MOTION RECOVERY OF PARALLEL MANIPULATORS USING GENERALIZED INVERSE AND PARTITIONED JACOBIAN MATRIX

VAHID NAZARI, LEILA NOTASH

Abstract. In this paper, the motion recovery of parallel manipulators due to joint failure is addressed. The failed joints have either zero velocities, locked joints, or nonzero velocities which are different from the desired velocities. For the Jacobian matrix of the failed leg, its generalized inverse, which satisfies the first, second, and fourth equations among four equations of Penrose, is utilized to characterize the minimum norm of the correctional velocity vector related to the healthy joints. The platform twist is fully recovered if the Jacobian matrix of the failed leg is of full row-rank or if the platform twist is in the range space of the Jacobian matrix of the failed leg. When the platform twist is in the orthogonal complement of the range space of the Jacobian matrix of the failed leg, or when the Jacobian matrix of the failed leg is not of full row-rank, the partial recovery of the platform twist is achieved using the partitioned Jacobian matrix, while the correctional joint velocity after the failure is minimized.

Key words: parallel manipulators, motion recovery, generalized inverse, Penrose equations.

1. INTRODUCTION

Solid-link parallel manipulators are regarded as closed form chains of links attached by active and passive joints. In parallel manipulators, the mobile platform is connected to the base by a number of legs/branches.

The failure in parallel manipulators could occur at the component level (active or passive joints, links, and sensors), the subsystem level (branch and mobile platform), and the system level (parallel device) [1]. If any of these failures degrades the performance of the manipulator so that the assigned task could not be completed, then the manipulator is considered a failed manipulator.

Redundant manipulators have the fault tolerance capability which enables them to execute the task when the failure happens. If there is redundancy left after the failure, the failed manipulator could continue the task while improving the kinematic performance, optimizing measures of fault tolerance, minimizing the

Queen's University, Kingston, Canada; E-mail: nazariv@me.queensu.ca, notash@me.queensu.ca

Rom. J. Techn. Sci. – Appl. Mechanics, Vol. 58, N^{os} 1–2, P. 33–48, Bucharest, 2013

energy consumption, and minimizing the correctional joint velocity or correctional joint force/torque.

From the kinematics point of view, failure recovery deals with retrieving the lost mobile platform motion caused by the failure in the manipulator through adjusting the velocity of the healthy joints. The analytical solution of the overall joint velocity vector after single locked joint failure of serial manipulators with minimum correctional joint velocity was derived in [2]. A procedure based on the projection of the lost joint motion onto the orthogonal complement of the null space of the Jacobian matrix associated with the failed leg in parallel manipulators was presented in [3, 4]. The motion recovery methodology resulted in the minimum 2-norm for the correctional velocity vector and for the overall joint velocity vector of the healthy joints. In [5] the least-squares approach to minimize the velocity jump in the mobile platform of the redundant serial manipulators was addressed when the multiple locked joint failures happened. The locked joint failures were studied using perturbation methodology to obtain the mobile platform velocity jump.

From the force point of view, failure analysis deals with the effect of full or partial loss of joint force/torque on the wrench capabilities of manipulators. In [6] torque redistribution and time regulation approaches for full or partial joint failure in closed-loop manipulators were investigated. The overall actuator torque was reduced by compensating for the lost torque and modifying the task time. A recovery methodology for retrieving the lost mobile platform wrench caused by zero or nonzero/limited joint force/torque was presented in [7]. Criteria based on projecting the lost platform wrench onto the orthogonal complement of the range space of the reduced Jacobian matrix for full or partial recovery of the lost wrench were established.

The objective of the paper is to formulate the overall velocity of the healthy joints in order to minimize the norm of the correctional velocity vector associated with the healthy joints. Velocity of all joints, including the failed ones, is known. When the Jacobian matrix of the failed leg is of full row-rank, the motion recovery considering the generalized inverse, which satisfies the first and fourth Penrose equations, is investigated. If the Jacobian matrix of the failed leg is not of full row-rank, the partitioned form of the Jacobian matrix is applied to partially recover the platform twist. This paper is laid out in the following manner. The motion recovery based on minimizing the correctional joint velocity after the failure is presented in Section 2. The simulation results are reported and compared with the motion recovery methodology of [4] in Section 3. Finally, the discussion and conclusion are given in Section 4.

2. FAILURE RECOVERY

In parallel manipulators, considering leg i, the $l\times 1$ joint velocity vector ${}^{i}\dot{\mathbf{q}}$ and $m\times 1$ platform twist $\mathbf{V}$ are related by the $m\times l$ Jacobian matrix ${}^{i}\mathbf{J}$ as

$$\mathbf{V} = {}^{i}\mathbf{J}\,{}^{i}\dot{\mathbf{q}} = \sum_{j=1}^{l} {}^{i}\mathbf{J}_{j}\,{}^{i}\dot{q}_{j}\ , \tag{1}$$

where each column of ${}^{i}\mathbf{J}$, ${}^{i}\mathbf{J}_{j}$, is a screw representing the axis of joint j in leg i, ${}^{i}\dot{q}_{j}$ is the velocity of joint j. Parallel manipulators consist of n_l legs and each leg has l active and passive joints and $l \geq m$. The platform velocity will be provided if $\mathbf{V}$ is in the range space of all ${}^{i}\mathbf{J}$, i = 1, 2, …, n_l. When the failure occurs, the manipulator joints may no longer provide the platform twist as desired. On the other hand, the velocity of the healthy joints should be adjusted to recover the platform twist. When g joints (active and passive) on leg i fail, their velocities, zero or nonzero, could be different than the desired velocities. The overall velocity vector of the healthy joints is recalculated to satisfy the following equation [4],

$$\mathbf{V}^{*} = \mathbf{V} - \sum_{gc} {}^{i}\mathbf{J}_{k}\,{}^{i}\dot{q}_{ck} = {}^{i}\mathbf{J}_{r}\,{}^{i}\dot{\mathbf{q}}_{tot\,r}\ , \tag{2}$$

where gc out of g failed joints have nonzero velocity; $\mathbf{V}^{*} \in R^{m}$ is the velocity of the platform, in which the portion of platform twist provided by the failed joints with nonzero velocity is deducted; ${}^{i}\mathbf{J}_{k}$ is the column of ${}^{i}\mathbf{J}$ associated with the gc failed joints with nonzero velocity; ${}^{i}\mathbf{J}_{r} \in R^{m\times(l-g)}$ is the Jacobian matrix of the failed leg in which the columns of ${}^{i}\mathbf{J}$ corresponding to g failed joints are eliminated; ${}^{i}\dot{\mathbf{q}}_{tot\,r} \in R^{l-g}$ is the overall velocity vector of the healthy joints after the failure, in which the entries corresponding to the failed joints are removed; and ${}^{i}\dot{q}_{ck} \neq 0$ is the velocity of the failed joint which is nonzero.

