

# ON THE GEOMETRY OF THE HOMOGENEOUS REPRESENTATION FOR THE GROUP OF PROPER RIGID-BODY DISPLACEMENTS

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*Abstract.* This work investigates the geometry of the homogeneous representation of the group of proper rigid-body displacements. In particular it is shown that there is a birational transformation from the Study quadric to the variety determined by the homogeneous representation. This variety is shown to be the join of a Veronese variety with a 2-plane. The rest of the paper looks at sub-varieties, first those which are sub-groups of the displacement group and then some examples defined by geometric constraints. In many cases the varieties are familiar as sub-varieties of the Study quadric, here their transforms to the homogeneous representation is considered. A final section deals with the map which sends each displacements to its inverse. This is shown to be a quadratic birational transformation.

*Key words:* rigid-body displacements, birational transformations, kinematics.

## 1. INTRODUCTION

Many problems in kinematics reduce to the problem of solving systems of algebraic equations. This might be enumerative problems like counting the number of assembly modes of a parallel manipulator or the number of postures of a serial robot. It could also refer to the problem of determining the motion of the coupler bar in a one degree-of-freedom mechanism. In these applications the main object of study is the position and orientation of rigid bodies articulated by simple joints.

A succinct representation of the group of rigid-body displacements was found by Eduard Study in the 1890s. In modern terminology the elements of the group are in correspondence with the points in a six dimensional projective quadric in  $\mathbb{P}^7$ . This correspondence is 1-to-1, except on a certain 3-plane in the quadric. This 3-plane will be labelled  $A_\infty$  in the following and the quadric will be referred to as the Study quadric.

The problems referred to above can now be viewed as intersections of sub-varieties. These sub-varieties are typically spaces of rigid-body displacements which can be

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achieved by the end-effector of some particular mechanism or linkage.

Now it is usually difficult to find the intersection of such varieties since they often contain the 3-plane of unphysical displacements  $A_\infty$ . Sometimes they are singular on this plane. Other constraint varieties intersect  $A_\infty$  in an imaginary 2-sphere. Hence, when studying intersections which are expected to be of dimension less than 3,  $A_\infty$  will occur as a component of the intersection indicating that the intersection of these varieties is not a complete intersection.

A standard approach to such a difficulty in Algebraic Geometry might be to ‘blow-up’ the intersection along the sub-variety  $A_\infty$ . This is a rather technical procedure, see [5] and [8], which is likely to produce an affine variety in a very large ambient space.

A more classical approach is taken to the problem in this work. In older texts on Algebraic Geometry particular birational transformations were studied individually, their geometry and applications to interesting geometrical problems was considered. This is the spirit of the present work. A birational transformation of the Study quadric is considered. It has the property that it commutes with the action of the group of rigid displacements  $SE(3)$  on the domain and codomain of the transformation. In other words the transformation is equivariant. Moreover the transformation has  $A_\infty$  as its exceptional set, that is the transformation is undefined on this sub-variety.

Other birational maps of the Study quadric have been considered, especially in the field of Computer Aided Design, see [3]. In fact it is easy to see that any Cayley map will give a birational map between the group and its Lie algebra. The transformation considered here is a map to a non-linear variety.

We begin by briefly reviewing the definition of the Study quadric as a way of fixing notation.

## 2. THE STUDY QUADRIC

A dual quaternion has the form

$$h = q_0 + \varepsilon q_1,$$

where  $q_0$  and  $q_1$  are ordinary quaternions. That is

$$q_0 = a_0 + a_1i + a_2j + a_3k \quad \text{and} \quad q_1 = c_0 + c_1i + c_2j + c_3k.$$

The dual unit  $\varepsilon$  satisfies the relation  $\varepsilon^2 = 0$  and commutes with the quaternion units  $i$ ,  $j$  and  $k$ .

Dual quaternions of the form

$$g = r + \frac{1}{2}\varepsilon tr,$$

can be used to represent rigid displacement. Here  $r$  is a quaternion representing a rotation and  $t$  is a pure quaternion representing the translational part of the displacement, that is  $t = t_1i + t_2j + t_3k$ , with  $t_i$  the components of the translation vector. In this description points in space are represented by dual quaternions of the form,

$$\hat{p} = 1 + \varepsilon p,$$

where  $p$  is a pure quaternion as above. The action of a rigid displacement on a point is given by

$$\hat{p}' = (r + \frac{1}{2}\varepsilon tr)(\hat{p}(r^- + \frac{1}{2}\varepsilon r^- t)).$$

In this equation the superscript  $-$  denotes the dual quaternion conjugate; the linear map which sends  $i, j, k$  to  $-i, -j, -k$  respectively but leaves 1 and  $\varepsilon$  unchanged.

In more detail we have

$$\begin{aligned} \hat{p}' &= (r + \frac{1}{2}\varepsilon tr)(1 + \varepsilon p)(r^- + \frac{1}{2}\varepsilon r^- t) = \\ &= rr^- + \varepsilon(rpr^- + \frac{1}{2}rr^- t + \frac{1}{2}trr^-) = \\ &= 1 + \varepsilon(rpr^- + t). \end{aligned}$$

As with the pure rotations,  $g$  and  $-g$  represent the same rigid displacement, that is the set of these dual quaternions double cover the group of rigid-body displacements.

Notice that not all dual quaternions represent rigid displacements. In fact the condition for a dual quaternion  $g$ , to be a rigid displacement, is simply

$$gg^- = 1.$$

This is easily checked using the form  $g = r + (1/2)\varepsilon tr$  given above and remembering that the rotation  $r$  satisfies  $rr^- = 1$  and that the translation  $t$  is a pure quaternion  $t^- = -t$ . It is a little harder to see that all dual quaternions satisfying this equation are rigid displacements.

Writing a general dual quaternion as,

$$g = (a_0 + a_1i + a_2j + a_3k) + \varepsilon(c_0 + c_1i + c_2j + c_3k),$$

the equation above can be separated into its dual and quaternion parts,

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 1, \\ a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 &= 0. \end{aligned}$$

Now suppose that the eight variables  $(a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)$  are actually homogeneous coordinate for a 7-dimensional projective space  $\mathbb{P}^7$ . This has the effect of identifying  $g$  and  $-g$  so that points of this space correspond to elements of the group

of rigid transformations, not the double cover of the group. The first equation above is no longer applicable - it is not homogeneous. The second equation however is homogeneous and defines a six dimensional quadric in  $\mathbb{P}^7$ ,

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0.$$

This is the Study quadric. Every rigid displacement corresponds to a single point on the quadric. On the other hand some points on the quadric do not correspond to rigid displacements. The 3-plane of ‘ideal points’, the points satisfying  $a_0 = a_1 = a_2 = a_3 = 0$  do not correspond to any rigid displacement. This 3-plane is  $A_\infty$  as introduced above.

The group of rigid-body displacements,  $SE(3)$  acts on the Study quadric by conjugation,

$$g_1 \mapsto gg_1g^{-1},$$

for any dual quaternion  $g$  representing a displacement. Notice that under this action, the 3-plane  $A_\infty$  is mapped to itself.

There are two families of 3-planes which lie entirely within the Study quadric. The families are usually referred to as  $A$ -planes and  $B$ -planes. They satisfy linear equations of the form,

$$(I_4 - M) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} - (I_4 + M) \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0,$$

where  $M$  is an orthogonal  $4 \times 4$  matrix. The  $A$ -planes and  $B$  planes are distinguished by the sign of the determinant of  $M$ . If  $\det(M) = 1$  the 3-plane is an  $A$ -plane while if  $\det(M) = -1$  it is a  $B$ -plane.

