NUMERICAL SIMULATION OF THE QUASISTATIC CONTACT PROBLEM WITH DAMAGE

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Abstract. This paper deals with the numerical analysis of a quasistatic contact problem with friction between an elastic body and a rigid obstacle. The contact is frictional and is modeled with a version of normal compliance condition and the Coulomb's law of dry friction, including the development of material damage, caused by opening and growth of micro-craks and micro-cavities, which results from internal compression or tension. The material damage is taken into account in the constitutive law. The damage field varies between one and zero at each point in body. The variational form of this problem is a coupled system which consists of a first-kind variational inequality for the displacement field and a linear parabolic variational equation for the evolution of damage field. The discrete scheme of the coupled system is introduced based on the finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivative, see [3, 4, 7]. The algorithm used is an iterative-alternative. The novelty of this work consists in the proposed numerical algorithm, and its implementations for a numerical solution of the quasistatic elastic contact problem with friction and damage.

Key words: quasistatic frictional contact, damage, liniarelastic material, weak solutions, finite elements, numerical simulations.

1. INTRODUCTION

This paper studies the numerical analysis of a coupled problem which consists from an elasto-quasistatic contact problem with friction, and an inclusion parabolic equation for the evolution of the damage field that results from internal compression or tension. The mathematical background of the theory of contact with friction is still incomplete.

The quasistatic model of contact with friction, without inertial effects has been proposed by Klarbring, Mikelic and Shillor [12]. This model consists of the incremental formulation obtained from the approximation with finite difference of the quasistatic variational inequation.

The proof of existence and uniqueness of solution for quasistatic frictional problem is based on the assumption that the displacements satisfy some constraint qualifications and the friction coefficient is sufficiently small.

The damage material is caused by the opening and growth of micro-cracks and micro-cavities with lead to decrease in the load carrying capacity of the body.

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The modeling of material damage uses the concept of damage field, see Frémand [6]. In this model, the damage field, ζ, may have values between one and zero at every point of the body. For $\zeta = 1$ the material is damage free, for $\zeta = 0$ the material is completely damaged, and for $0 < \zeta < 1$ the material is partially damaged.

The evolution of the damaged field is described by an inclusion parabolic equation with a damage source function. This source function results from a mecanical compression or tension provided from elastic contact problem.

The novelty of this paper consists in the proposed algorithm and its implementation for a quasistatic elastic contact problem with friction and damage.

2. CLASICAL AND VARIATIONAL FORMULATION

We will consider an elastic body which occupies the domain $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, with the boundary $\Gamma = \partial\Omega$ which we consider Lipschitz continuous, and let the time interval of interest be $[0,T], T > 0$. The boundary $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous, and it is divided into three disjunct measurable parts Γ_D, Γ_N, Γ_C, where $\text{meas}(\Gamma_D) \neq \varnothing$. The volume force $\mathbf{f}$ acts on $\Omega_T = \Omega \times (0,T)$. For $x \in \Gamma$, we denote by $\mathbf{v}(x)$ and $\boldsymbol{\tau}(x)$ the unit outward and tangential vector to Γ.

The body is clamped on Γ_D and, in consequence the field of displacement will vanish there, the volume forces density $\mathbf{f}$ acts in Ω_T, the surface tractions with intensity $\mathbf{f}_N$ will act on Γ_N and on Γ_C the body is supposed to be in unilateral or bilateral contact with friction on a foundation, see Campo *et al.* [3]. We will denote by $\mathbf{u}$ the displacement field, $\boldsymbol{\sigma}$ the stress tensor and $\boldsymbol{\varepsilon}(\mathbf{u})$ the linearized tensor of deformation. We denote by ζ the damage field, defined on Ω_T which measures the intensity of the micro-cracks in the body.

The body is assumed to be elastic with the following constitutive law:

$$\boldsymbol{\sigma} = \zeta \mathcal{A}\left(\boldsymbol{\varepsilon}(\mathbf{u})\right),$$

where $\mathcal{A}$ is stipulated linear function. The damage field, which measures the decrease in the strength of the material, will be denoted $\zeta = \zeta(x,t)$ which is the ratio $\zeta = \zeta(x,t) = \dfrac{E_{\text{eff}}}{E}$, where E_{eff} is the effective elasticity modulus, at the time t and E is the elasticity modulus of the damage free body. From this definition results that the damage field is restrained to the values $0 \le \zeta \le 1$. The evolution of microscopic cracks and cavities responsible for the damage is described by a parabolic (inclusion) differential equation, according to Frémand and Nedjar [7].