From the kinematics point of view, $\mathcal{R}({}^{i}\mathbf{J}_{r})$ is the range space (image space) of ${}^{i}\mathbf{J}_{r}$, which is a subspace in the m-dimensional space of $\mathbf{V}^{*}$. The platform velocity of the manipulator is provided through an appropriate choice of ${}^{i}\dot{\mathbf{q}}_{tot\,r}$ if and only if $\mathbf{V}^{*} \in \mathcal{R}({}^{i}\mathbf{J}_{r})$. The orthogonal complement of the range space, $\mathcal{R}({}^{i}\mathbf{J}_{r})^{\perp}$, indicates the subspace made up of all of kinematically unrealizable twist $\mathbf{V}^{*}$. The subspace of $\mathcal{N}({}^{i}\mathbf{J}_{r})$ represents the variety of solutions in the $(l-g)-$dimensional space of ${}^{i}\dot{\mathbf{q}}_{tot\,r}$, in which ${}^{i}\mathbf{J}_{r}\,{}^{i}\dot{\mathbf{q}}_{tot\,r} = \mathbf{0}$. The orthogonal complement of the null space of ${}^i\mathbf{J}_r$, $\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$, is the subspace in the $(l-g)$-dimensional space of ${}^i\dot{\mathbf{q}}_{tot\,r}$, which maps ${}^i\dot{\mathbf{q}}_{tot\,r}$ onto the range space of ${}^i\mathbf{J}_r$ [8].

If ${}^i\dot{\mathbf{q}}_{tot\,r,1}$ and ${}^i\dot{\mathbf{q}}_{tot\,r,2}$ are in $\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$ and ${}^i\mathbf{J}_r\,{}^i\dot{\mathbf{q}}_{tot\,r,1}= {}^i\mathbf{J}_r\,{}^i\dot{\mathbf{q}}_{tot\,r,2}$, i.e., ${}^i\dot{\mathbf{q}}_{tot\,r,1}$ and ${}^i\dot{\mathbf{q}}_{tot\,r,2}$ are mapped onto the same point in $\mathcal{R}({}^i\mathbf{J}_r)$, therefore ${}^i\mathbf{J}_r({}^i\dot{\mathbf{q}}_{tot\,r,1}-{}^i\dot{\mathbf{q}}_{tot\,r,2})=\mathbf{0}$. In other words, ${}^i\dot{\mathbf{q}}_{tot\,r,1}-{}^i\dot{\mathbf{q}}_{tot\,r,2}\in\mathcal{N}({}^i\mathbf{J}_r)$. Because ${}^i\dot{\mathbf{q}}_{tot\,r,1}-{}^i\dot{\mathbf{q}}_{tot\,r,2}\in\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$, it can be concluded that ${}^i\dot{\mathbf{q}}_{tot\,r,1}-{}^i\dot{\mathbf{q}}_{tot\,r,2}\in\mathcal{N}({}^i\mathbf{J}_r)^{\perp}\cap\mathcal{N}({}^i\mathbf{J}_r)$. Since $\mathcal{N}({}^i\mathbf{J}_r)$ and $\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$ are orthogonal, ${}^i\dot{\mathbf{q}}_{tot\,r,1}={}^i\dot{\mathbf{q}}_{tot\,r,2}$, which indicates that ${}^i\mathbf{J}_r$ is one-to-one mapping of $\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$ onto $\mathcal{R}({}^i\mathbf{J}_r)$.

It follows that if $\mathbf{V}^*\in\mathcal{R}({}^i\mathbf{J}_r)$, equation (2) has a solution in $\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$ which is unique. Then the general solution is given as

$$ {}^i\dot{\mathbf{q}}_{tot\,r}={}^i\dot{\mathbf{q}}_{tot\,r,p}+{}^i\dot{\mathbf{q}}_{tot\,r,h}\,, \tag{3} $$

where ${}^i\dot{\mathbf{q}}_{tot\,r,p}$ is the particular solution which belongs to $\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$ and ${}^i\dot{\mathbf{q}}_{tot\,r,h}$ is the homogenous solution in $\mathcal{N}({}^i\mathbf{J}_r)$. Since $\mathcal{N}({}^i\mathbf{J}_r)\perp\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$, then $\left\|{}^i\dot{\mathbf{q}}_{tot\,r,p}+{}^i\dot{\mathbf{q}}_{tot\,r,h}\right\|_2=\left\|{}^i\dot{\mathbf{q}}_{tot\,r,p}\right\|_2+\left\|{}^i\dot{\mathbf{q}}_{tot\,r,h}\right\|_2$ and therefore

$$ \left\|{}^i\dot{\mathbf{q}}_{tot\,r}\right\|_2=\left\|{}^i\dot{\mathbf{q}}_{tot\,r,p}\right\|_2+\left\|{}^i\dot{\mathbf{q}}_{tot\,r,h}\right\|_2 . \tag{4} $$

Equation (4) indicates that $\left\|{}^i\dot{\mathbf{q}}_{tot\,r}\right\|_2>\left\|{}^i\dot{\mathbf{q}}_{tot\,r,p}\right\|_2$ unless ${}^i\dot{\mathbf{q}}_{tot\,r}={}^i\dot{\mathbf{q}}_{tot\,r,p}$. Therefore, the solution which minimizes the 2-norm of the overall velocity vector associated with the healthy joints in equation (2) is unique and lies in $\mathcal{N}({}^i\mathbf{J}_r)^{\perp}$ [9].

2.1. PENROSE EQUATIONS

For the Jacobian matrix ${}^i\mathbf{J}_r$, there exists a unique matrix which satisfies the following four Penrose equations

$$ {}^i\mathbf{J}_r\,{}^i\mathbf{J}_r^{\#}\,{}^i\mathbf{J}_r={}^i\mathbf{J}_r\,, \tag{5} $$

$$ {}^i\mathbf{J}_r^{\#}\,{}^i\mathbf{J}_r\,{}^i\mathbf{J}_r^{\#}={}^i\mathbf{J}_r^{\#}\,, \tag{6} $$

$$({}^{i}\mathbf{J}_{r}{}^{i}\mathbf{J}_{r}^{\#})^{T} = {}^{i}\mathbf{J}_{r}^{i}\mathbf{J}_{r}^{\#}, \tag{7}$$

$$({}^{i}\mathbf{J}_{r}^{\#}{}^{i}\mathbf{J}_{r})^{\mathrm{T}} = {}^{i}\mathbf{J}_{r}^{\#}{}^{i}\mathbf{J}_{r}. \tag{8}$$

This unique matrix is called the Moore-Penrose generalized inverse and is denoted by ${}^{i}\mathbf{J}_{r}^{\#}$. In other generalized inverses, on the other hand, some (not all) of the Penrose equations are satisfied. If ${}^{i}\mathbf{J}_{r}$ is square and nonsingular, then the ordinary inverse of ${}^{i}\mathbf{J}_{r}^{-1}$ meets four equations (5)–(8). Therefore, for such a matrix, the Moore-Penrose generalized inverse is the ordinary inverse. For any matrix ${}^{i}\mathbf{J}_{r} \in R^{m\times(l-g)}$, the matrix ${}^{i}\mathbf{J}_{r}^{(1,2,\ldots,4)} \in R^{(l-g)\times m}$ denotes the set of matrices which satisfy equations $(1),(2),\ldots,(4)$ among equations (5)–(8). Matrix ${}^{i}\mathbf{J}_{r}^{(1,2,\ldots,4)}$ is also called an $\{1,2,\ldots,4\}$ – inverse of ${}^{i}\mathbf{J}_{r}$. It follows that ${}^{i}\mathbf{J}_{r}^{(1,3)}$ represents the generalized inverse, which meets the first and third Penrose equations and ${}^{i}\mathbf{J}_{r}^{(1,4)}$ denotes the generalized inverse, satisfying equations (5) and (8). It should be noted that as it was presented in [10], if there is a {1}-inverse, there will exist a {1,2}-inverse of ${}^{i}\mathbf{J}_{r}$. It follows that if there are ${}^{i}\mathbf{J}_{r}^{(1,3)}$ and ${}^{i}\mathbf{J}_{r}^{(1,4)}$, there will exist ${}^{i}\mathbf{J}_{r}^{(1,2,3)}$ and ${}^{i}\mathbf{J}_{r}^{(1,2,4)}$, respectively and that ${}^{i}\mathbf{J}_{r}^{(1,2,3,4)}$ has both properties of ${}^{i}\mathbf{J}_{r}^{(1,3)}$ and ${}^{i}\mathbf{J}_{r}^{(1,4)}$. When a reduction in the error of the platform twist caused by the failure is desired, one can use ${}^{i}\mathbf{J}_{r}^{(1,3)}$. Also ${}^{i}\mathbf{J}_{r}^{(1,4)}$ minimizes the overall joint velocity after the failure.