In general an  $A$ -plane will meet a  $B$ -plane in a single point. Generally,  $A$ -planes do not meet other  $A$ -planes, but there are exceptions. For example, the  $A$ -plane of rotations about a point meets the  $A$ -plane of rigid displacements of a plane in a line; the line of rotations about the axis normal to the plane and passing through the point. Two  $B$ -planes do not generally meet, again with exceptions.

Within the family of  $A$ -planes we can distinguish two types: those that do not meet  $A_\infty$  and those that meet this 3-plane in a line – a third type is  $A_\infty$  itself, which clearly meets itself in a 3-plane. These properties of the  $A$ -planes are invariant under conjugations by rigid displacements. Moreover,  $A$ -planes which contain the identity element, the dual quaternion 1, are sub-groups of  $SE(3)$ . The sub-groups not meeting  $A_\infty$  are sub-groups of rotations about a fixed point, conjugate to  $SO(3)$ . The sub-groups which meet  $A_\infty$  in a line are planar sub-groups, conjugate to  $SE(2)$ . Further details

can be found in [10, §11.2.2 and 11.2.3].

### 3. THE HOMOGENEOUS REPRESENTATION

In Computer Aided Design and several other areas it is common to extend the standard representation of  $SE(3)$  to an action on the projective space  $\mathbb{P}^3$ , see for example, [7]. The homogeneous coordinates of  $\mathbb{P}^3$  are  $p_1 : p_2 : p_3 : p_4$  for example. Then the action of the group can be written as

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{R} & \tilde{\mathbf{t}} \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}.$$

The elements of the matrix  $\tilde{R}$  together with the components of  $\tilde{\mathbf{t}}$  and the element  $\Delta$  can now be thought of as homogeneous coordinates

in a 12-dimensional projective space  $\mathbb{P}^{12}$ . In the following sections an explicit birational map is given between this variety and the Study quadric.

#### 3.1. A BIRATIONAL MAP FROM THE STUDY QUADRIC

The mapping from the Study quadric to  $\mathbb{P}^{12}$  is given by

$$\Delta = a_0^2 + a_1^2 + a_2^2 + a_3^2, \quad (1)$$

the rotation matrix is

$$\tilde{R} = \Delta I_3 + 2a_0 A + 2A^2, \quad (2)$$

here the matrix  $A$  is the  $3 \times 3$  anti-symmetric matrix,

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

So in terms of coordinates, the rotation matrix can be written

$$\begin{pmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \tilde{r}_{13} \\ \tilde{r}_{21} & \tilde{r}_{22} & \tilde{r}_{23} \\ \tilde{r}_{31} & \tilde{r}_{32} & \tilde{r}_{33} \end{pmatrix} = \begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2(a_1 a_2 - a_0 a_3) & 2(a_1 a_3 + a_0 a_2) \\ 2(a_1 a_2 + a_0 a_3) & a_0^2 - a_1^2 + a_2^2 - a_3^2 & 2(a_2 a_3 - a_0 a_1) \\ 2(a_1 a_3 - a_0 a_2) & 2(a_2 a_3 + a_0 a_1) & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}. \quad (3)$$

The translational part of the mapping is given by

$$\tilde{\mathbf{t}} = 2(a_0\mathbf{c} - c_0\mathbf{a} + \mathbf{a} \times \mathbf{c}), \quad (4)$$

where  $\mathbf{a} = (a_1, a_2, a_3)^T$  and  $\mathbf{c} = (c_1, c_2, c_3)^T$ . Explicitly the mapping is given by

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \end{pmatrix} = 2 \begin{pmatrix} a_0c_1 - a_1c_0 + a_2c_3 - a_3c_2 \\ a_0c_2 - a_1c_3 - a_2c_0 + a_3c_1 \\ a_0c_3 + a_1c_2 - a_2c_1 - a_3c_0 \end{pmatrix}. \quad (5)$$

There are several properties worth noticing about this map. First of all it is easy to see that when  $a_0 = a_1 = a_2 = a_3 = 0$ , the map is undefined. Hence  $A_\infty$  lies in the exceptional set of the mapping. Moreover it is possible to show that this is the only circumstance where the map is undefined hence  $A_\infty$  is exactly the exceptional set for the mapping.

Now when  $c_0 = c_1 = c_2 = c_3 = 0$ , clearly  $\tilde{\mathbf{t}} = \mathbf{0}$  and the map reduces to the veronese map from  $\mathbb{P}^3 \rightarrow \mathbb{P}^9$ . To see this notice that all the degree two monomials in the  $a_i$ s can be written as linear functions in the elements of  $\tilde{R}$  and  $\Delta$ ,

$$\begin{aligned} a_0^2 &= \frac{1}{4}(\Delta + \tilde{r}_{11} + \tilde{r}_{22} + \tilde{r}_{33}), & a_0a_1 &= \frac{1}{4}(\tilde{r}_{32} - \tilde{r}_{23}), \\ a_0a_2 &= \frac{1}{4}(\tilde{r}_{13} - \tilde{r}_{31}), & a_0a_3 &= \frac{1}{4}(\tilde{r}_{21} - \tilde{r}_{12}), \\ a_1^2 &= \frac{1}{4}(\Delta + \tilde{r}_{11} - \tilde{r}_{22} - \tilde{r}_{33}), & a_1a_2 &= \frac{1}{4}(\tilde{r}_{12} + \tilde{r}_{21}), \\ a_1a_3 &= \frac{1}{4}(\tilde{r}_{13} + \tilde{r}_{31}), & a_2^2 &= \frac{1}{4}(\Delta - \tilde{r}_{11} + \tilde{r}_{22} - \tilde{r}_{33}), \\ a_2a_3 &= \frac{1}{4}(\tilde{r}_{23} + \tilde{r}_{32}), & a_3^2 &= \frac{1}{4}(\Delta - \tilde{r}_{11} - \tilde{r}_{22} + \tilde{r}_{33}). \end{aligned} \quad (6)$$

These linear functions are linearly independent, so the mapping on the 3-plane of pure rotations is given by the Veronese map, see [5]. This reference also shows that the Veronese variety, the image of this map, has degree 8.

Next consider the image of the 3-plane given by  $a_1 = a_2 = a_3 = c_0 = 0$ . The image of this 3-plane lies in the 3-plane given by  $4 \times 4$  matrices of the form

$$\begin{pmatrix} a_0I_3 & 2\mathbf{c} \\ 0 & a_0 \end{pmatrix}.$$

Within this 3-plane lies the 2-plane given by matrices of the form

$$\begin{pmatrix} 0 & 2\mathbf{c} \\ 0 & 0 \end{pmatrix}.$$

Strictly speaking this 2-plane does not lie in the image of the mapping since we have cancelled a factor of  $a_0$ , however it does lie in the closure of the mapping. This 2-plane will be referred to as  $T_\infty$  in the following. It is easy to see that  $T_\infty$  is disjoint from the  $\mathbb{P}^9$  containing the Veronese variety considered above. Moreover it is easy

to see that any  $4 \times 4$  matrix in the image of the map (or more exactly, in the closure of the map), lies on a line joining a point in the Veronese variety with a point in the 2-plane; that is a linear combination of a rotation matrix with a translation vector. Another way to say this

is that the image of the map is the join of the two varieties: the Veronese variety and the 2-plane. This image variety will be called  $\tilde{V}$  in what follows. According to Harris [5], the degree of such a variety is the product of the degrees of the varieties being joined. In this case the degree of the Veronese variety is 8 and the degree of the 2-plane is simply 1 so the image of the Study quadric;  $\tilde{V}$ , is a variety of degree  $8 \times 1 = 8$ .