Because of some technical reasons we will choose a positive small constant ζ_* in order to restrain the damage function to the values $\zeta_* \le \zeta \le 1$. If we have ζ approaching zero the body will have dense micro cracks and modeling this elastic body is senseless. That is, we will use the truncating operator

$$\eta_*(\zeta) = \begin{cases} 1 & \text{if } 1 \le \zeta \\ \zeta & \text{if } \xi_* \le \zeta \le 1 \\ \xi_* & \text{if } \zeta \le \xi_* \end{cases}$$

specifying that there will be no difference between ζ and η_*. Let us denote by S^d the space of the second order symmetric tensors on $\mathbb{R}^d$, by "$\cdot$" the inner product on $\mathbb{R}^d$ or S^d and $|\cdot|$ the Euclidean norms on these spaces.

We also define the following variational spaces:

$$H = \left[L^2(\Omega)\right]^d, \quad Y = L^2(\Omega), \quad \mathcal{K} = \left\{\xi \in H^1(\Omega);\ 0 \le \xi \le 1 \ a.e. \in \Omega\right\}$$

$$V = \left\{\mathbf{v} \in \left[H^1(\Omega)\right]^2;\ \ \mathbf{v} = 0 \ \text{on } \Gamma_D, \ \ \mathbf{v}_\nu = \mathbf{v}\cdot\boldsymbol{\nu} = 0 \ \text{on } \Gamma_C\right\}$$

$$Q = \left\{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^2 \in \left[L^2(\Omega)\right]^{2\times 2};\ \ \tau_{ij} = \tau_{ji},\ i,j = 1,2\right\}.$$

Moreover, for a Banach space *X*, let $(\cdot,\cdot)_X$ denote its inner product and $\|\cdot\|_X$ its associated norm.

Let $\mathbf{u}_0$ and ζ_0 be the initial values of the displacement and damage fields, respectively, and assume that the inertia effects are negligible and so the process is quasistatic. The classical form of the coupled system of the quasistatc contact problem with friction and of the evolution of micro-cavities and of cracks responsible for the damage is:

Problem P. Find a displacement field $\mathbf{u}: \Omega\times[0,T]\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: \Omega\times[0,T]\to S^d$, and a damage field $\zeta: \Omega\times[0,T]\to[0,1]$ such that,

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f} = 0 \quad \text{in } \Omega_T, \tag{1}$$

$$\boldsymbol{\sigma} = \zeta\mathcal{A}\big(\boldsymbol{\varepsilon}(\mathbf{u})\big) \quad \text{in } \Omega_T, \tag{2}$$

$$\dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) \in \Phi\big(\boldsymbol{\varepsilon}(\mathbf{u}),\zeta\big) \quad \text{in } \Omega_T, \tag{3}$$

$$\frac{\partial\zeta}{\partial\boldsymbol{\nu}} = 0 \quad \text{on } \Gamma\times(0,T), \tag{4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_D\times(0,T), \tag{5}$$

$$\boldsymbol{\sigma}\mathbf{v} = \mathbf{f}_N \quad \text{on } \Gamma_N \times (0,T). \tag{6}$$

The initial conditions are given by

$$\mathbf{u}(x,0) = \mathbf{u}_0, \quad \zeta(x,0) = \zeta_0 \quad \text{in } \Omega. \tag{7}$$

The contact condition are given by:

$$\sigma_\nu = -c_\nu (u_\nu - g)_+^{m_\nu} \quad \text{on } \Gamma_C \times (0,T), \tag{8}$$

where if

$$u_\nu = g \;\Rightarrow\; \sigma_\tau = 0, \tag{9}$$

and if $u_\nu > g$ it results

$$\begin{cases} |\sigma_\tau| \le c_\tau (u_n - g)_+^{m_\tau} \;\Rightarrow\; \dot{u}_\tau = 0 \\ |\sigma_\tau| \le c_\tau (u_\nu - g)_+^{m_\tau} \;\Rightarrow\; (\exists)\, \lambda > 0 \;\text{ s.t. }\; \dot{u}_\tau = -\lambda \sigma_\tau , \end{cases} \tag{10}$$

