha) < \mu\lambda. \quad (13)$$

If  $(1 + \mu)\sqrt{\ddot{\theta}^2 + \dot{\theta}^4} < \mu\lambda$  is satisfied, the inequation (13) is held. Another side of the inequation (11) can be verified by using the same approach. So this completes the proof.

### C. Feedback Controller

Since the cart motion is unactuated and the control input  $\tau$  directly controls the rotation of the inverted pendulum, the control strategy of the PDC is to control the pendulum to track a proper trajectory that can drive the cart to move in one desired direction. A feedback controller using partial feedback linearization has been designed for the tracking control of the pendulum in 2. The tracking controller is presented as below.

$$\tau = \alpha(\theta)u + \beta(\theta, \dot{\theta}), \quad (14)$$

$$u = \ddot{\theta}_d - K_v(\dot{\theta} - \dot{\theta}_d) - K_p(\theta - \theta_d), \quad (15)$$

where

$$\alpha = ml^2 - \frac{m^2 l^2}{M + m} \cos \theta (\cos \theta + \mu \sin \theta),$$

$$\beta = \frac{ml \cos \theta}{M + m} [ml(\sin \theta - \mu \cos \theta)\dot{\theta}^2 + \mu(M + m)g] - mgl \sin \theta,$$

$K_v$  and  $K_p$  are linear gains,  $\ddot{\theta}_d$ ,  $\dot{\theta}_d$ , and  $\theta_d$  are desired trajectories of angular acceleration, velocity, and position, respectively.

The next task is to find the optimal trajectory of the pendulum that drives the cart in an optimum manner.

#### D. Initializations

The system parameters are given in Table 1. Before the optimization, the following initial definitions and assumptions are given.

Table 1

The PDC Parameters 1

$M$	$m$	$l$	$\mu$	$g$
0.5 kg	0.05 kg	0.3 m	0.01	9.81 m/s <sup>2</sup>

#### Definitions:

- 1) Forward stroke: one forward stroke of the pendulum is a cycle which rotates from its initial angle  $\theta_0$  to  $\theta_1$ .
- 2) Backward stroke: one backward stroke of the pendulum is a cycle which reverses back from  $\theta_1$  to its initial angle  $\theta_0$ .
- 3) Full stroke: one full stroke of the pendulum is a cycle which includes a forward stroke and a backward stroke.
- 4) Fast motion: rotating the pendulum fast to generate a large internal thrust horizontally that leads to  $\dot{x} \neq 0$ .
- 5) Slow motion: rotating the pendulum slowly to generate a small internal thrust that leads to  $\dot{x} = 0$ .

#### Assumptions:

- a) The pendulum rotates in upper plane of  $[-\pi/2, \pi/2]$ .
- b) For forward and backward strokes, the pendulum starts and ends with still.
- c) The initial position of the pendulum is  $-\pi/2$  for each full stroke.
- d) For each full stroke, the cart moves and ends with still.

### 3. OPTIMIZATION

In this section, the following optimization issues will be investigated:

- The furthest displacement of the cart in one full stroke;
- The fastest average speed of the cart in one full stroke;
- The most efficient control input in one full stroke.

#### A. Phase and Time-Domain Trajectories

Considering the boundaries of Lemma 1 and Lemma 2, we have the maximal angular acceleration and velocity at  $\lambda$  and  $\sqrt{\lambda}$ , the minimal angular acceleration and velocity at  $\mu\lambda/(1+\mu)$  and  $\sqrt{\mu\lambda/(1+\mu)}$ , respectively. Based on Assumption (a), the pendulum phase trajectory for the fast motion is contained in the region

$$\Omega_f = \left\{ (\theta, \dot{\theta}) : |\theta| \leq \frac{\pi}{2}, |\dot{\theta}| \leq \sqrt{\lambda}, |\ddot{\theta}| \leq \sqrt{\lambda^2 - \dot{\theta}^4} \right\}$$

and the phase trajectory for the slow motion is contained in the region

$$\Omega_s = \left\{ (\theta, \dot{\theta}) : |\theta| \leq \frac{\pi}{2}, |\dot{\theta}| \leq \sqrt{\frac{\mu}{1+\mu}\lambda}, |\ddot{\theta}| \leq \sqrt{\left(\frac{\mu}{1+\mu}\lambda\right)^2 - \dot{\theta}^4} \right\}$$