### 3.2. IMPLICIT DEFINITION OF THE IMAGE VARIETY

The usual equations for defining the variety  $\tilde{V}$  would be

$$\tilde{R}\tilde{R}^T = \tilde{R}^T\tilde{R} = \Delta^2 I_3 \quad \text{and} \quad \det \tilde{R} = \Delta^3, \quad (7)$$

since the rotation matrix  $(1/\Delta)\tilde{R}$  is orthogonal with determinant  $+1$ . Note that the equations above comprise 13 homogeneous equations in 10 unknowns, 12 quadratic equations and one cubic.

On the other hand the standard implicit equations for the Veronese variety comprise 21 quadratic equations. These are given by the vanishing of the linearly independent  $2 \times 2$  sub-determinants of the matrix

$$\begin{pmatrix} a_0^2 & a_0a_1 & a_0a_2 & a_0a_3 \\ a_0a_1 & a_1^2 & a_1a_2 & a_1a_3 \\ a_0a_2 & a_1a_2 & a_2^2 & a_2a_3 \\ a_0a_3 & a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}.$$

For example, the top-left  $2 \times 2$  determinant gives the identity

$$\begin{vmatrix} a_0^2 & a_0a_1 \\ a_0a_1 & a_1^2 \end{vmatrix} = 0.$$

Expanding the determinant and substituting for the degree 2 monomials in the  $a_i$ s using (6) gives the quadratic

$$\tilde{r}_{11}^2 - \tilde{r}_{22}^2 - \tilde{r}_{23}^2 + 2\tilde{r}_{23}\tilde{r}_{32} - \tilde{r}_{32}^2 - 2\tilde{r}_{22}\tilde{r}_{33} - \tilde{r}_{33}^2 + 2\tilde{r}_{11}\Delta + \Delta^2 = 0.$$

The question arises: Do the two sets of equations determine the same ideal? The answer depends on the ground field. When working over the real numbers it is well known that the equations given in (7) define the special orthogonal group  $SO(3)$  and

hence we expect the answer to be: The varieties defined by the vanishing of these two systems of equations are the same so the ideals they generate must be the same or one is the radical of the other perhaps.

However, it is usually more convenient to work over the field of complex numbers so that Bézout's theorem and other results based on the idea of degree can be used. In this case, as above, it is easy to see that the image variety of the transformation lies in the variety defined by the equations in (7).

But there are also other components which satisfy the equations in (7). Suppose  $M$  is an element of the Lie group  $SO(2, 1)$ , that is  $M$  is a  $3 \times 3$  matrix with unit determinant which satisfies the equations

$$M^T J M = J \text{ and } M J M^T = J,$$

with

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The matrix

$$\tilde{R}' = \Delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix} M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix},$$

where  $i$  is the complex unit ( $i^2 = -1$ ), can easily be seen to satisfy the equations in (7). Since, the group  $SO(2, 1)$  has two connected components, we see that the complex variety defined in (7) has at least three components. Only the real component, which satisfies the equations for the Veronese variety, is the one we are interested in here.

### 3.3. THE INVERSE TRANSFORMATION

From equation (2) above the following can be found

$$\tilde{R} - \tilde{R}^T = 4a_0 A,$$

since  $A$  is anti-symmetric and hence  $A^2$  is symmetric. Also, since  $\Delta = a_0^2 + (\mathbf{a} \cdot \mathbf{a})$  and  $\text{Tr}(\tilde{R}) = 3\Delta - 4(\mathbf{a} \cdot \mathbf{a})$ , we have

$$\Delta + \text{Tr}(\tilde{R}) = 4a_0^2.$$

These relations essentially give the inverse to the Veronese map from  $\mathbb{P}^3$  to  $\mathbb{P}^9$ ; the homogeneous version of the standard representation of the rotation group  $SO(3)$ . To find the translational part of the inverse let us consider

$$((\Delta + \text{Tr}(\tilde{R}))I_3 - \tilde{R} + \tilde{R}^T)\tilde{\mathbf{t}} = (4a_0^2 I_3 - 4a_0 A)\tilde{\mathbf{t}}.$$



Using equation (4) above this can be written as

$$((\Delta + \text{Tr}(\tilde{R}))I_3 - \tilde{R} + \tilde{R}^T)\tilde{\mathbf{t}} = 8a_0^2(a_0\mathbf{c} - c_0\mathbf{a} + \mathbf{a} \times \mathbf{c}) - 8a_0(a_0\mathbf{a} \times \mathbf{c} + \mathbf{a} \times (\mathbf{a} \times \mathbf{c})),$$

the right-hand-side of the above can be simplified to give

$$((\Delta + \text{Tr}(\tilde{R}))I_3 - \tilde{R} + \tilde{R}^T)\tilde{\mathbf{t}} = 8a_0((a_0^2 + \mathbf{a} \cdot \mathbf{a})\mathbf{c} - (a_0c_0 + \mathbf{a} \cdot \mathbf{c})\mathbf{a}).$$

The coefficient of  $\mathbf{a}$  in the above is the relation satisfied by the Study quadric that is this quantity vanishes on the Study quadric. Also since  $\Delta = a_0^2 + \mathbf{a} \cdot \mathbf{a}$ , we get

$$((\Delta + \text{Tr}(\tilde{R}))I_3 - \tilde{R} + \tilde{R}^T)\tilde{\mathbf{t}} = 8a_0\Delta\mathbf{c}.$$

To find  $c_0$  recall that  $\tilde{R} - \tilde{R}^T = 4a_0A$  is an anti-symmetric  $3 \times 3$  matrix. Now write the 3-vector corresponding to the matrix  $\tilde{R} - \tilde{R}^T$  as  $\tilde{\mathbf{r}}$  so that  $\tilde{\mathbf{r}} = 4a_0\mathbf{a}$ . So consider the expression  $-\tilde{\mathbf{r}} \cdot \tilde{\mathbf{t}}$

$$-\tilde{\mathbf{r}} \cdot \tilde{\mathbf{t}} = -8a_0\mathbf{a} \cdot (a_0\mathbf{c} - c_0\mathbf{a} + \mathbf{a} \times \mathbf{c}),$$

this simplifies to

$$-\tilde{\mathbf{r}} \cdot \tilde{\mathbf{t}} = 8a_0\Delta c_0.$$

The inverse map can now be written explicitly as a

$$\begin{aligned} a_0 &= -2(\Delta + \tilde{r}_{11} + \tilde{r}_{22} + \tilde{r}_{33})\Delta \\ a_1 &= 2(\tilde{r}_{23} - \tilde{r}_{32})\Delta \\ a_2 &= 2(\tilde{r}_{31} - \tilde{r}_{13})\Delta \\ a_3 &= 2(\tilde{r}_{12} - \tilde{r}_{21})\Delta, \end{aligned} \tag{8}$$

$$\begin{aligned} c_0 &= (\tilde{r}_{23} - \tilde{r}_{32})\tilde{t}_1 + (\tilde{r}_{31} - \tilde{r}_{13})\tilde{t}_2 + (\tilde{r}_{12} - \tilde{r}_{21})\tilde{t}_3 \\ c_1 &= (\Delta + \tilde{r}_{11} + \tilde{r}_{22} + \tilde{r}_{33})\tilde{t}_1 - (\tilde{r}_{12} - \tilde{r}_{21})\tilde{t}_2 + (\tilde{r}_{31} - \tilde{r}_{13})\tilde{t}_3 \\ c_2 &= (\tilde{r}_{12} - \tilde{r}_{21})\tilde{t}_1 + (\Delta + \tilde{r}_{11} + \tilde{r}_{22} + \tilde{r}_{33})\tilde{t}_2 - (\tilde{r}_{23} - \tilde{r}_{32})\tilde{t}_3 \\ c_3 &= -(\tilde{r}_{31} - \tilde{r}_{13})\tilde{t}_1 + (\tilde{r}_{23} - \tilde{r}_{32})\tilde{t}_2 + (\Delta + \tilde{r}_{11} + \tilde{r}_{22} + \tilde{r}_{33})\tilde{t}_3. \end{aligned}$$

The exceptional set for this mapping consists of the intersection with a pair of 8 dimensional planes

$$\Delta = \tilde{t}_1 = \tilde{t}_2 = \tilde{t}_3 = 0,$$

and

$$(\Delta + \tilde{r}_{11} + \tilde{r}_{22} + \tilde{r}_{33}) = (\tilde{r}_{23} - \tilde{r}_{32}) = (\tilde{r}_{31} - \tilde{r}_{13}) = (\tilde{r}_{12} - \tilde{r}_{21}) = 0.$$

The intersection of the second 8-plane here with the group gives the set of displacements with rotation angle  $\pi$  radians but arbitrary translational part. It is also easy to see that this 8-plane contains the 2-plane  $T_\infty$  defined above.