where $c_\nu = c_\nu(s)$, $c_\tau = c_\tau(s)$, m_ν and m_τ are parameters which characterize the contact interface. These parameters can be deduced experimentally. The parameter $g,\ g \ge 0$ is the initial gap between Γ_C and the foundation measured along the outward normal direction to Γ_C. We denoted the positive part of the argument by $(\cdot)_+$. The friction law from **Problem P**, is a generalization of the Coulomb's law, which is recovered if $m_\nu = m_\tau$. In such case, $\mu = c_\tau / c_\nu$ is the usual coefficient of friction, positive and enough small. This law describes a dependence of the friction coefficient on normal contact pressure. Furthermore in order to have well defined integrals on the boundary, Γ_C, it is necessary for the following relations to hold $c_n(s),\ c_T(s) \in L^\infty(\Gamma_C)$, $1 \le m_\nu, m_\tau < \infty$ if $d = 2$ and $1 \le m_\nu, m_\tau \le 3$ if $d = 3$. Here, eq. (8) represent the normal compliance law, where $u_\nu = \mathbf{u} \cdot \boldsymbol{\tau}$ denotes the normal displacement, $\boldsymbol{\sigma}_\tau = (\boldsymbol{\tau} \cdot \boldsymbol{\sigma}\mathbf{v})\boldsymbol{\tau}$ and $\dot{\mathbf{u}}_\tau = (\boldsymbol{\tau} \cdot \dot{\mathbf{u}})\boldsymbol{\tau}$ are the tangential components of the stress and velocity fileds. The constant $\kappa > 0$ is the damage diffusion constant, Φ is the damage source function, $\partial I_{[0,1]}$ denotes the subdifferential of the indicator function $I_{[0,1]}$ which enforces the constraint $0 \le \zeta \le 1$. Assuming that it does not exist damage flux on boundary Γ, that is, $\partial\zeta / \partial\nu = 0$ on Γ where η is the exterior normal unit vector on the boundary Γ, see [13,14]. In the study of the mechanical problem (1)-(7), we assume essentially that the elastic operator $\mathcal{A}$ and the damage source function Φ are Lipschitz continuous operators and that $\mathcal{A}$ is strictly monotone. Let the body forces and surface tractions have the regularity

$\mathbf{f} \in C([0,T];H)$ and $\mathbf{f}_N \in C\left([0,T];[L^2(\Gamma_N)]^d\right)$ define the element $\mathbf{F}(t) \in V$ given by

$$(\mathbf{F}(t),\mathbf{v})_V = (\mathbf{f}(t),\mathbf{v})_H + (\mathbf{f}_N(t),\mathbf{v})_{[L^2(\Gamma_N)]^d},$$

and let $\mu : \Gamma_C \to [0,+\infty)$ be such that $\mu \in L^\infty(\Gamma_C)$, the friction coefficient. Let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form

$$a(\xi_1,\xi_2) = \kappa \int_\Omega \nabla\xi_1 \cdot \nabla\xi_2 \mathrm{d}x, \quad \forall\ \xi_1,\xi_2 \in H^1(\Omega),$$

and we denote by $j_\nu : V \to \mathbb{R}$ and $j_\tau : V \to \mathbb{R}$ the functionals

$$j_\nu(u,v) = \int_{\Gamma_C} c_\nu (u_\nu - g)_+^{m_\nu} \,|\, v_\nu \,|\, \mathrm{d}S, \quad \forall\, \mathbf{v} \in V;$$

$$j_\tau(u,v) = \int_{\Gamma_C} c_\tau (u_n - g)_+^{m_\tau} \,|\, v_\tau \,|\, \mathrm{d}S, \quad \forall\, \mathbf{v} \in V.$$

By choosing test functions from V and $\mathcal{K}$, applying the Green's formula and using the conditions and the notation above, we obtain the following formulation.

Problem VP. Find a displacement field $\mathbf{u} : \Omega_T \to V$, a stress field $\boldsymbol{\sigma} : \Omega_T \to Q$ and a damage field $\xi : \Omega_T \to \mathcal{K}$, such that $\mathbf{u}(0) = \mathbf{u}_0$, $\xi(0) = \xi_0$ and for a.e. $t \in [0,T]$,

$$\boldsymbol{\sigma} = \zeta \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u})),$$

$$\left(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{w} - \dot{\mathbf{u}}(t))\right)_Q + j_\nu\left(\mathbf{u}(t), \mathbf{w} - \dot{\mathbf{u}}\right) + j_\tau\left(\mathbf{u}(t), \mathbf{w} - \dot{\mathbf{u}}\right) \geq \left(\mathbf{F}(t), \mathbf{w} - \dot{\mathbf{u}}(t)\right)_V, \ \forall \mathbf{w} \in V,$$

$$\left(\dot{\zeta}(t), \xi - \zeta(t)\right)_Y + a\left(\zeta(t), \xi - \zeta(t)\right) \geq \left(\Phi(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)), \xi - \zeta(t)\right)_Y, \ \forall\, \xi \in \mathcal{K}.$$

The existence of a unique solution to **Problem VP** and its regularity are summarized in the following theorem.