2.2. MINIMUM NORM FOR THE CORRECTIONAL JOINT VELOCITY VECTOR

In this section, the generalized inverse of ${}^{i}\mathbf{J}_{r}$ which satisfies the first and fourth Penrose equations, is formulated and applied for the minimum norm correctional joint velocity. Equation (2) has an exact solution if and only if for some ${}^{i}\mathbf{J}_{r}^{(1)}$, ${}^{i}\mathbf{J}_{r}{}^{i}\mathbf{J}_{r}^{(1)}\mathbf{V}^{*} = \mathbf{V}^{*}$. In this case the general solution for equation (2) is

$${}^{i}\dot{\mathbf{q}}_{tot\,r} = {}^{i}\mathbf{J}_{r}^{(1)}\mathbf{V}^{*} + (\mathbf{I} - {}^{i}\mathbf{J}_{r}^{(1)}{}^{i}\mathbf{J}_{r}){}^{i}\boldsymbol{k}, \tag{9}$$

where ${}^{i}\mathbf{J}_{r}^{(1)}$ meets ${}^{i}\mathbf{J}_{r}{}^{i}\mathbf{J}_{r}^{(1)}{}^{i}\mathbf{J}_{r} = {}^{i}\mathbf{J}_{r}$ and ${}^{i}\boldsymbol{k}$ is an arbitrary $(l-g)\times 1$ vector. In general, ${}^{i}\mathbf{J}_{r}^{(1)}$ is not a uniquely defined matrix. Therefore, the particular solution in equation (9) has infinite solutions. Among all particular solutions in equation (9), the generalized inverse which produces the minimum particular solution is desired.

As shown in [9], for any ${}^{i}\mathbf{J}_{r}^{(1,4)}$ there exists a unique solution ${}^{i}\dot{\mathbf{q}}_{tot\,r,p} = {}^{i}\mathbf{J}_{r}^{(1,4)}\mathbf{V}^{*}$, which lies in $\mathcal{N}({}^{i}\mathbf{J}_{r})^{\perp}$ and, thus, is the unique minimum norm solution.

Considering the generalized inverse of ${}^{i}\mathbf{J}_{r}$ satisfying the first and fourth Penrose equations, the general solution to equation (2) could be written as

$$ {}^{i}\dot{\mathbf{q}}_{tot\,r} = {}^{i}\mathbf{J}_{r}^{(1,4)}\mathbf{V}^{*} + {}^{i}\mathbf{N}\,{}^{i}\boldsymbol{\lambda}\,. \tag{10} $$

The first term on the right-hand side of equation (10) is the minimum norm solution (particular solution), and the second term is a homogeneous solution that belongs to the null space of ${}^{i}\mathbf{J}_{r}$ and does not affect the platform motion, ${}^{i}\mathbf{N}$ is the $(l-g)\times(l-g-m)$ matrix whose columns span the kernel of ${}^{i}\mathbf{J}_{r}$, and ${}^{i}\boldsymbol{\lambda}$ is an arbitrary $(l-g-m)\times 1$ vector.

The correctional velocity of the healthy joints is

$$ {}^{i}\Delta\dot{\mathbf{q}}_{r} = {}^{i}\dot{\mathbf{q}}_{tot\,r} - {}^{i}\dot{\mathbf{q}}_{f\,r}\,, \tag{11} $$

where ${}^{i}\dot{\mathbf{q}}_{f\,r} \in R^{l-g}$ is the velocity of the healthy joints before the failure in which the entries associated with the failed joints that have zero and nonzero velocities are removed.

Substituting equation (10) into equation (11) leads to

$$ {}^{i}\Delta\dot{\mathbf{q}}_{r} = {}^{i}\mathbf{J}_{r}^{(1,4)}\mathbf{V}^{*} + {}^{i}\mathbf{N}\,{}^{i}\boldsymbol{\lambda} - {}^{i}\dot{\mathbf{q}}_{f\,r}\,. \tag{12} $$

To minimize the correctional velocity of the healthy joints, the objective function is defined as the square of the Euclidean norm of the vector of the correctional velocity of the healthy joints, $({}^{i}\Delta\dot{\mathbf{q}}_{r})^{\mathrm{T}}({}^{i}\Delta\dot{\mathbf{q}}_{r})$, and the linear constraint equation in terms of the overall velocity vector is $\mathbf{V}^{*} - {}^{i}\mathbf{J}_{r}\,{}^{i}\dot{\mathbf{q}}_{tot\,r} = \mathbf{0}$ [4]. The objective function is rearranged by applying equation (12) in terms of vector ${}^{i}\boldsymbol{\lambda}$ as

$$ f({}^{i}\boldsymbol{\lambda}) = ({}^{i}\mathbf{J}_{r}^{(1,4)}\mathbf{V}^{*} + {}^{i}\mathbf{N}\,{}^{i}\boldsymbol{\lambda} - {}^{i}\dot{\mathbf{q}}_{f\,r})^{\mathrm{T}}({}^{i}\mathbf{J}_{r}^{(1,4)}\mathbf{V}^{*} + {}^{i}\mathbf{N}\,{}^{i}\boldsymbol{\lambda} - {}^{i}\dot{\mathbf{q}}_{f\,r})\,, \tag{13} $$

subjected to the following equality constraint

$$ \mathbf{V}^{*} - {}^{i}\mathbf{J}_{r}({}^{i}\mathbf{J}_{r}^{(1,4)}\mathbf{V}^{*} + {}^{i}\mathbf{N}\,{}^{i}\boldsymbol{\lambda}) = \mathbf{0}\,. \tag{14} $$

The equality constraint (14) is always satisfied provided ${}^{i}\mathbf{J}_{r}$ is of full row-rank. Since $\mathbf{J}_{r}\,{}^{i}\mathbf{J}_{r}^{(1,4)} = \mathbf{I}\,, (\mathbf{I} \in R^{m\times m}),\ {}^{i}\mathbf{J}_{r}\,{}^{i}\mathbf{N} = \mathbf{0}$, then equation (14) is simplified as $\mathbf{V}^{*}(\mathbf{I} - {}^{i}\mathbf{J}_{r}\,{}^{i}\mathbf{J}_{r}^{(1,4)}) - {}^{i}\mathbf{J}_{r}\,{}^{i}\mathbf{N}\,{}^{i}\boldsymbol{\lambda} = \mathbf{0}$. If ${}^{i}\mathbf{J}_{r}$ is not of full row-rank, the equality constraint (14) cannot be satisfied, and the partitioned Jacobian matrix in Section