## 4. TRANSFORMS OF SUB-GROUPS

In this section we look at sub-varieties in  $\tilde{V}$  defined by the images of some sub-groups of  $SE(3)$  and some related sub-varieties of the Study quadric.

### 4.1. ROTATIONS AND TRANSLATIONS

In the Study quadric sub-groups of rotations about a fixed axis or translation in a fixed direction are given by lines through the identity. To be specific, the group of rotations about the  $z$ -axis can be parametrised as a dual quaternion

$$r_z(c, s) = c + sk,$$

where  $c$  and  $s$  could be taken as homogeneous coordinates in a  $\mathbb{P}^1$ , a projective line, but could also be viewed as the cosine and sine of half the rotation angle.

Similarly the translations in the  $z$ -direction can be written

$$t_z = \mu + \varepsilon\lambda k,$$

again  $\lambda$  and  $\mu$  can be viewed as homogeneous parameters or  $\lambda/\mu$  gives half the translation distance.

Rotational and translational sub-groups are distinguished by how they meet  $A_\infty$ . Translation sub-groups are lines which meet  $A_\infty$ , in the above example, when  $\mu = 0$ . On the other hand, lines which represent rotations do not. Since  $A_\infty$  is also the set of exceptional points for the quadratic birational map we can see that rotational sub-groups will transform to conic curves in  $\tilde{V}$  while translational sub-groups will transform to lines. In general, since the degree of the transform is two, the transform of a degree  $n$  curve will have degree  $2n - m$ , where  $m$  is number of times the curve meets the exceptional set (properly counted). Indeed applying the map to the two examples above gives

$$\begin{pmatrix} c^2 - s^2 & -2sc & 0 & 0 \\ 2sc & c^2 - s^2 & 0 & 0 \\ 0 & 0 & c^2 + s^2 & 0 \\ 0 & 0 & 0 & c^2 + s^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 2\lambda \\ 0 & 0 & 0 & \mu \end{pmatrix}.$$

Notice that the conic representing the rotational sub-group meets the invariant set  $\Delta = 0$ ,  $\tilde{\mathbf{t}} = \mathbf{0}$  in two complex conjugate points, that is when  $s = \pm ic$ . On the other hand, the line representing the one-parameter sub-group of translations meets this 8-plane in a single point, when  $\mu = 0$ .

## 4.2. CYLINDER SUB-GROUPS

The cylinder sub-groups are isomorphic to  $SO(2) \times \mathbb{R}$ , the product of a rotation sub-group with a translation sub-group. They are the sub-groups which preserve a fixed line in space. For example the sub-group which preserves the  $z$ -axis is the product of rotations about this axis with the sub-group of translations parallel to the axis

$$\text{Cyl}_z(c, s; \mu, \lambda) = (c + sk)(\mu + \varepsilon \lambda k) = (\mu c + \mu sk) + \varepsilon(-\lambda s + \lambda ck).$$

This can easily be seen to be a parametrisation of the Segre variety  $\mathbb{P}^1 \times \mathbb{P}^1$  which is well known to be isomorphic to the non-singular quadric surface in  $\mathbb{P}^3$ . This surface meets the 3-plane  $A_\infty$  in a line, when  $\mu = 0$ .

Transforming to  $\tilde{V}$  we get the parametrised matrix

$$\begin{pmatrix} \mu(c^2 - s^2) & -2\mu sc & 0 & 0 \\ 2\mu sc & \mu(c^2 - s^2) & 0 & 0 \\ 0 & 0 & \mu(c^2 + s^2) & 2\lambda(c^2 + s^2) \\ 0 & 0 & 0 & \mu(c^2 + s^2) \end{pmatrix}.$$

This variety in  $\mathbb{P}^{12}$  lies on a 3-plane given by the 9 linear equations;  $\tilde{r}_{13} = \tilde{r}_{23} = \tilde{r}_{31} = \tilde{r}_{32} = \tilde{t}_1 = \tilde{t}_2 = 0$ ,  $\tilde{r}_{11} = \tilde{r}_{22}$ ,  $\tilde{r}_{12} = -\tilde{r}_{21}$  and  $\tilde{r}_{33} = \Delta$ . The parametrisation also clearly satisfies the quadratic equation

$$(\tilde{r}_{11} + \Delta)(\tilde{r}_{11} - \Delta) = \tilde{r}_{12}\tilde{r}_{21},$$

so we can conclude that these cylinder sub-groups are 2-dimensional quadrics in  $\tilde{V}$ .

Notice that the 2-dimensional quadric meets the 2-plane  $T_\infty$ , defined by the linear equations  $\tilde{R} = 0$ ,  $\Delta = 0$  in a single point, given parametrically by  $\mu = 0$ . This point is clearly the unique singular point in the quadric. Hence, this quadric is a cone with vertex in  $T_\infty$ .

## 4.3. A-PLANES AND B-PLANES

As usual we can find the characteristics of these sub-varieties by looking at standard examples and then appeal to the action of the group to ensure that these properties hold for general cases. So we begin by examining the case of the  $A$ -plane of rotations about the origin. In the Study quadric this  $A$ -plane is given by the intersection of the 4 hyperplanes  $c_0 = c_1 = c_2 = c_3 = 0$ . The image of this  $A$ -plane in the variety  $\tilde{V}$  is simply the Veronese variety given as the intersection of  $\tilde{V}$  with the 9-plane  $\tilde{t}_1 = \tilde{t}_2 = \tilde{t}_3 = 0$ .

The  $A$ -plane representing planar displacements parallel to the  $xy$ -plane is given in the Study quadric by the intersection of 4 hyperplanes,  $a_1 = a_2 = c_0 = c_3 = 0$ . In  $\tilde{V}$  this sub-group will be given by matrices of the form

$$\begin{pmatrix} \gamma(c^2 - s^2) & -2\delta sc & 0 & \alpha(c^2 + s^2) \\ 2\gamma sc & \gamma(c^2 - s^2) & 0 & \beta(c^2 + s^2) \\ 0 & 0 & \gamma(c^2 + s^2) & 0 \\ 0 & 0 & 0 & \gamma(c^2 + s^2) \end{pmatrix}.$$

These matrices clearly lie on 8 hyperplanes,  $\tilde{r}_{13} = \tilde{r}_{23} = \tilde{r}_{31} = \tilde{r}_{32} = \tilde{t}_3 = 0$ ,  $\tilde{r}_{11} = \tilde{r}_{22}$ ,  $\tilde{r}_{12} = -\tilde{r}_{21}$  and  $\tilde{r}_{33} = \Delta$ ; together with the quadric hyper-surface;  $(\tilde{r}_{11} + \Delta)(\tilde{r}_{11} - \Delta) = \tilde{r}_{12}\tilde{r}_{21}$ . When  $\gamma = 0$  this 3-dimensional quadric meets the 2-plane  $T_\infty$  in a line.