Theorem 1. *If the initial conditions are accomplished such that* $\mathbf{u}_0 \in V$ *and* $\xi_0 \in \mathcal{K}$*, then* ***Problem VP*** *has a unique solution for* $u \in C^1([0,T];V)$ *and* $\xi \in H^1(0,T;Y) \cap L^2\left(0,T;H^1(\Omega)\right)$.

The proof of Theorem 1 is obtained following [9] and it is based on monotone operator theory and classical results on parabolic equations.

3. NUMERICAL APPROXIMATIONS

In this section we use a finite element algorithm for solving **Problem VP** and obtain an error estimate on the approximate solutions. For convenience, we rewrite the variational **problem VP** in terms of the velocity field $\mathbf{v}(t) = \dot{\mathbf{u}}(t)$ given by

$$\mathbf{u}(t) = \int_0^t \mathbf{v}(s)\mathrm{d}s + \mathbf{u}_0 .$$

For this, we will approximate this variational problem in two steps. First, we consider two finite-dimensional spaces $V^h \subset V$ and $B^h \subset H^1(\Omega)$ approximating the spaces V and $H^1(\Omega)$, respectively, and let $\mathcal{K}^h = \mathcal{K} \cap B^h$. Here, $h > 0$ denotes the discretization parameter. The unknowns are the velocity field and the damage field. Secondly, we will discretize the time derivatives, by considering an uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$ and let k be the time step size, $k = T/N$. For a continuous function $g(t)$, let $g_n = g(t_n)$ and, for a sequence $\{w_n\}_{n=0}^N$, we let $\delta w_n = (w_n - w_{n-1})/k$ be its corresponding divided differences. The fully discrete approximation of **Problem VP**, based on the forward Euler scheme, is as follows:

Problem VP^{hk}. Find $\mathbf{v}^{hk} = \{\mathbf{v}_n^{hk}\}_{n=0}^N \subset V^h$ and $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$, such that $\xi_0^{hk} = \xi_0^h$ and for all $\xi^h \in \mathcal{K}^h$, $\mathbf{w}^h \in V^h$ and $n = 1,2,\ldots,N$,

$$\left(\xi_{n-1}^{hk}\mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk})), \boldsymbol{\varepsilon}(\mathbf{w}^h - \mathbf{v}_n^{hk})\right)_Q + j_\nu\left(\mathbf{u}_{n-1}^{hk}, \mathbf{w}^h - \mathbf{v}_n^{hk}\right) - j_\tau\left(\mathbf{u}_{n-1}^{hk}, \mathbf{w}^h - \mathbf{v}_n^{hk}\right) \geq$$

$$\geq \left(\mathbf{F}_n, \mathbf{w}^h - \mathbf{v}_n^{hk}\right)_V ,$$

$$\left(\delta\xi_n^{hk}, \xi^h - \xi_n^{hk}\right)_Y + a\left(\xi_n^{hk}, \xi^h - \xi_n^{hk}\right) \geq \left(\Phi(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \xi_{n-1}^{hk}), \xi^h - \xi_n\right)_Y ,$$

where the discrete displacement fields $\mathbf{u}^{hk} = \{\mathbf{u}_n^{hk}\}_{n=0}^N \subset V^h$ are defined by $\mathbf{u}_n^{hk} = \mathbf{u}_{n-1}^{hk} + k\,\mathbf{v}_n^{hk}$ for $n = 1,\ldots,N$, and $\mathbf{u}_0^{hk} = \mathbf{u}_0^h$ and ξ_0^h are appropriate approximations of the initial conditions. Similarly, the discrete damage fields $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$ are defined by $\xi_n^{hk} = \xi_{n-1}^{hk} + k\,\xi_n^{hk}$ for $n = 1,\ldots,N$. Using standard arguments for variational inequalities [5,8], we deduce the existence and uniqueness of the solution to **Problem VP^{hk}**.