2.2 is applied to partially recover the platform twist. To optimize $f({}^i\boldsymbol{\lambda})$, the derivative of equation (13) with respect to ${}^i\boldsymbol{\lambda}$ should be equal to zero, *i.e.*,

$$\begin{aligned}&\frac{\partial f({}^i\boldsymbol{\lambda})}{\partial\, {}^i\boldsymbol{\lambda}} = 2\,{}^i\mathbf{N}^T({}^i\mathbf{J}_r^{(1,4)}\mathbf{V}^* + {}^i\mathbf{N}\,{}^i\boldsymbol{\lambda} - {}^i\dot{\mathbf{q}}_{f\,r}) = \mathbf{0}\\ &\Rightarrow {}^i\boldsymbol{\lambda} = ({}^i\mathbf{N}^T\,{}^i\mathbf{N})^{-1}\,{}^i\mathbf{N}^\mathrm{T}({}^i\dot{\mathbf{q}}_{f\,r} - {}^i\mathbf{J}_r^{(1,4)}\mathbf{V}^*)\,.\end{aligned} \tag{15}$$

Substituting ${}^i\boldsymbol{\lambda}$ into equations (10) and (12) leads to

$${}^i\dot{\mathbf{q}}_{tot\,r} = {}^i\mathbf{J}_r^{(1,4)}\mathbf{V}^* + {}^i\mathbf{N}({}^i\mathbf{N}^\mathrm{T}\,{}^i\mathbf{N})^{-1}\,{}^i\mathbf{N}^\mathrm{T}({}^i\dot{\mathbf{q}}_{f\,r} - {}^i\mathbf{J}_r^{(1,4)}\mathbf{V}^*)\,, \tag{16}$$

$${}^i\Delta\dot{\mathbf{q}}_r = {}^i\mathbf{J}_r^{(1,4)}\mathbf{V}^* - {}^i\dot{\mathbf{q}}_{f\,r} + {}^i\mathbf{N}({}^i\mathbf{N}^\mathrm{T}\,{}^i\mathbf{N})^{-1}\,{}^i\mathbf{N}^\mathrm{T}({}^i\dot{\mathbf{q}}_{f\,r} - {}^i\mathbf{J}_r^{(1,4)}\mathbf{V}^*)\,. \tag{17}$$

Considering the minimum norm least-square solution, these formulations are valid for the Moore-Penrose generalized inverse if the $\{1,4\}$-inverse of ${}^i\mathbf{J}_r$ is replaced by ${}^i\mathbf{J}_r^{\#}$. In this case, equations (16) and (17) will be the same as [4] when the homogeneous solution is excluded.

Since the $\{1,2,4\}$-inverse of ${}^i\mathbf{J}_r$ represents the same properties as the $\{1,4\}$-inverse, the $\{1,2,4\}$-inverse of ${}^i\mathbf{J}_r$, ${}^i\mathbf{J}_r^{(1,2,4)}$, meets the first, second, and fourth equations of Penrose [11].

$${}^i\mathbf{J}_r^{(1,2,4)} = {}^i\mathbf{J}_r^\mathrm{T}({}^i\mathbf{J}_r\,{}^i\mathbf{J}_r^\mathrm{T})^{(1)}\,. \tag{18}$$

The $\{1\}$-inverse of $({}^i\mathbf{J}_r\,{}^i\mathbf{J}_r^\mathrm{T})$ is constructed by transferring $({}^i\mathbf{J}_r\,{}^i\mathbf{J}_r^\mathrm{T})$ into a reduced row echelon form. The reduced row echelon form can be computed by Gaussian elimination method. For an elementary row operations matrix ${}^i\mathbf{E}\in R^{m\times m}$ and permutation matrix ${}^i\mathbf{U}\in R^{m\times m}$ so as ${}^i\mathbf{E}({}^i\mathbf{J}_r\,{}^i\mathbf{J}_r^\mathrm{T})\,{}^i\mathbf{U} = \begin{bmatrix}\mathbf{I}_n & {}^i\mathbf{L}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$, ${}^i\mathbf{L}\in R^{(m-n)\times(m-n)}$, then ${}^i\mathbf{U}\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ \mathbf{0} & {}^i\mathbf{M}\end{bmatrix}{}^i\mathbf{E}$ is a $\{1\}$-inverse of $({}^i\mathbf{J}_r\,{}^i\mathbf{J}_r^\mathrm{T})$ for any ${}^i\mathbf{M}\in R^{(m-n)\times(m-n)}$ [9], n is the rank of ${}^i\mathbf{J}_r$ and identity matrix $\mathbf{I}_n\in R^{n\times n}$.

Considering the reduced row echelon form of matrix ${}^i\mathbf{J}_r$, the basis for the null space of ${}^i\mathbf{J}_r$ is formulated. If ${}^i\mathbf{J}_r\in R^{m\times(l-g)}$ has rank n and ${}^i\mathbf{E}_k, {}^i\mathbf{E}_{k-1}, \ldots, {}^i\mathbf{E}_1$ are elementary row operations, and ${}^i\mathbf{Q}\in R^{(l-g)\times(l-g)}$ is a permutation matrix, ${}^i\mathbf{J}_r$ could be brought into the partitioned form as

$$ {}^{i}\mathbf{E}\,{}^{i}\mathbf{J}_r\,{}^{i}\mathbf{Q} = \begin{bmatrix} \mathbf{I}_n & {}^{i}\mathbf{K} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{19} $$

where ${}^{i}\mathbf{E} = {}^{i}\mathbf{E}_k\,{}^{i}\mathbf{E}_{k-1} \ldots {}^{i}\mathbf{E}_1 \in R^{m\times m}$, k is the number of row operations to form partitioned matrix (19), and ${}^{i}\mathbf{K}$ is the $n \times (l-g-n)$ matrix, which is used to form the null space of ${}^{i}\mathbf{J}_r$. The elementary row operations matrix ${}^{i}\mathbf{E}$ and permutation matrix ${}^{i}\mathbf{Q}$ are square and nonsingular, and their inverses exist. Multiplying both sides of equation (19) by ${}^{i}\mathbf{E}^{-1}$ and ${}^{i}\mathbf{Q}^{-1}$ leads to

$$ {}^{i}\mathbf{J}_r = {}^{i}\mathbf{E}^{-1} \begin{bmatrix} \mathbf{I}_n & {}^{i}\mathbf{K} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} {}^{i}\mathbf{Q}^{-1}. \tag{20} $$

It follows from equation (20) that the columns of ${}^{i}\mathbf{N} = {}^{i}\mathbf{Q}\begin{bmatrix} -{}^{i}\mathbf{K} \\ \mathbf{I}_{l-g-n} \end{bmatrix}$ $\in R^{(l-g)\times(l-g-n)}$ are a basis for the null space of ${}^{i}\mathbf{J}_r$, *i.e.*, ${}^{i}\mathbf{J}_r\,{}^{i}\mathbf{N} = \mathbf{0}$. If the generalized inverse ${}^{i}\mathbf{J}_r^{(1,4)}$ formulated in equation (18) is used in equation (16), the overall velocity of the healthy joints is written as

$$ \begin{aligned} {}^{i}\dot{\mathbf{q}}_{tot\,r} = {}& {}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)}\mathbf{V}^* + {}^{i}\mathbf{N}({}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\mathbf{N})^{-1}\,{}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\dot{\mathbf{q}}_{f\,r} \\ & - {}^{i}\mathbf{N}({}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\mathbf{N})^{-1}\,{}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)}\mathbf{V}^*. \end{aligned} \tag{21} $$

Since the columns of ${}^{i}\mathbf{N}$ are linearly independent, then ${}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\mathbf{N}$ is identity. The last term on the right-hand side of equation (21) becomes zero as ${}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\mathbf{J}_r^{\mathrm{T}} = ({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{N})^{\mathrm{T}} = \mathbf{0}$. The simplified form of equation (21) is

$$ {}^{i}\dot{\mathbf{q}}_{tot\,r} = {}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)}\mathbf{V}^* + {}^{i}\mathbf{N}\,{}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\dot{\mathbf{q}}_{f\,r}. \tag{22} $$