Next we turn to the  $B$ -planes. There is a single  $B$ -plane which meets  $A_\infty$  in a 2-plane. This is the sub-group of all translations given by  $a_1 = a_2 = a_3 = c_0 = 0$ . In  $\tilde{V}$  this corresponds to the 3-plane,  $\tilde{r}_{12} = \tilde{r}_{23} = \tilde{r}_{31} = \tilde{r}_{21} = \tilde{r}_{13} = \tilde{r}_{32} = 0$ ,  $\tilde{r}_{11} = \tilde{r}_{22} = \tilde{r}_{33} = \Delta$ . This 3-plane contains  $T_\infty$ .

Finally, all other  $B$ -planes will meet  $A_\infty$  in a single point. These can be translated to coincide with the space of all rotations about lines in the  $xy$ -plane. This plane is given by  $a_3 = c_0 = c_1 = c_2 = 0$ . In Section 3.1, the transformation from the Study quadric, restricted to the 3-plane  $c_0 = c_1 = c_2 = c_3 = 0$ , was found to be the Veronese map from  $\mathbb{P}^3$  to  $\mathbb{P}^9$ . Here, where  $a_3 = 0$ , this map is clearly restricted to a 2-plane and the transformation is the Veronese map from  $\mathbb{P}^2$  to  $\mathbb{P}^5$ . The image of this map is usually known as the Veronese surface and is well known to be an irreducible projective variety of degree 4 (see [5]). Removing the restriction on the  $c_i$ s, it is easy to see that the closure of the image of this  $B$ -plane must consist of a single point in  $T_\infty$ , since on the  $B$ -plane  $c_0 = c_1 = c_2 = 0$ . Hence the image of a  $B$ -plane in  $\tilde{V}$  must be a cone over the Veronese surface. By the remarks in Section 3.1 this cone will be a degree 4 variety whose vertex lies on  $T_\infty$ .

#### 4.4. SCHOENFLIES SUB-GROUPS

A Schoenflies sub-group is generated by all translations and the rotations about a single axis. Such sub-groups have sometimes been described as the possible motions of a waiter's tray. The waiter can translate the tray in any direction and may rotate it about any vertical axis but may not tip the tray; a rotation about a horizontal axis. In the Study quadric these 4-dimensional sub-groups lie in the intersection of the Study quadric with a 5-plane. Such 5-planes contain the 3-plane of unphysical elements  $A_\infty$ .

For the homogeneous representation the elements of a Schoenflies sub-group can be parametrised as an arbitrary translation followed by a rotation, for a Schoenflies

sub-group of displacements parallel to the  $xy$ -plane this would be

$$\begin{pmatrix} \delta(c^2 - s^2) & -2\delta sc & 0 & \alpha(c^2 + s^2) \\ 2\delta sc & \delta(c^2 - s^2) & 0 & \beta(c^2 + s^2) \\ 0 & 0 & \delta(c^2 + s^2) & \gamma(c^2 + s^2) \\ 0 & 0 & 0 & \delta(c^2 + s^2) \end{pmatrix}.$$

These displacements clearly lie on the variety determined by the 7 hyperplanes,  $\tilde{r}_{13} = \tilde{r}_{23} = \tilde{r}_{31} = \tilde{r}_{32} = 0$ ,  $\tilde{r}_{11} = \tilde{r}_{22}$ ,  $\tilde{r}_{12} = -\tilde{r}_{21}$  and  $\tilde{r}_{33} = \Delta$ ; together with the quadric hypersurface;  $(\tilde{r}_{11} + \Delta)(\tilde{r}_{11} - \Delta) = \tilde{r}_{12}\tilde{r}_{21}$ . These sub-groups are therefore represented in the homogeneous representation by 4-dimensional quadrics, singular over the 2-plane  $T_\infty$ .

## 5. SOME DISPLACEMENT SUB-VARIETIES

Next we look at some more sub-varieties, this time defined by geometric problems. It may well have been Study who first looked at these types of sub-variety.

### 5.1. POINT-PLANE CONSTRAINTS

In [11] it was shown that the set of rigid displacements which maintain the incidence of an arbitrary point on a given plane, lie in the intersection of the Study quadric with another 6-dimensional quadric in  $\mathbb{P}^7$ . The point-plane constraint varieties all contain the 3-plane  $A_\infty$ . See also [14] and [16] for recent work in this area.

Rather than substitute the definition of the inverse mapping into the equation for the point-plane constraint, it will involve less computation to derive the equations determining this variety afresh.

Let the plane be denoted by a 4-dimensional vector

$$\begin{pmatrix} \mathbf{n} \\ -d \end{pmatrix},$$

where  $\mathbf{n}$  is the unit 3-vector in the direction of the normal to the plane and  $d$  is the perpendicular distance to the origin. Now the incidence relation between the point and the plane is given by

$$(\mathbf{n}^T, -d) \begin{pmatrix} \mathbf{p} \\ p_4 \end{pmatrix} = 0,$$

so the possible displacements of the point which preserve incidence will satisfy,

$$(\mathbf{n}^T, -d) \begin{pmatrix} \tilde{R} & \tilde{\mathbf{t}} \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ p_4 \end{pmatrix} = 0. \quad (9)$$

Notice that this equation is linear in the coordinate  $\tilde{r}_{ij}, \tilde{t}_i$  and  $\Delta$ , hence the equation represents a hyperplane in  $\mathbb{P}^{12}$ . Intersecting 6 such hyperplanes with the degree eight group variety  $\tilde{V}$  will give 8 solutions in general. This result was derived in a much more complicated way in [11].

## 5.2. POINT-SPHERE CONSTRAINTS

The observation that point-sphere constraints were quadrics in the Study quadric was a key ingredient in Husty's solution to the forward kinematics of the Stewart-Gough platform, [6].

Again, rather than substitute the inverse map found above into quadratic equation for a point-sphere constraint found in [11] it is simpler and more informative to re-derive the required equation.

A point with 3-dimensional position vector  $\mathbf{p}$  lies on a sphere with centre  $\mathbf{c}$  and radius  $r$  if and only if

$$(\mathbf{p} - \mathbf{c})^2 = r^2.$$

This can be written as the scalar product of a pair of 5-dimensional vectors

$$(-2\mathbf{c}^T, |\mathbf{c}|^2 - r^2, 1) \begin{pmatrix} \mathbf{p} \\ 1 \\ |\mathbf{p}|^2 \end{pmatrix} = 0.$$

Notice here that the coordinates of the point and the sphere have been separated into different vectors.

The action of the group of rigid displacements on the 5-vectors representing points can be written in partitioned form as

$$\begin{pmatrix} \mathbf{p} \\ 1 \\ |\mathbf{p}|^2 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{R} & \tilde{\mathbf{t}} & 0 \\ 0 & 1 & 0 \\ 2\tilde{\mathbf{t}}^T \tilde{R} & |\tilde{\mathbf{t}}|^2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \\ |\mathbf{p}|^2 \end{pmatrix}.$$

Note that this 5-dimensional representation of  $SE(3)$  appears in [4].

Think of the components of the 5-vectors as homogeneous coordinates in a  $\mathbb{P}^4$ . A general point in this projective space will have coordinates,  $(\mathbf{p}^T, \lambda, \mu)^T$ . The points in this  $\mathbb{P}^4$  corresponding to points in space satisfy the equation

$$\mathbf{p}^T \mathbf{p} - \lambda \mu = 0.$$

The set of zeros for this equation form a 3-dimensional sphere, often called the horosphere. There is a 1-to-1 correspondence between points of this sphere and the points in space, with the exception of a single point on the sphere  $(\mathbf{0}^T, 0, 1)^T$ . This point is

usually called the point at infinity. The horosphere is the 1 point compactification of  $\mathbb{R}^3$ . Contrast this with the compactification of  $\mathbb{R}^3$  represented by  $\mathbb{P}^3$ , where there is a whole projective plane at infinity.