The trajectory of Lemma 1 in time domain is obtained by solving numerical integration of the following differential equations

$$\begin{cases} \dot{\theta} = y & \theta(0) = -\frac{\pi}{2} \\ \dot{y} = \sqrt{\lambda^2 - y^4} & \dot{\theta}(0) = 0 \end{cases} \quad (16)$$

Similarly, by solving the boundary of Lemma 2

$$\begin{cases} \dot{\theta} = y & \theta(0) = -\frac{\pi}{2} \\ \dot{y} = \sqrt{\left(\frac{\mu}{1+\mu}\lambda\right)^2 - y^4} & \dot{\theta}(0) = 0 \end{cases} \quad (17)$$

the trajectory of Lemma 2 in time domain is obtained. The velocity boundaries of the fast motion and the slow motion in time domain can be piecewise approximately represented by a first-order polynomial

$$\dot{\theta}(t) = kt + b \quad t \in [t_i, t_{i+1}], \quad (18)$$

where  $i = 0, 1, 2, \dots, 6$ . The plot of the boundary is shown in Fig. 4, and the coefficients of the trajectory are given in Table 2 and Table 3, where time is in seconds and angular velocity is in radian per second.

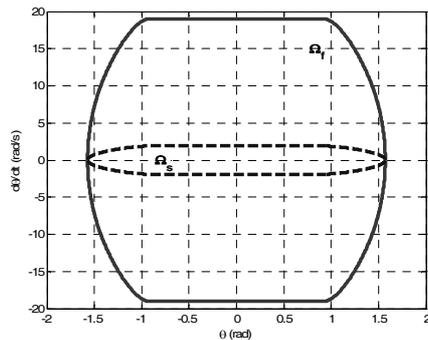


Fig. 3 – Phase trajectory of the pendulum (solid line in red shows fast motion boundary; dash line in blue shows slow motion boundary).

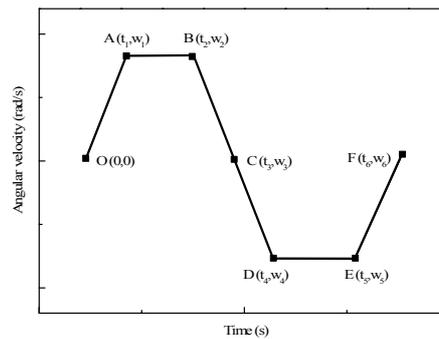


Fig. 4 – The boundaries of Lemma 1 and Lemma 2 ( $\Omega_f$  and  $\Omega_s$ ) in time domain.

Note that the boundary trajectories  $\Omega_f$  and  $\Omega_s$  in time domain will be used in the rest of the paper for different optimization purposes.

Table 2

The configuration of the fast motion boundary  $\Omega_f$ 

$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$
0.06	0.16	0.22	0.28	0.38	0.45
$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$
18.97	18.97	0	-18.97	-18.97	0

Table 3

The configuration of the slow motion boundary  $\Omega_s$ 

$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$
0.62	1.7	2.32	2.18	2.51	3.13
$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$
1.85	1.85	0	-1.85	-1.85	0

### B. Cart Motion

It is known from the equations of motion that the PDC is a holonomic underactuated mechanical system 4 which has a completely integrable second-order constraint (1). So integrating (1) once using Assumptions (b) and (d) gives

$$(M + m) \int_0^t \ddot{x} dt - ml \int_0^t (\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta) dt + \mu \sigma(\dot{x}) \int_0^t [(M + m)g - ml\ddot{\theta} \sin \theta - ml\dot{\theta}^2 \cos \theta] dt = 0.$$