The ${}^{i}\mathbf{N}\,{}^{i}\mathbf{N}^{\mathrm{T}}$ in the second term on the right-hand side of equation (22) represents the null space projector, which maps ${}^{i}\dot{\mathbf{q}}_{f\,r}$ onto the null space of ${}^{i}\mathbf{J}_r$. It was shown in [12] that, among all least-squares solutions to equation (2), ${}^{i}\dot{\mathbf{q}}_{tot\,r} = {}^{i}\mathbf{J}_r^{(1,4)}\,{}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{(1,3)}\mathbf{V}^*$ is the one that is unique and minimum. Therefore, ${}^{i}\mathbf{J}_r^{(1,4)}\,{}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{(1,3)}$ is the Moore-Penrose generalized inverse of ${}^{i}\mathbf{J}_r$ and ${}^{i}\mathbf{J}_r^{(1,3)}$ is formulated as $({}^{i}\mathbf{J}_r^{\mathrm{T}}\,{}^{i}\mathbf{J}_r)^{(1)}\,{}^{i}\mathbf{J}_r^{\mathrm{T}}$ [9]. The velocity of the healthy joints before the failure can be written as

$$^{i}\dot{\mathbf{q}}_{f\,r} = {}^{i}\mathbf{J}_r^{(1,4)}\,{}^{i}\mathbf{J}\,{}^{i}\mathbf{J}^{(1,3)}\mathbf{V}^{*}. \tag{23}$$

Using equations (18) and (23), the overall velocity of the healthy joints after the failure in equation (22) can be calculated as

$$^{i}\dot{\mathbf{q}}_{tot\,r} = {}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)}\mathbf{V}^{*} + {}^{i}\mathbf{N}\,{}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)}\,{}^{i}\mathbf{J}\,{}^{i}\mathbf{J}^{(1,3)}\mathbf{V}^{*}. \tag{24}$$

The second term on the right-hand side of equation (24) is zero because ${}^{i}\mathbf{N}^{\mathrm{T}}\,{}^{i}\mathbf{J}_r^{\mathrm{T}} = ({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{N})^{\mathrm{T}} = \mathbf{0}$. Therefore, the optimal velocity of the healthy joints after the failure in order to minimize the correctional joint velocity is the particular solution of equation (10), which is

$$^{i}\dot{\mathbf{q}}_{tot\,r} = {}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)}(\mathbf{V} - \sum\nolimits_{gc}{}^{i}\mathbf{J}_k\,{}^{i}\dot{q}_{ck}). \tag{25}$$

Equation (25) is the same as the expression derived in [4] when ${}^{i}\mathbf{J}_r$ is full row-rank. When the vector of the healthy joint velocities is not physically consistent, i.e., leg i has a combination of revolute and prismatic joints, and when ${}^{i}\mathbf{J}_r$ is of full row-rank, then a weighting metric would be required to avoid inconsistency. Therefore, the generalized inverse is calculated as ${}^{i}\mathbf{J}_r^{(1,4)} = {}^{i}\mathbf{W}\,{}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{W}\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)}$. The weighting metric ${}^{i}\mathbf{W}$ is chosen so that ${}^{i}\dot{\mathbf{q}}_{tot\,r}^{\mathrm{T}}({}^{i}\mathbf{W}^{-1}\,{}^{i}\dot{\mathbf{q}}_{tot\,r})$ becomes physically consistent, e.g., to minimize/maximize the kinetic energy of leg i [4].

It should be noted that ${}^{i}\mathbf{J}_r^{(1,4)}$ is unique if and only if the Jacobian matrix of the failed leg, ${}^{i}\mathbf{J}_r$, is full row-rank. It is clear that for full row-rank ${}^{i}\mathbf{J}_r$, since $\mathrm{rank}({}^{i}\mathbf{J}_r) = \mathrm{rank}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})$, then $({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})$ is nonsingular and $\{1\}$-inverse of $({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})$ would be the same as the ordinary inverse, *i.e.*, $({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{-1}$ and ${}^{i}\mathbf{J}_r^{(1,4)} = {}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{(1)} = {}^{i}\mathbf{J}_r^{\mathrm{T}}({}^{i}\mathbf{J}_r\,{}^{i}\mathbf{J}_r^{\mathrm{T}})^{-1} = {}^{i}\mathbf{J}_r^{\#}$. Therefore, ${}^{i}\mathbf{J}_r^{(1,4)}\mathbf{V}^{*}$ is the minimum 2-norm, which leads to the least-squares solution for equation (2).

2.3. PARTITIONED JACOBIAN MATRIX

If ${}^{i}\mathbf{J}_r$ is neither full row-rank nor full column-rank, *i.e.*, $\mathbf{V}^{*}$ does not belong to the range space of ${}^{i}\mathbf{J}_r$, all components of the platform twist would not be retrieved. So it is desired to recover as many components of the platform twist as possible while the correctional velocity of the healthy joints is minimized. Since ${}^{i}\mathbf{J}_r$ is of rank n, $n < \min(m, l-g)$ it has at least one nonsingular $n \times n$ submatrix

${}^i\mathbf{J}_{r,11}$. By rearrangement of rows and columns using the permutation matrices ${}^i\mathbf{P} \in \mathrm{R}^{m\times m}$ and ${}^i\mathbf{Q} \in \mathrm{R}^{(l-g)\times(l-g)}$, the partitioned Jacobian matrix can be derived as

$$ {}^i\mathbf{P}\, {}^i\mathbf{J}_r\, {}^i\mathbf{Q} = \begin{bmatrix} {}^i\mathbf{J}_{r,11} & {}^i\mathbf{J}_{r,12} \\ {}^i\mathbf{J}_{r,21} & {}^i\mathbf{J}_{r,22} \end{bmatrix}, \qquad (26) $$

where permutation matrices ${}^i\mathbf{P}$ and ${}^i\mathbf{Q}$ are orthogonal, *i.e.*, ${}^i\mathbf{P}^{-1} = {}^i\mathbf{P}^{\mathrm{T}}$ and ${}^i\mathbf{Q}^{-1} = {}^i\mathbf{Q}^{\mathrm{T}}$, and their columns are unit vectors, in which one entry is identity and the rest are zero. When the columns of ${}^i\mathbf{J}_r$ are interchanged, it is post multiplied by the permutation matrix ${}^i\mathbf{Q}$, and premultiplication of the matrix by ${}^i\mathbf{P}$ leads to the interchange of the rows. Matrix ${}^i\mathbf{J}_{r,11} \in R^{n\times n}$ is square and nonsingular, and its inverse exists. Also, there are no submatrices ${}^i\mathbf{J}_{r,21}$ and ${}^i\mathbf{J}_{r,22}$ when ${}^i\mathbf{J}_r$ is of full row-rank and no submatrices ${}^i\mathbf{J}_{r,12}$ and ${}^i\mathbf{J}_{r,22}$ where ${}^i\mathbf{J}_r$ is of full column-rank.