It is possible to extend the 5-dimensional representation of  $SE(3)$  to a homogeneous representation on  $\mathbb{P}^4$ . This representation preserves the horosphere

$$\begin{pmatrix} \mathbf{p} \\ \lambda \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} \Delta \tilde{R} & \Delta \tilde{\mathbf{t}} & 0 \\ 0 & \Delta^2 & 0 \\ 2\tilde{\mathbf{t}}^T \tilde{R} & |\tilde{\mathbf{t}}|^2 & \Delta^2 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \lambda \\ \mu \end{pmatrix}.$$

Hence the equation for displacements which move the point in such a way that it remains on the sphere can be written

$$(-2\mathbf{c}^T, |\mathbf{c}|^2 - r^2, 1) \begin{pmatrix} \Delta \tilde{R} & \Delta \tilde{\mathbf{t}} & 0 \\ 0 & \Delta^2 & 0 \\ 2\tilde{\mathbf{t}}^T \tilde{R} & |\tilde{\mathbf{t}}|^2 & \Delta^2 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \\ |\mathbf{p}|^2 \end{pmatrix} = 0.$$

In this equation it is clear that the coordinates  $\tilde{r}_{ij}$ ,  $\tilde{t}_i$  and  $\Delta$  of  $\mathbb{P}^{12}$  appear with degree 2, hence this equation represents a quadric hyper-surface in  $\mathbb{P}^{12}$ . Notice also that this equation will be satisfied identically if we set  $\tilde{\mathbf{t}} = \mathbf{0}$  and  $\Delta = 0$ . This represents an 8-plane in  $\mathbb{P}^{12}$  and, as seen in Section 3.3 above, the intersection of this 8-plane with the group variety is the exceptional set of the map to the Study quadric. Notice that this implies that the quadric is singular, since a smooth 11-dimensional quadric cannot contain an 8-dimensional plane.

The intersection of 4 or more of these point-sphere varieties will all contain this 8-plane as a component. Hence, it is, unfortunately, not possible to use Bézout's theorem to count the number of intersection of 6 such varieties.

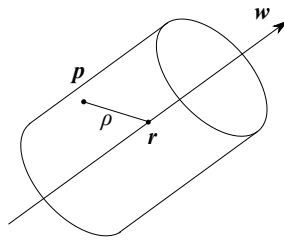


Fig. 1 – A point on a cylinder.

### 5.3. THE CS DYAD

The variety of displacements generated by a cylindrical and a spherical joint connected in series can also be thought of as the displacements that move a point, the

centre of the spherical joint, in such a way that it remains on a right circular cylinder. In [13] it was shown that this variety is the intersection of the Study quadric with a quartic hyper-surface in  $\mathbb{P}^7$ . Moreover the variety contains the 3-plane  $A_\infty$  and is singular on  $A_\infty$ .

Again this example can be treated in a similar fashion to the previous ones. Consider a line with Plücker coordinates  $(\boldsymbol{\omega}^T, \mathbf{v}^T)$ , where  $\boldsymbol{\omega}$  is the direction of the line and  $\mathbf{v}$  its moment. For lines these Plücker coordinates satisfy  $\boldsymbol{\omega} \cdot \mathbf{v} = 0$ , which can be ensured by writing the moment as  $\mathbf{v} = \mathbf{q} \times \boldsymbol{\omega}$ , for some point  $\mathbf{q}$  on the line.

Given a point in space  $\mathbf{p}$  the perpendicular distance between the line and this point will be the length of a vector  $\mathbf{p} - \mathbf{r}$ , where  $\mathbf{r}$  is the point on the line at the foot of the perpendicular, see Figure 1. Taking the vector product of the perpendicular with  $\boldsymbol{\omega}$  gives

$$(\mathbf{p} - \mathbf{r}) \times \boldsymbol{\omega} = \mathbf{p} \times \boldsymbol{\omega} - \mathbf{v} = \rho |\boldsymbol{\omega}| \mathbf{e},$$

where  $\rho$  is the perpendicular distance from the point to the line and  $\mathbf{e}$  is a unit vector perpendicular to both  $\boldsymbol{\omega}$  and the perpendicular vector. Taking the scalar product of this equation with itself gives the equation of a cylinder with the line as axis

$$(\mathbf{p} \times \boldsymbol{\omega} - \mathbf{v})^T (\mathbf{p} \times \boldsymbol{\omega} - \mathbf{v}) = \rho^2 |\boldsymbol{\omega}|^2.$$

This equation can be rearranged into the following matrix equation

$$(\mathbf{p}^T, 1) \begin{pmatrix} \Omega^2 & (\boldsymbol{\omega} \times \mathbf{v}) \\ (\boldsymbol{\omega} \times \mathbf{v})^T & \rho^2 |\boldsymbol{\omega}|^2 - |\mathbf{v}|^2 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = 0.$$

Here  $\Omega$  is the  $3 \times 3$  anti-symmetric matrix representing  $\boldsymbol{\omega}$ , so that  $\Omega \mathbf{p} = \boldsymbol{\omega} \times \mathbf{p}$  and  $\Omega^2 \mathbf{p} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p})$ .

The homogeneous equation for rigid displacements, which preserve the incidence between the point and the cylinder, can now be seen to be simply

$$(\mathbf{p}^T, p_4) \begin{pmatrix} \tilde{R} & \tilde{\mathbf{t}} \\ 0 & \Delta \end{pmatrix}^T \begin{pmatrix} \Omega^2 & (\boldsymbol{\omega} \times \mathbf{v}) \\ (\boldsymbol{\omega} \times \mathbf{v})^T & \rho^2 |\boldsymbol{\omega}|^2 - |\mathbf{v}|^2 \end{pmatrix} \begin{pmatrix} \tilde{R} & \tilde{\mathbf{t}} \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ p_4 \end{pmatrix} = 0.$$

This is clearly quadratic in the coordinates  $\tilde{r}_{ij}$ ,  $\tilde{t}_i$  and  $\Delta$ .

#### 5.4. LINE IN A COMPLEX

In this section we look at the set of group elements which move a line in such a way that it remains in a given line complex. This problem appears in Blaschke [1].

A line complex is a set of lines whose Plücker coordinates satisfy a homogeneous linear equation

$$\mathbf{m} \cdot \boldsymbol{\omega} + \mathbf{f} \cdot \mathbf{v} = 0,$$



where  $\mathbf{m}$  and  $\mathbf{f}$  are vectors of coefficients. This equation is easier to handle in a partitioned vector form

$$\mathscr{W}^T \mathbf{s} = (\mathbf{m}^T, \mathbf{f}^T) \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0.$$

The 6-vector  $\mathscr{W}$ , which represents the coefficients in the equation for the complex, can be thought of as a wrench. The 6-vector  $\mathbf{s}$ , of Plücker coordinates can be thought of as a twist, see [10] for more details.

The group of rigid-body displacements acts on lines, and twists via the adjoint representation of the group. This representation can be extended to a representation of  $\tilde{V}$  as follows

$$\begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} \longrightarrow \begin{pmatrix} \Delta \tilde{R} & 0 \\ \tilde{T} \tilde{R} & \Delta \tilde{R} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}.$$

Here  $\tilde{R}$  is the un-normalised rotation matrix as above and  $\tilde{T}$  is the  $3 \times 3$  anti-symmetric matrix corresponding to the 3-vector  $\tilde{\mathbf{t}}$ .