The cart speed is given as

$$\dot{x} = \frac{ml\dot{\theta}[\cos \theta + \mu\sigma(\dot{x})\sin \theta]}{M + m} - \mu\sigma(\dot{x})gt. \quad (19)$$

Integrating (1) twice gives

$$(M + m) \int_0^t \int_0^t \ddot{x} dt - ml \int_0^t \int_0^t (\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta) dt + \mu \sigma(\dot{x}) \int_0^t \int_0^t [(M + m)g - ml\ddot{\theta} \sin \theta - ml\dot{\theta}^2 \cos \theta] dt = 0. \quad (20)$$

Since the initial angle of the pendulum is fixed (Assumption (c)), the cart displacement is rewritten as

$$x = \frac{ml[\sin \theta - \mu\sigma(\dot{x})\cos \theta + 1]}{M + m} - \frac{1}{2}\mu\sigma(\dot{x})gt^2. \quad (21)$$

Next, we consider the three optimization issues aforementioned in this section.

### C. Optimization of Cart Displacement

From (21), the cart can achieve the furthest displacement if the pendulum rotates the full range from  $\pi/2$  to  $\pi/2$  using the shortest time in the forward stroke. So a rest-to-rest movement is required for the cart, *i.e.* move forward in forward stroke and stay still in backward stroke. As shown in Fig. 5, consider a series of motions as below:

*Forward stroke:*

- 1)  $OF_A$ : the motion with the maximal fast-motion acceleration of the pendulum leads to the acceleration of the cart;
- 2)  $F_A F_B$ : the pendulum rotates using the maximal fast-motion angular velocity while the cart keeps accelerating;
- 3)  $F_B F_C$ : the motion with the maximal fast-motion deceleration of the pendulum leads to the deceleration of the cart.

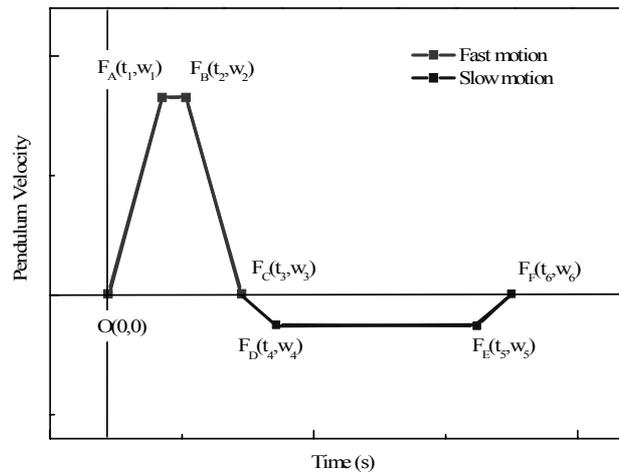


Fig. 5 – The velocity trajectories for the furthest displacement of the cart.

*Backward stroke:*

- 4)  $F_C F_D$ : after the pendulum rotates to  $\pi/2$  at  $F_C$ , it starts backward stroke using the maximal acceleration of the slow motion;
- 5)  $F_D F_E$ : the pendulum rotates using the maximal angular velocity of the slow motion while keeps the cart still;
- 6)  $F_E F_F$ : the pendulum decelerates using the maximal deceleration of the slow motion until it reaches to its original position  $-\pi/2$  at  $F_F$ .

Table 4

The configuration of the trajectory for the furthest displacement

$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$
0.06	0.166	0.226	0.846	1.924	2.544
$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$
18.97	18.97	0	-1.85	-1.85	0

The series of motions is a combination of the fast motion and the slow motion. So we can obtain the trajectory using the configurations in Table 2 and Table 3. The final configuration of the pendulum trajectory for the furthest displacement is presented in Table 4.