4. NUMERICAL SOLUTION OF PROBLEM VP^{hk} FOR TWO DIMENSIONAL CASE

Let $n \in \{1,\ldots,N\}$ and assume that $\mathbf{u}_{n-1}^{hk}$ and ξ_{n-1}^{hk} are known, and for the next step is used a penalty-duality algorithm introduced in [2], in order to obtain the discrete damage field, the discrete velocity field, strain field and stress field, by using numerical algorithms available in the literature [1,8,9,10].

In the two-dimensional case, the elastic stress tensor $\boldsymbol{\sigma} = \zeta\mathcal{A}\left(\boldsymbol{\varepsilon}(\mathbf{u})\right)$, under plane stress hypothesis has the following form:

$$(\mathcal{A}\boldsymbol{\tau})_{\alpha\beta} = \frac{Er}{1-r^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+r}\tau_{\alpha\beta}, \quad 1 \leq \alpha,\ \beta \leq 2,\ \tau \in S_2 ,$$

where E and r represent the Young's modulus and the Poisson's ratio of the material, respectively, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. The damage source function used here has the form

$$\Phi\left(\boldsymbol{\varepsilon}(\mathbf{u}),\xi\right)=\lambda_d\left(\frac{1-\xi}{\eta_*(\xi)}\right)-\frac{1}{2}\lambda_u\left|(\boldsymbol{\varepsilon}(\mathbf{u}))\right|^2+\lambda_w, \tag{11}$$

where λ_d, λ_u and λ_w are process parameters, and if the value of $\left|(\boldsymbol{\varepsilon}(\mathbf{u}))\right|^2>q^*$ then $\left|(\boldsymbol{\varepsilon}(\mathbf{u}))\right|^2=q^*$. A truncation values of $q^*=1000$, $\xi_*=0.01$ are considered as lower limit for the damage. The incremental form is obtained by the variational formulation of the **Problem VP^{hk}** through the approximation of the temporal derivatives of the displacements with finite differences. The quasistatic problem will be solved step by step, such that at each step we shall calculate small strains, small displacements and damage field, and we will add to the previously calculated result, for small changes of the applied forces. Obviously, both the contact area and the contact state are changing (open contact, fixed contact and sliding contact). In the paper we use the penalized augumented Lagrangian method which combines penalty-duality method in order to impose on the contact boundaries intense tensions which serve as penalties, for the case when the contact condition are violated, with Lagrange's multipliers method, together with Uzawa's algorithms type is one of the most used methods for solving the discretized contact problems with friction, using the finite element method [11, 15, 20, 21]. Also, Newton-Raphson's method is ideal for solving the iterative and incremental contact linearized problem with friction, presented in the form of the penalized Lagrangian [16], and for solving the mechanical problem of quasistatic damage evolution in an elastic body.

EXAMPLE 1. THE CONTACT PROBLEM BETWEEN A PLANE PLATE AND A RIGID FOUNDATION

This example, presented in Fig. 1, was considered by Raous [17] and has the advantage that is simple and contains all the contact states (open, stick and sliding). Five variants are presented with different loads and different friction coefficients. Also, an increasing traction is acting on a rectangular plate plane and we study the stress distribution and damage, at time $t=1$, in the center of the body and near the contact boundary. The following data were used in this example:

$T=1\,\mathrm{s}$, $F=20\,\mathrm{daN/mm^2}$, $E=1000\,\mathrm{daN/mm^2}$, $r=0.3$, $\lambda_d=0.1$, $\lambda_u=1000$, $\lambda_w=0$, $\xi_*=0.01$, $\kappa=0.01$, $\xi_0=1$.

Next, we use a triangulation with finite triangular elements, with linear interpolation functions, and the contact surface was modeled with finite contact elements with 3 nodes [11, 18], and with liniar triangular finite element for the damage field.

Is observed the linear convergence of the algorithm with respect to $h+k$.

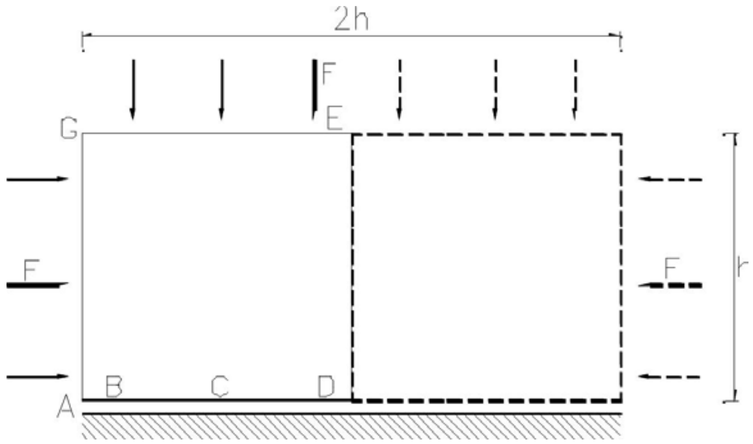

Fig. 1 – The geometry ($h = 40$ mm) and the loading.