So given a line  $\mathbf{s}^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$  lying in a complex defined by a wrench  $\mathscr{W}^T = (\mathbf{m}^T, \mathbf{f}^T)$ , the rigid displacements which move the line in such a way that it remains in the complex will satisfy the equation

$$(\mathbf{m}^T, \mathbf{f}^T) \begin{pmatrix} \Delta \tilde{R} & 0 \\ \tilde{T} \tilde{R} & \Delta \tilde{R} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0. \quad (10)$$

The equation clearly has degree 2 in  $\Delta$  and the components of  $\tilde{R}$  and  $\tilde{T}$  and hence defines a quadric hyper-surface in  $\mathbb{P}^{12}$ . The intersection of this quadric with  $\tilde{V}$  is a 5-dimensional sub-variety of the group. It is also clear from this representation, that, given any line and corresponding wrench, the quadric will contain the 8-plane given by  $\Delta = 0$ ,  $\tilde{\mathbf{t}} = \mathbf{0}$  and hence must be a singular quadric. These singular quadrics will also contain the 3-plane given by  $\tilde{R} = 0$ .

Finally here, we investigate how this sub-variety transforms to the Study quadric. In most of the examples in this work the sub-varieties studied are well known as sub-varieties of the Study quadric. Here we go the other way and map this sub-variety to the  $\mathbb{P}^7$  in which the Study quadric lies.

To find the equation of this sub-variety in  $\mathbb{P}^7$  we use equations (2) and (4) to substitute into equation (10) above. In particular, we look at the bottom left-hand corner of the matrix

$$\tilde{T} \tilde{R} = \Delta \tilde{T} + 2a_0 \tilde{T} A + 2\tilde{T} A^2.$$

Now to use (4) to substitute for  $\tilde{T}$  we note that the  $3 \times 3$  anti-symmetric matrix corresponding to the vector  $\mathbf{a} \times \mathbf{c}$  can be written  $AC - CA$  where as usual,  $A$  and  $C$  are the anti-symmetric matrices corresponding to  $\mathbf{a}$  and  $\mathbf{c}$  respectively. Simplifying the above expression, using the fact that  $\Delta = a_0^2 + \mathbf{a} \cdot \mathbf{a}$ , gives

$$\tilde{T} \tilde{R} = 2\Delta(a_0 C + c_0 A + AC + CA) - 4a_0(a_0 c_0 + \mathbf{a} \cdot \mathbf{c})A - 4(a_0 c_0 + \mathbf{a} \cdot \mathbf{c})A^2.$$

The Study quadric itself is given by  $a_0c_0 + \mathbf{a} \cdot \mathbf{c} = 0$ , hence we can delete the last two terms here. Also, all blocks of the matrix in (10) now contain a factor of  $\Delta$ , which can be canceled as we are working in homogeneous coordinates. The result is

$$(\mathbf{m}^T, \mathbf{f}^T) \begin{pmatrix} \Delta I_3 + 2a_0A + 2A^2 & 0 \\ 2(a_0C + c_0A + AC + CA) & \Delta I_3 + 2a_0A + 2A^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0.$$

Since this is clearly a quadratic equation in the homogeneous coordinates of  $\mathbb{P}^7$ , the sub-variety is given by the intersection of the Study quadric with the quadric represented by the above equation.

A general quadric in  $\mathbb{P}^7$  is usually written as a symmetric  $8 \times 8$  matrix, that is any homogeneous quadratic equation can be put in the form

$$\mathbf{g}^T Q \mathbf{g} = 0,$$

where  $\mathbf{g} = (a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)^T$  is the column vector of homogeneous coordinates. The equation above, for the quadric can be rearranged into this form, the result is most easily stated using partitioned forms of the matrix

$$Q = \begin{pmatrix} \Xi & \Upsilon \\ \Upsilon & 0 \end{pmatrix},$$

where the two  $4 \times 4$  symmetric matrices  $\Xi$  and  $\Upsilon$  are in turn given in a partitioned form

$$\Xi = \begin{pmatrix} 0 & (\boldsymbol{\omega} \times \mathbf{m} + \mathbf{v} \times \mathbf{f})^T \\ (\boldsymbol{\omega} \times \mathbf{m} + \mathbf{v} \times \mathbf{f}) & M\Omega + \Omega M + FV + VF \end{pmatrix},$$

and

$$\Upsilon = \begin{pmatrix} 0 & (\boldsymbol{\omega} \times \mathbf{f})^T \\ (\boldsymbol{\omega} \times \mathbf{f}) & \Omega F + F\Omega \end{pmatrix}.$$

To simplify these equations it has been assumed that the line initially lies in the complex, that is that  $\mathbf{m} \cdot \boldsymbol{\omega} + \mathbf{f} \cdot \mathbf{v} = 0$ . As usual,  $M$ ,  $\Omega$  and so forth, represent the  $3 \times 3$  anti-symmetric matrices corresponding to  $\mathbf{m}$ ,  $\boldsymbol{\omega}$  and so on.

Notice that from this presentation of the quadric, it is easy to see that the quadric contains  $A_\infty$ , the  $A$ -plane of unphysical displacements. In this respect, these quadrics are very similar to the quadric representing point-plane constraints, hence we might hope to treat intersections of several of these constraints in a similar fashion to the way several point-plane constraints were analysed in [11].

## 5.5. CAMERA CALIBRATION

A common problem in robot vision is to find the rigid body displacement between

the tool frame of the robot and the coordinate frame used by a camera attached to the robot's end-effector. Some information can be found by subjecting the robot's end-effector to a known displacement – by moving the robot, and then analysing the change of image captured by the camera to compute the displacement in the camera's coordinate frame, see for example [9].

The equation to be solved is usually given as

$$AX = XB,$$

where  $A, B$  are known rigid displacements, written as  $4 \times 4$  homogeneous matrices and  $X$  is an unknown rigid displacement to be found. To avoid confusion with the notation in the rest of this work, the equation to be solved will be written as

$$M_a X = X M_b, \quad (11)$$

where now  $M_a$  and  $M_b$  are the known rigid displacements.

Rearranging the equation to

$$M_a = X M_b X^{-1}$$

shows that, for consistency,  $M_a$  and  $M_b$  must be conjugate displacements. This implies that the pitch and angles of  $M_a$  and  $M_b$  must be the same. Hence, all we can infer from this data is that the displacement must move the axis of  $M_b$ , call it  $\mathbf{s}_b = (\boldsymbol{\omega}_b^T, (\mathbf{p}_b \times \boldsymbol{\omega}_b)^T)^T$ , to the axis of  $M_a$ ,  $\mathbf{s}_a = (\boldsymbol{\omega}_a^T, (\mathbf{p}_a \times \boldsymbol{\omega}_a)^T)^T$ . Here  $\mathbf{p}_a$  and  $\mathbf{p}_b$  are arbitrary points on the axes of  $M_a$  and  $M_b$  respectively. Using the adjoint representation of the group this gives the equation

$$\begin{pmatrix} \boldsymbol{\omega}_a \\ \mathbf{p}_a \times \boldsymbol{\omega}_a \end{pmatrix} = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega}_b \\ \mathbf{p}_b \times \boldsymbol{\omega}_b \end{pmatrix},$$

where  $R$  is the rotation matrix of  $X$  and  $T$  is the anti-symmetric form of the translation vector. Looking at the rotational part of this equation gives

$$R\boldsymbol{\omega}_b = \boldsymbol{\omega}_a,$$

as a homogeneous equation this can be written

$$\tilde{R}\boldsymbol{\omega}_b - \Delta\boldsymbol{\omega}_a = 0.$$

This represents three linear equations in  $\mathbb{P}^{12}$ . Looking at the translational equations gives

$$\mathbf{t} \times R\boldsymbol{\omega}_b + R(\mathbf{p}_b \times \boldsymbol{\omega}_b) = \mathbf{p}_a \times \boldsymbol{\omega}_a.$$

This can be rearranged using the rotational equation to give the homogeneous equa-

tion

$$(\tilde{\mathbf{t}} + \tilde{\mathbf{R}}\mathbf{p}_b - \Delta\mathbf{p}_a) \times \boldsymbol{\omega}_a = 0.$$

It is not too difficult to parametrise the solutions to these linear equations, however this problem has been essentially solved above in Section 4.2. The displacements sought are simply the set of displacements which move one axis to the other. This set can be thought of as the set of symmetries of the first line composed with any displacement which moves the line to its final position. That is, the set can be parametrised as the left-translation of the quadric surface found in Section 4.2 above. Left translation is a linear map of  $\tilde{\mathcal{V}}$  and hence we may conclude that the set of displacements satisfying equation (11) above is still a quadric cone. The vertex of the cone will remain on the 2-plane  $T_\infty$  since this plane is clearly invariant with respect to left translations.