A simulation result is shown in Fig. 6. It is seen that the cart was driven in the fast motion which is also the forward stroke of the pendulum, and was kept still during the slow motion which is the backward stroke of the pendulum. It can be obtained from the simulation that the cart achieved the maximal displacement 0.052 m in 0.226 seconds and kept stationary until 2.544 seconds when a full stroke cycle was finished. Note that, it is not surprise that the simulation result is consistent with the result calculated by equation (21) which indicates that the furthest displacement can be achieved by enlarging the final angle of the fast motion while minimizing the duration of the full stroke cycle. The phase trajectory of the pendulum in one full stroke is presented in Fig. 7, and the cart position as a function of the pendulum angle is shown in Fig. 8.

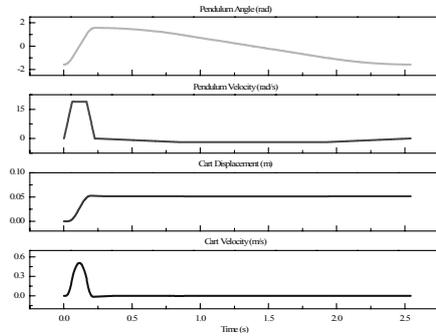


Fig. 6 – The furthest displacement of the cart in one full stroke.

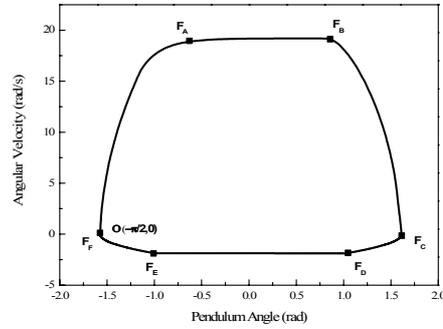


Fig. 7 – The phase trajectory of the pendulum for the furthest cart displacement.

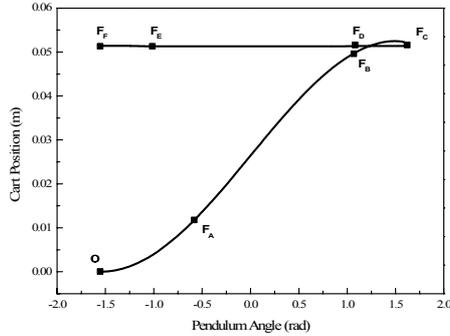


Fig. 8 – The cart position as a function of the pendulum angle for the furthest cart displacement.

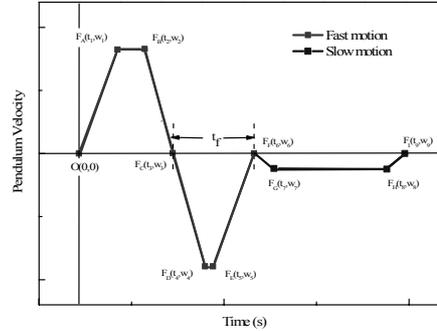


Fig. 9 – One full stroke of the pendulum angular trajectory for the fastest average cart speed.

#### D. Optimization of Average Cart Speed

Consider the movement shown in Fig. 9 as below.

*Forward stroke:*

1)  $OF_A$ : the motion with the maximal fast-motion acceleration of the pendulum leads to the acceleration of the cart;

2)  $F_A F_B$ : the pendulum rotates using the maximal fast-motion angular velocity while the cart keeps accelerating;

3)  $F_B F_C$ : the motion with the maximal fast-motion deceleration of the pendulum leads to the deceleration of the cart.

*Backward stroke:*

4)  $F_C F_D$ : after the pendulum rotates to  $\pi/2$  at  $E_C$ , it starts backward stroke using the maximal fast-motion acceleration which leads to the acceleration of the cart in backward direction;

5)  $F_D F_E$ : the pendulum rotates using the angular velocity  $-w_l$  while the cart keeps accelerating;

6)  $F_E F_F$ : the pendulum decelerates using the maximal deceleration of the fast motion which leads to the deceleration of the cart;

7)  $F_F F_G$ : the motion with the maximal slow-motion acceleration while the cart keeps still;

8)  $F_G F_H$ : the pendulum rotates using the maximal angular velocity of the slow motion while keeps the cart still;

9)  $F_H F_I$ : the motion with the maximal slow-motion deceleration until it stops at  $-\pi/2$ .