In equation (26) columns comprising ${}^i\mathbf{J}_{r,12}$ and ${}^i\mathbf{J}_{r,22}$ are linear combinations of the columns formed by ${}^i\mathbf{J}_{r,11}$ and ${}^i\mathbf{J}_{r,21}$, *i.e.*,

$$ \begin{bmatrix} {}^i\mathbf{J}_{r,12} \\ {}^i\mathbf{J}_{r,22} \end{bmatrix} = \begin{bmatrix} {}^i\mathbf{J}_{r,11} \\ {}^i\mathbf{J}_{r,21} \end{bmatrix} {}^i\mathbf{T}, \qquad (27) $$

where ${}^i\mathbf{T} = {}^i\mathbf{J}_{r,11}^{-1}\, {}^i\mathbf{J}_{r,12}$ and ${}^i\mathbf{T} \in R^{n\times(l-g-n)}$. Also, rows comprising ${}^i\mathbf{J}_{r,21}$ and ${}^i\mathbf{J}_{r,22}$ are linear combinations of the rows of ${}^i\mathbf{J}_{r,11}$ and ${}^i\mathbf{J}_{r,12}$, *i.e.*,

$$ \begin{bmatrix} {}^i\mathbf{J}_{r,21} & {}^i\mathbf{J}_{r,22} \end{bmatrix} = {}^i\mathbf{S} \begin{bmatrix} {}^i\mathbf{J}_{r,11} & {}^i\mathbf{J}_{r,12} \end{bmatrix}, \qquad (28) $$

where ${}^i\mathbf{S} = \mathbf{J}_{r,21}\, {}^i\mathbf{J}_{r,11}^{-1}$ and ${}^i\mathbf{S} \in R^{(m-n)\times n}$. The four partitioned Jacobians in equation (26) can be defined in terms of ${}^i\mathbf{J}_{r,11}$ as

$$ \begin{aligned} {}^i\mathbf{J}_{r,12} &= {}^i\mathbf{J}_{r,11}\, {}^i\mathbf{T} \\ {}^i\mathbf{J}_{r,21} &= {}^i\mathbf{S}\, {}^i\mathbf{J}_{r,11} \\ {}^i\mathbf{J}_{r,22} &= {}^i\mathbf{S}\, {}^i\mathbf{J}_{r,12} = {}^i\mathbf{S}\, {}^i\mathbf{J}_{r,11}\, {}^i\mathbf{T}\,. \end{aligned} \qquad (29) $$

From equations (26) and (29), the partitioned Jacobian matrix of ${}^i\mathbf{J}_r$ can be formulated as follows:

$$ {}^{i}\mathbf{J}_{r} = {}^{i}\mathbf{P}^{\mathrm{T}} \begin{bmatrix} {}^{i}\mathbf{J}_{r,11} & {}^{i}\mathbf{J}_{r,11}\,{}^{i}\mathbf{T} \\ {}^{i}\mathbf{S}\,{}^{i}\mathbf{J}_{r,11} & {}^{i}\mathbf{S}\,{}^{i}\mathbf{J}_{r,11}\,{}^{i}\mathbf{T} \end{bmatrix} {}^{i}\mathbf{Q}^{\mathrm{T}} = {}^{i}\mathbf{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{I}_{r} \\ {}^{i}\mathbf{S} \end{bmatrix} {}^{i}\mathbf{J}_{r,11} \begin{bmatrix} \mathbf{I}_{r} & {}^{i}\mathbf{T} \end{bmatrix} {}^{i}\mathbf{Q}^{\mathrm{T}} . \tag{30} $$

The basis for the null space of ${}^{i}\mathbf{J}_{r}$ so that ${}^{i}\mathbf{J}_{r}{}^{i}\mathbf{N} = \mathbf{0}$ is formed by the columns of the $(l-g)\times(l-g-n)$ matrix ${}^{i}\mathbf{N} = {}^{i}\mathbf{Q}\begin{bmatrix} -{}^{i}\mathbf{T} \\ \mathbf{I}_{l-g-n} \end{bmatrix}$. The partitioned matrix in equation (30) is applied to formulate ${}^{i}\mathbf{J}_{r}^{(1,2,4)}$ as [9].

$$ {}^{i}\mathbf{J}_{r}^{(1,2,4)} = {}^{i}\mathbf{Q} \begin{bmatrix} \mathbf{I}_{r} \\ {}^{i}\mathbf{T}^{\mathrm{T}} \end{bmatrix} (\mathbf{I}_{r} + {}^{i}\mathbf{T}\,{}^{i}\mathbf{T}^{\mathrm{T}})^{-1} \begin{bmatrix} {}^{i}\mathbf{J}_{r,11}^{-1} & \mathbf{0} \end{bmatrix} {}^{i}\mathbf{P} . \tag{31} $$

The generalized inverse in equation (31) is easily verified to satisfy ${}^{i}\mathbf{J}_{r}{}^{i}\mathbf{J}_{r}^{(1,2,4)}{}^{i}\mathbf{J}_{r} = {}^{i}\mathbf{J}_{r}$ and $({}^{i}\mathbf{J}_{r}^{(1,2,4)}\,{}^{i}\mathbf{J}_{r})^{\mathrm{T}} = {}^{i}\mathbf{J}_{r}^{(1,2,4)}\,{}^{i}\mathbf{J}_{r}$. If the Jacobian matrix of the failed leg is neither full row-rank nor full column-rank, the full recovery of the platform twist may not be provided. More components of the platform twist with the minimum 2-norm of the correctional joint velocity vector are recovered if permutation matrices ${}^{i}\mathbf{P}$ and ${}^{i}\mathbf{Q}$ in equation (31) are correctly selected. It should be noted that the generalized inverse for any partitioned matrix ${}^{i}\mathbf{J}_{r}$ of rank n satisfying the first, second and fourth Penrose equations depends on the permutation matrix ${}^{i}\mathbf{P}$. On the other hand, creating different column combinations of ${}^{i}\mathbf{J}_{r}$ by choosing matrix ${}^{i}\mathbf{Q}$ leads to a unique generalized inverse. Therefore, if ${}^{i}\mathbf{J}_{r}$ is of full column-rank, $m > l-g = n$, its generalized inverse could be simplified as ${}^{i}\mathbf{J}_{r}^{(1,2,4)} = \begin{bmatrix} {}^{i}\mathbf{J}_{r,11}^{-1} & 0 \end{bmatrix} {}^{i}\mathbf{P}$ and $\mathbf{0} \in \mathrm{R}^{n\times(m-n)}$, which is changed by permutation matrix ${}^{i}\mathbf{P}$, *i.e.*, selecting different rows of ${}^{i}\mathbf{J}_{r}$ to form ${}^{i}\mathbf{J}_{r,11}$ results in different generalized inverses ${}^{i}\mathbf{J}_{r}^{(1,2,4)}$. However, as already mentioned, a full row-rank matrix ${}^{i}\mathbf{J}_{r}$ has a unique generalized inverse which is the Moore-Penrose generalized inverse.