Table 1

Contact states for different loading cases

μ	F [daN/mm^2]	f [daN/mm^2]	Open zone AB [mm]	Sliding zone BC [mm]	Stick zone CD [mm]
1	10	–5	3.75	20	16.25
1	15	–5	5	20.75	7.5
0.2	10	–5	0	40	0
0.2	10	–15	0	22.5	17.5
0.2	10	–25	0	5	35

The area near the contact boundary was finest discretizated for a better approximation and a more precise delimitation of the various contact states. These are in good agreement with the one from [17].

EXAMPLE 2. THE CONTACT PROBLEM BETWEEN TWO CYLINDERS AND A RIGID FOUNDATION

This example represents an axial, symmetric contact problem, which consists of two cylinders which come in contact at the bases. The partial geometry and the loading are shown in Fig. 2.

The following data were used in this example: $T = 1\text{s}$, $F = 5000\,\text{daN/mm}^2$, $E = 21000\,\text{daN/mm}^2$, $r = 0.3$, $\lambda_d = 0.25$, $\lambda_u = 1000$, $\lambda_w = 0$, $\xi_* = 0.01$, $\kappa = 0.01$, $\xi_0 = 1$.

The most damage area there is near the contact boundary with foundation.

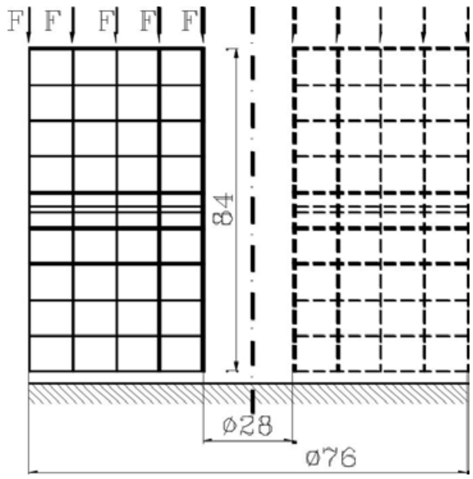

Fig. 2 – The geometry, the mesh and loading.

Table 2

Calculated *versus* experimental normal displacements

μ	F [daN/mm^2]	calculated u_N [mm]	experimental u_N [mm]
0.05	5000	0.0160	0.0165
0.25	5000	0.022	0.0265
0.2	5000	0.021	0.023
0.25	5000	0.034	0.037
0.1	5000	0.0200	0.0205
0.2	5000	0.0344	0.0347

5. CONCLUSIONS

The elastic frictional quasistatic contact problem with damage is modeled using a coupled system which consists from a elasto-quasistatic equation with boundary coditions and contact conditions with friction, and an inclusion parabolic equation for the evolution of the damage field.

In this paper, a quasistatic elastic contact problem was studied. The contact was modelled using the Coulomb's friction law. According to [15,16,22], the effect of the damage was included into the model. The variational formulation led to a coupled system of two nonlinear parabolic variational inequalities. Then, a fully discrete scheme, namely **Problem VP^{hk}**, was introduced using the finite element method and the Euler scheme for approximating the spatial variable and the time derivative, respectively, error estimates were provided according to [3, 18, 19], from which, under suitable regularity assumptions, the linear convergence of the scheme was derived. The main contribution of this paper concerned the numerical resolution of **Problem VP**^{hk}, where a penalty of the frictional term was used. Then, a penalty-duality algorithm, introduced in [2], was employed for solving the penalized problem. Finally, two numerical examples were performed to show the accuracy of the algorithm.

From the numerical examples we conclude that the solution of the mechanical problem of quasistatic damage evolution is considerably more regular with only a minor increase in the assumptions on the regularity of the data.

It is obvious that the algorithm can be also divergent, when the conditions are not satisfied. In the case when the stick contact surface, of two bodies in contact, is much larger than the sliding contact zone, then the algorithm converges in a relatively small number of iterations.

Critical situations may occur if we change the sliding contact with a rigid one or vice versa, this being the most important change in the behavior of the solution. Those difficulties can be overcamed if we reduce the incremental step (quasistatic problem) until two successive terms are sufficiently close.

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