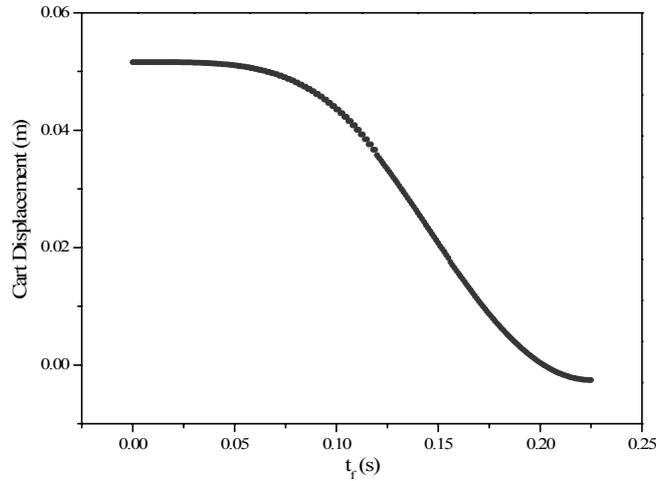


Fig. 10 – The duration of the fast motion in the backward stroke ( $t_f$ ) as a function of cart displacement.

As discussed in Section III.C, the furthest displacement is achieved at  $F_C$ , the fast motion  $F_{C-F}$  in the backward stroke will bring the cart back a short distance which depends on the duration of the fast motion  $t_f$  in the backward stroke. Furthermore, as it is known that the pendulum rotates from  $\pi/2$  to  $-\pi/2$  in the backward stroke, it is easily found that the duration of the slow motion  $F_{F-I}$  can be a function of  $t_f$ . Therefore, we use  $t_f$  as a control parameter of the cart displacement and the average cart speed in order to find the optimum  $t_f$  that gives the fastest

average speed of the cart. The simulation results of the cart displacement and the average speed as a function of  $t_f$  are shown in Fig. 10 and Fig. 11, respectively.

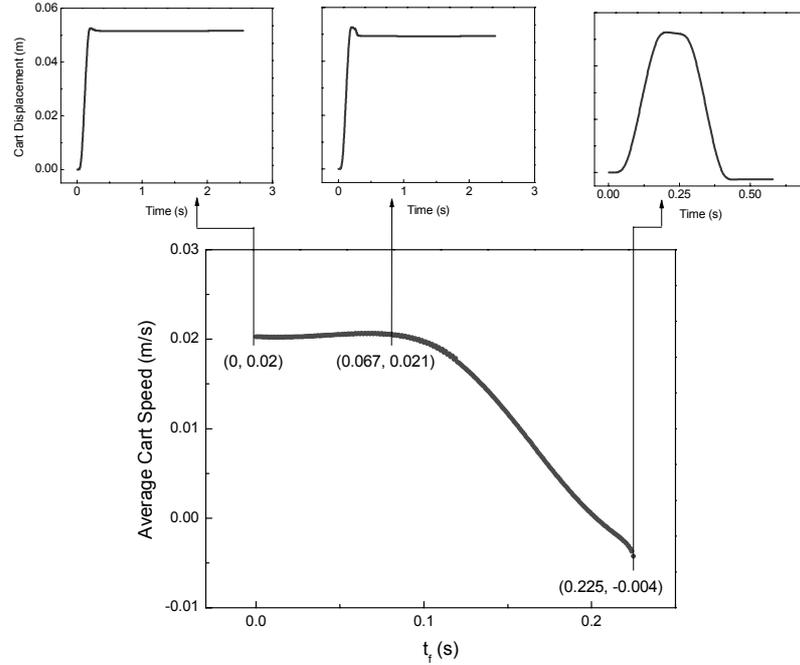


Fig. 11 – The duration of the fast motion in the backward stroke ( $t_f$ ) as a function of average cart speed.