3. CASE STUDY

In this section, the manipulator modeled and investigated in [4] is used as a case study, and the results are compared. As depicted in Fig. 1, each leg is kinematically redundant with 3 active prismatic joints and two passive revolute joints. The coordinates of the base attachment points in the fixed coordinate system are $(-2 \quad -1.5)$, $(2 \quad -1.5)$ and $(0 \quad 1.5)$, respectively. The position of connection points on the platform is set at a constant radius of 0.25 meters. The angular coordinates of the leg connections to the mobile platform are −150°, −30° and 90°, respectively. As reported in [4], when the mobile platform pose is $\mathbf{p} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$ meter and $\varphi = -30°$, leg 3 is in the Y direction with the joint displacements of ${}^{3}\mathbf{q} = \begin{bmatrix} 0.0 & 0.125 & 90 & 1.283 & -30 \end{bmatrix}^{\mathrm{T}}$. Then, the Jacobian matrix of leg 3 will be

$$ {}^{3}\mathbf{J} = \begin{bmatrix} 0 & 1.0 & 1.5 & 0 & 0.217 \\ 1 & 0 & -0.125 & -1.0 & -0.125 \\ 0 & 0 & 1.0 & 0 & 1.0 \end{bmatrix}. \tag{32} $$

For the platform twist of $\mathbf{V} = \begin{bmatrix} 1 & 0.5 & 0 \end{bmatrix}^{\mathrm{T}}$ the minimum 2-norm least-squares solution is ${}^{3}\dot{\mathbf{q}} = {}^{3}\mathbf{J}^{\mathrm{T}}({}^{3}\mathbf{J}\,{}^{3}\mathbf{J}^{\mathrm{T}})^{-1}\mathbf{V}^{*} = \begin{bmatrix} 0.250 & 0.548 & 0.352 & -0.250 & -0.352 \end{bmatrix}^{\mathrm{T}}$. When the second active joint (h=2) of leg 3 (i=3) is locked, *i.e.*, ${}^{3}\dot{q}_{c2} = 0$, the remaining healthy joints have the velocity of ${}^{1}\dot{\mathbf{q}}_{f} = \begin{bmatrix} 0.250 & 0.352 & -0.250 & -0.352 \end{bmatrix}^{\mathrm{T}}$, and the platform velocity is calculated as ${}^{3}\mathbf{J}_{r}\,{}^{3}\dot{\mathbf{q}}_{f\,r} = \begin{bmatrix} 0.452 & 0.5 & 0.0 \end{bmatrix}^{\mathrm{T}}$.

Since ${}^{3}\mathbf{J}_{r}$ is of full row-rank, ${}^{3}\mathbf{J}_{r}^{(1,4)}$ is unique and identical to the Moore-Penrose generalized inverse, *i.e.*, ${}^{3}\mathbf{J}_{r}^{(1,4)} = {}^{3}\mathbf{J}_{r}^{\#} = {}^{3}\mathbf{J}_{r}^{\mathrm{T}}({}^{3}\mathbf{J}_{r}\,{}^{3}\mathbf{J}_{r}^{\mathrm{T}})^{-1}$. To minimize the vector of the correctional velocity of the healthy joints, the overall velocity of the healthy joints is calculated as ${}^{3}\dot{\mathbf{q}}_{tot\,r} = {}^{3}\mathbf{J}_{r}^{(1,4)}(\mathbf{V} - {}^{3}\mathbf{J}_{2}\,{}^{3}\dot{q}_{c2})$ $= \begin{bmatrix} 0.250 & 0.779 & -0.250 & -0.779 \end{bmatrix}^{\mathrm{T}}$, which is the same as [4]. Because the platform twist is in the range space of ${}^{3}\mathbf{J}_{r}$ and also in the range space of the Jacobian matrix of the healthy legs, the motion of the failed joint is fully recovered by adjusting the velocity of the two passive revolute joints.

In another example, the first three joints of leg 3 are jammed (g = 3) for the joint displacement of $\begin{bmatrix} 0.0 & 0.125 & 60 & 1.283 & -30 \end{bmatrix}^{\mathrm{T}}$ and the platform twist of $\mathbf{V}^{*} = \begin{bmatrix} 1 & 0.5 & 0.24 \end{bmatrix}^{\mathrm{T}}$. The Jacobian matrix of leg 3 will be

$$ {}^{3}\mathbf{J} = \begin{bmatrix} 0 & 1.0 & 1.236 & -0.5 & 0.125 \\ 1 & 0 & -0.858 & -0.866 & -0.217 \\ 0 & 0 & 1.0 & 0 & 1.0 \end{bmatrix}. \tag{33} $$

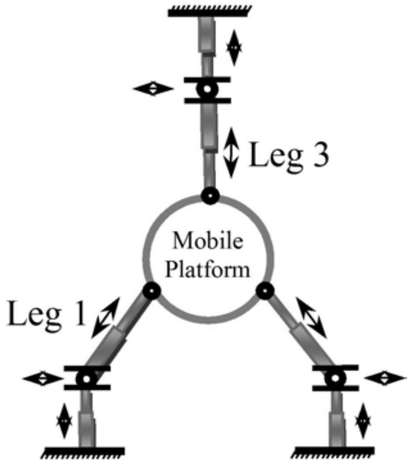

Fig. 1 – Redundant planar parallel manipulator.

If the Jacobian matrix of leg 3 after the failure is not of full row-rank, the platform twist cannot be fully recovered. As mentioned, equation (31) will be simplified as

$$ {}^3\mathbf{J}_r^{1,2,4} = \begin{bmatrix} {}^3\mathbf{J}_{r,11}^{-1} & \mathbf{0} \end{bmatrix} {}^3\mathbf{P} = \begin{bmatrix} {}^3\mathbf{J}_{r,11}^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}^3\mathbf{P}_1 \\ {}^3\mathbf{P}_2 \end{bmatrix} = {}^3\mathbf{J}_{r,11}^{-1}\, {}^3\mathbf{P}_1 , \tag{34} $$

where ${}^3\mathbf{P}_1 \in \mathbf{R}^{2\times 3}$. If ${}^3\mathbf{J}_{r,11}$ is a nonsingular and square matrix of rank 2, there will be $\begin{pmatrix} m \\ l-g \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$ different submatrices to form ${}^3\mathbf{P}_1$ as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and also, three submatrices to form ${}^3\mathbf{P}_2$ as $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Premultiplying ${}^3\mathbf{J}_r$, the reduced form of ${}^3\mathbf{J}$ in equation (33), by ${}^3\mathbf{P}_1$ forms ${}^3\mathbf{J}_{r,11}$. For full-rank ${}^3\mathbf{J}_{r,11}$, the submatrix ${}^3\mathbf{P}_1$ forms a set of equations ${}^3\dot{\mathbf{q}}_{tot\,r} = {}^3\mathbf{J}_{r,11}^{-1}\, {}^3\mathbf{P}_1 \mathbf{V}^*$. Then, the solution for this set of equations, *i.e.*, the overall velocity of the healthy joints, is used in ${}^3\mathbf{P}_2\, {}^3\mathbf{J}_r$ to calculate the deviation of the platform twist from the desired value. For the first permutation submatrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, ${}^3\mathbf{J}_{r,11} = {}^3\mathbf{P}_1\, {}^3\mathbf{J}_r = \begin{bmatrix} -0.5 & 0.125 \\ -0.866 & -0.217 \end{bmatrix}$. Applying this submatrix in ${}^3\dot{\mathbf{q}}_{tot\,r} = {}^3\mathbf{J}_{r,11}^{-1}\, {}^3\mathbf{P}_1 \mathbf{V}^*$ and equation (11) leads to ${}^3\Delta\dot{\mathbf{q}}_r = \begin{bmatrix} -0.807 & 2.870 \end{bmatrix}^{\mathrm{T}}$ and its

2-norm of 2.981, which recovers the first two components of the platform twist, and the 2-norm of the lost platform twist is 2.605. The second permutation submatrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ results in ${}^{3}\mathbf{J}_{r,11} = {}^{3}\mathbf{P}_1\, {}^{3}\mathbf{J}_r = \begin{bmatrix} -0.5 & 0.125 \\ 0 & 1 \end{bmatrix}$, ${}^{3}\Delta\dot{\mathbf{q}}_r = \begin{bmatrix} -1.459 & 0.264 \end{bmatrix}^{\mathrm{T}}$ and the 2-norm of the correctional joint velocity vector 1.482, which recovers the first and third components of the platform twist, and the 2-norm of the lost platform twist is 1.128. Also, ${}^{3}\mathbf{J}_{r,11} = \begin{bmatrix} -0.866 & -0.217 \\ 0 & 1 \end{bmatrix}$, ${}^{3}\Delta\dot{\mathbf{q}}_r = \begin{bmatrix} -0.156 & 0.264 \end{bmatrix}^{\mathrm{T}}$ and the 2-norm of correctional joint velocity vector using the third permutation submatrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 0.307. Among these results, the submatrix that produces the lowest 2-norm of the correctional joint velocity vector is chosen.