It is shown from Fig. 10 that, as  $t_f$  is increasing, the cart displacement is decreasing due to the backward movement of the cart in the backward stroke. Note that  $t_f$  varies from 0 to 0.226 seconds which  $t_f = 0$  second means there is only one slow motion in the backward stroke (equals to the motion of the furthest displacement) and  $t_f = 0.226$  seconds means there is no slow motion, but only one fast motion is used in the backward stroke. An optimum parameter is observed at  $t_f = 0.067$  seconds in Fig. 11 which shows the average cart speed as a function of  $t_f$ . It is seen that, when  $t_f = 0$ , the furthest displacement of the cart is shown corresponding to an averaged cart speed of 0.02 m/s. When  $t_f = 0.067$ , the fastest average cart speed is obtained at 0.021 m/s. Finally, when  $t_f = 0.225$ , the pendulum executed a fast motion in the forward stroke and in the backward stroke respectively which led to an average speed of  $-0.004$  m/s. The phase trajectory of the pendulum and the cart position as a function of the pendulum angle for the fastest average cart speed are presented in Fig. 12 and Fig. 13, respectively.

#### E. Optimization of Energy Consumption

In order to minimize the control energy in one full stroke, the energy consumption at per unit displacement is considered as below

$$E = \left( \int_0^T \tau^2 dt \right) / x(T), \quad (22)$$

where  $T$  is the period time for one full stroke. Next, consider the pendulum motion as below:

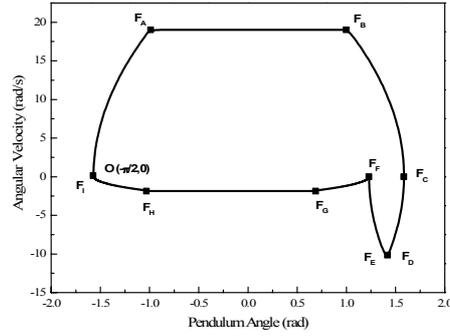


Fig. 12 – The phase trajectory of the pendulum for the fastest average cart speed.

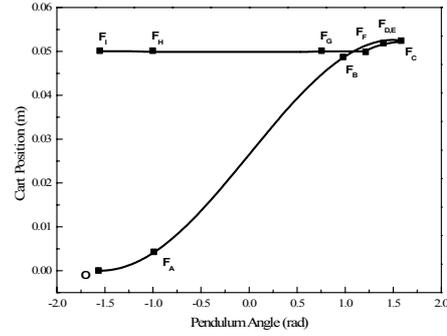


Fig. 13 – The cart position as a function of the pendulum angle for the fastest average speed.

*Forward stroke:*

- 1)  $OE_A$ : the motion with the maximal fast-motion acceleration of the pendulum leads to the acceleration of the cart;
- 2)  $E_AE_B$ : the pendulum rotates using the angular velocity  $w_1$  while the cart keeps accelerating;
- 3)  $E_BE_C$ : the motion with the maximal fast-motion deceleration of the pendulum leads to the deceleration of the cart.

*Backward stroke:*

- 4)  $E_CE_D$ : after the pendulum rotates to  $\pi/2$  at  $E_C$ , it starts backward stroke using the maximal fast-motion acceleration which leads to the acceleration of the cart in backward direction;
- 5)  $E_DE_E$ : the pendulum rotates using the angular velocity  $-w_1$  while the cart keeps accelerating;
- 6)  $E_EE_F$ : the pendulum decelerates using the maximal deceleration of the fast motion which leads to the deceleration of the cart;
- 7)  $E_FE_G$ : the motion with the maximal slow-motion acceleration while the cart keeps still;
- 8)  $E_GE_H$ : the pendulum rotates using the maximal angular velocity of the slow motion while keeps the cart still;
- 9)  $E_HE_I$ : the motion with the maximal slow-motion deceleration until it stops at  $-\pi/2$ .