In Table 1, the results for the third permutation submatrix are shown and compared with failure recovery methodology in [4]. The Moore-Penrose generalized inverse of the Jacobian matrix is used in [4] to minimize the 2-norms of both correctional and overall joint velocity vectors, as well as the 2-norm of the error in the platform velocity vector.

As shown in Table 1, the 2-norm of the correctional joint velocity vector and the overall joint velocity vector using the third partitioned submatrix, ${}^{3}\mathbf{P}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the generalized inverse of the reduced form of ${}^{3}\mathbf{J}$ in equation (33), ${}^{3}\mathbf{J}_r^{1,4} = {}^{3}\mathbf{J}_{r,11}^{-1}\, {}^{3}\mathbf{P}_1 = \begin{bmatrix} 0 & -1.155 & -0.25 \\ 0 & 0 & 1 \end{bmatrix}$, is lower than that reported in [4]. Also, two components of the platform twist are recovered, and the 2-norm of the lost platform velocity in this example is larger than that in [4] because minimizing the least square solution to equation (2) is not considered in this paper.

The lost platform velocity could be fully recovered if and only if ${}^{3}\mathbf{P}_2\mathbf{V}^* = {}^{3}\mathbf{S}\, {}^{3}\mathbf{P}_1\mathbf{V}^*$, *i.e.*, the first component of $\mathbf{V}^*$ is a linear combination of the last two components of $\mathbf{V}^*$ when the third partitioned submatrix is used. In other words, the joint velocity calculated from ${}^{3}\dot{\mathbf{q}}_{tot\,r} = {}^{3}\mathbf{J}_{r,11}^{-1}\, {}^{3}\mathbf{P}_1\mathbf{V}^*$ can recover the first component of $\mathbf{V}^*$, if and only if ${}^{3}\dot{\mathbf{q}}_{tot\,r}$ satisfies ${}^{i}\mathbf{J}_{r,21}\, {}^{3}\dot{\mathbf{q}}_{tot\,r} = {}^{3}\mathbf{P}_2\mathbf{V}^*$, which is equivalent to ${}^{i}\mathbf{J}_{r,21}\, {}^{3}\mathbf{J}_{r,11}^{-1}\, {}^{3}\mathbf{P}_1\mathbf{V}^* = {}^{3}\mathbf{P}_2\mathbf{V}^*$ or ${}^{3}\mathbf{P}_2\mathbf{V}^* = {}^{3}\mathbf{S}\, {}^{3}\mathbf{P}_1\mathbf{V}^*$. For this example of having failure in the first three joints, ${}^{3}\mathbf{P}_2\mathbf{V}^* = 1$ and ${}^{3}\mathbf{S}\, {}^{3}\mathbf{P}_1\mathbf{V}^* = 0.349$, which are not equal. The full recovery of the platform twist could be achieved when the first component of the platform twist is 0.349.

Table 1

Example locked joint failure for leg 3 in the manipulator of Fig. 1

	Partitioned Jacobian matrix ${}^{3}\mathbf{J}_r$	Methodology in [4]
Correctional joint velocity of the healthy joints	$[-0.156 \quad 0.264]^{\mathrm{T}}$	$[-0.496 \quad 0.381]^{\mathrm{T}}$
2-norm of the correctional joint velocity	0.307	0.626
Overall velocity vector of the healthy joints	$[-0.637 \quad 0.24]^{\mathrm{T}}$	$[-0.978 \quad 0.357]^{\mathrm{T}}$
2-norm of the overall velocity vector	0.681	1.04
Lost platform velocity	$[0.651 \quad 0 \quad 0]^{\mathrm{T}}$	$[0.467 \quad -0.269 \quad -0.117]^{\mathrm{T}}$
2-norm of the lost platform velocity	0.651	0.551

4. DISCUSSION AND CONCLUSION

In this paper, the generalized inverse that meets the first, second, and fourth Penrose conditions was applied for motion recovery. It was shown that if the lost platform twist was in the range space of the Jacobian matrix associated with the failed leg, the generalized inverse satisfying the first, second, and fourth Penrose equations led to a unique solution, which was minimal. These formulations indicated that, when there was full recovery, *i.e.*, when the Jacobian matrix was of full row-rank, the generalized inverse satisfying the first, second and fourth Penrose equations led to the same results as [4], in which the Moore-Penrose generalized inverse (satisfying all four Penrose conditions) was used. The partitioned Jacobian matrix was applied if the full recovery of the platform motion was not achieved. It required checking all possible permutation matrices first and then selecting the matrix that resulted in the lowest correctional joint velocity.

Received on January 7, 2013

REFERENCES

1. NOTASH, Leila, HUANG, Li, *On the Design of Fault Tolerant Parallel Manipulators*, Mech. Mach. Theory, **38**, *1*, pp. 85-101, 2003.

2. JING, Zhao, CHENG, Fang, *On the Joint Velocity Jump during Fault Tolerant Operations for Manipulators with Multiple Degrees of Redundancy*, Mech. Mach. Theory, **44**, *6*, pp. 1201–1210, 2009.
3. NOTASH, Leila, *Motion Recovery after Joint Failure in Parallel Manipulators*, Spec. ed. Trans. Can. Soc. Mech. Eng., **35**, *4*, pp. 559-571, 2011.
4. NOTASH, Leila, *On the Twist Recovery Methodologies after Failure*, Latest Adv. in Rob. Kin. (editors: J. Lenarcic and M. Husty), Springer, 2012, pp. 11-18.
5. ABDI, Hamid, NAHAVANDI, Saeid, FRAYMAN, Yakov, MACIEJEWSKI, Anthony A., *Optimal Mapping of Joint Faults into Healthy Joint Velocity Space for Fault Tolerant Redundant Manipulators*, Robotica, **30**, *4*, pp. 635-648, 2012.
6. TING, Yung, TOSUNOGLU, Sabri, FREEMAN Robert, *Torque Redistribution and Time Regulation Methods for Actuator Saturation Avoidance of Fault-tolerant Parallel Robots*, Rob. Syst., **12**, *12*, pp. 807-820, 1995.
7. NOTASH, Leila, *A Methodology for Actuator Failure Recovery in Parallel Manipulators*, Mech. Mach. Theory, **46**, *4*, pp. 454-465, 2011.
8. NAKAMURA, Y., *Advanced Robotics: Redundancy and Optimization*, Addison Wesley, 1991.
9. BEN-ISRAEL, A., GREVILLE, T.N.E., *Generalized Inverses: Theory and Applications*, 2nd edition, Springer, New York, 2003.
10. BJERHAMMAR, Arne, *A Generalized Matrix Algebra*, Trans. Roy. Inst. Tech., **124**, Stockholm, 1958.
11. URQUHART, N. Scott, *Computation of Generalized Inverse Matrices which Satisfy Specified Conditions*, Soc. Indus. App. Math., **10**, *2*, pp. 216-218, 1968.
12. PENROSE, Roger, TODD, John Arthur, *On Best Approximate Solutions of Linear Matrix Equations*, Math. Proc. of the Cambridge Philo. Soc., **52**, *1*, pp. 17-19, 1956.