The detailed pendulum trajectory is shown in Fig. 14. To find the optimal configuration of the pendulum trajectory for the most efficient control consumption, the energy consumption at per unit displacement represented by equation (22) is computed by varying the maximal fast-motion angular velocity  $w_1$  and the duration of the fast motion in the backward stroke  $t_f$ . In the simulation,  $w_1$  is varying from the maximal slow-motion angular velocity 1.85 rad/s to the maximal fast-motion angular velocity 18.97 rad/s, and  $t_f$  is varying from 0 s to 0.225 s.

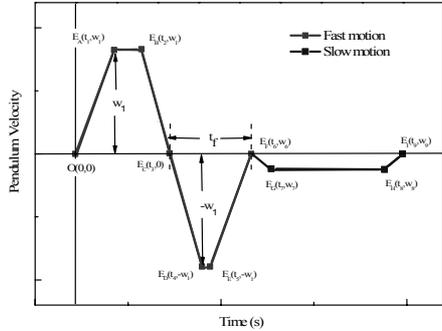


Fig. 14 – One full stroke of the pendulum angular trajectory for the most efficient control consumption.

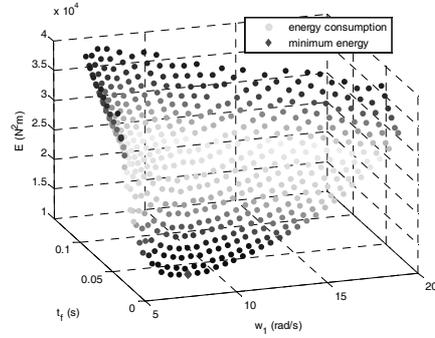


Fig. 15 – The energy efficiency as a function of  $w_1$  and  $t_f$ .

The simulation result for the energy efficiency as a function of  $w_1$  and  $t_f$  is shown in Fig. 15. From the result, the minimum energy consumption at per unit displacement is obtained when  $(w_1, t_f) = (7.35, 0)$  which means the maximal fast-motion speed is 7.35 rad/s and there is no fast motion in the backward stroke. The phase trajectory of the pendulum and the cart position as a function of the pendulum angle are shown in Fig. 16 and Fig. 17, respectively.

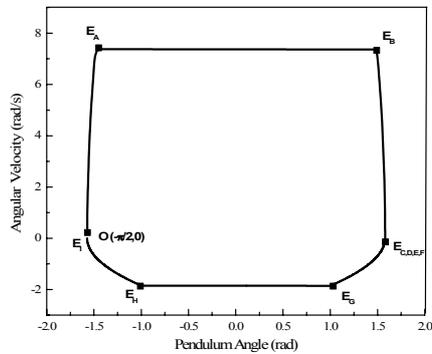


Fig. 16 – The phase trajectory of the pendulum for the most efficient energy consumption.

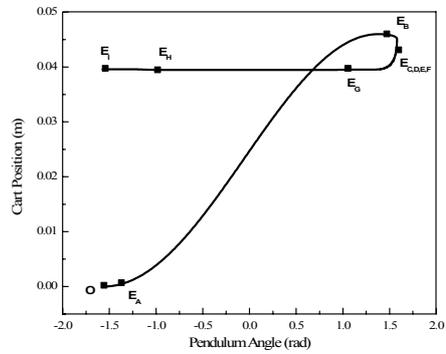


Fig. 17 – The cart position as a function of the pendulum angle for the most efficient energy consumption.

#### 4. CONCLUSIONS

The paper has investigated the trajectory planning problem of the pendulum-driven cart. Three optimization problems have been studied: 1) the furthest cart displacement; 2) the fastest average cart speed; 3) the most efficient control consumption. The optimizations were undertaken by varying a piecewise linear

trajectory of the pendulum approximately obtained from the system constraints. The optimum trajectories were found by extensive simulation studies.

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