Of course the above does not solve the original calibration problem. In reality measurements from real systems will contain errors and are thus unlikely to satisfy the consistency conditions: the measured values for  $M_a$  and  $M_b$  are unlikely to have exactly the same pitches and rotation angles. A statistical approach is required to solve this problem but it is not clear how best to do this at present. However, it is hoped that this geometry will inform the process, it is expected that any solution procedure will respect the group structure of the problem.

## 6. INVERSE OF A DISPLACEMENT AS A BIRATIONAL TRANSFORM

In the Study quadric the inverse of a group element is given by its quaternion conjugate; a linear map. In the homogeneous representation things are a little more complicated. Classically this has led to a distinction being made between a rigid-body motion and its inverse motion. For example the distinction between the Darboux and Mannheim motions as discussed in [2]. Below we look at the geometry of this situation.

The map which sends a group element to its inverse is a quadratic transformation of  $\tilde{\mathcal{V}}$ . It is given by

$$\text{inv} : \begin{pmatrix} \tilde{\mathbf{R}} & \tilde{\mathbf{t}} \\ 0 & \Delta \end{pmatrix} \longmapsto \begin{pmatrix} \Delta\tilde{\mathbf{R}}^T & -\tilde{\mathbf{R}}^T\tilde{\mathbf{t}} \\ 0 & \Delta^2 \end{pmatrix}.$$

A little computation is needed to show it, but, as one should expect, this transformation is self-inverse. Hence, this is a birational transformation of  $\tilde{\mathcal{V}}$  to itself.

The exceptional set of the transformation is given by the intersection of  $\tilde{\mathcal{V}}$  with the two linear spaces  $T_\infty$  the 2-plane given by,  $\Delta = 0$ ,  $\tilde{\mathbf{R}} = 0$ , and the 8-plane given by  $\Delta = 0$ ,  $\tilde{\mathbf{t}} = \mathbf{0}$ .

It is possible to find the inverses of some of the sub-varieties found above. Classically, these were thought of as, for example, the possible displacements of a plane

which keep it incident on a given point. As opposed to a point-plane constraint, the sub-variety of displacements of a point under which it remains on a given plane.

To find the inverse of a point-plane constraint consider equation (9) above, replacing the  $4 \times 4$  displacement matrix by its inverse gives

$$(\mathbf{n}^T, -d) \begin{pmatrix} \Delta \tilde{R}^T & -\tilde{R}^T \tilde{\mathbf{t}} \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ p_4 \end{pmatrix} = 0.$$

This matrix is clearly quadratic in the coordinates of  $\mathbb{P}^{12}$  and hence this represents a quadric hyper-surface. Recall that the original point-plane constraint was represented by a hyperplane in  $\mathbb{P}^{12}$ .

To find the inverses of other sub-varieties we need to know how to transform other combinations of  $\tilde{R}$  and  $\tilde{\mathbf{t}}$ . For example, it is easy to see that

$$\tilde{R}^T \tilde{\mathbf{t}} \mapsto (\Delta \tilde{R})(-\tilde{R}^T \tilde{\mathbf{t}}) = -\Delta^3 \tilde{\mathbf{t}},$$

since  $\tilde{R}\tilde{R}^T = \Delta^2 I_3$ , from above. With results such as this it can be seen that the inverse of the point-sphere constraint is given by

$$(-2\mathbf{c}^T, |\mathbf{c}|^2 - r^2, 1) \begin{pmatrix} \Delta \tilde{R}^T & -\tilde{R}^T \tilde{\mathbf{t}} & 0 \\ 0 & \Delta^2 & 0 \\ 2\Delta \tilde{\mathbf{t}}^T & |\tilde{\mathbf{t}}|^2 & \Delta^2 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \\ |\mathbf{p}|^2 \end{pmatrix} = 0,$$

where a factor  $\Delta^2$  has been cancelled since the equation is homogeneous. So the inverse of a point-sphere constraint is again a quadric hyper-surface.

The inverse of the line-in-a-complex constraint can be shown to be given by

$$(\mathbf{m}^T, \mathbf{f}^T) \begin{pmatrix} \Delta \tilde{R}^T & 0 \\ -\tilde{R}^T \tilde{T} & \Delta \tilde{R}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0.$$

This relies on the transformation

$$\tilde{R} \tilde{T} \mapsto -\Delta^2 \tilde{R}^T \tilde{T},$$

and then cancelling  $\Delta^2$ . Once again a quadric hyper-surface is transformed into another quadric hyper-surface.

Finally, by contrast, it is easy to see that the quadric hyper-surface determined by the CS dyad found in Section 5.3 above, is transformed to a degree-4 hyper-surface

$$(\mathbf{p}^T, p_4) \begin{pmatrix} \Delta \tilde{R}^T & -\tilde{R}^T \tilde{\mathbf{t}} \\ 0 & \Delta^2 \end{pmatrix}^T \begin{pmatrix} \Omega^2 & (\boldsymbol{\omega} \times \mathbf{v}) \\ (\boldsymbol{\omega} \times \mathbf{v})^T & \rho^2 |\boldsymbol{\omega}|^2 - |\mathbf{v}|^2 \end{pmatrix} \begin{pmatrix} \Delta \tilde{R}^T & -\tilde{R}^T \tilde{\mathbf{t}} \\ 0 & \Delta^2 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ p_4 \end{pmatrix} = 0.$$

Finally here another small example of the utility of this type of geometry, consider the inverse to the Wunderlich motion. The Wunderlich motion is a rational cubic

motion that can be parametrised as

$$W(c, s) = \begin{pmatrix} c(c^2 + s^2) & -2sc^2 & 0 & -\alpha c(c^2 - s^2) + 2\beta sc^2 + \gamma s(c^2 + s^2) \\ 2sc^2 & c(c^2 - s^2) & 0 & 0 \\ 0 & 0 & c(c^2 + s^2) & \alpha' c(c^2 - s^2) + 2\beta' sc^2 + \gamma' s(c^2 + s^2) \\ 0 & 0 & 0 & c(c^2 + s^2) \end{pmatrix},$$

see [15]. Here,  $\alpha, \alpha', \beta, \beta'$  and  $\gamma, \gamma'$  are constants of the motion with  $\gamma$  and  $\gamma'$  not both zero. The quantities  $s$  and  $c$  are the homogeneous parameters which can be taken as  $c = \cos \theta/2$  and  $s = \sin \theta/2$ , where  $\theta$  is the rotation angle. A very small computation reveals that this motion does not meet  $\Delta = 0, \tilde{\mathbf{t}} = \mathbf{0}$  but does meet  $T_\infty$  in a single point: where  $c = 0$ . Hence the degree of the inverse motion will be  $2 \times 3 - 1 = 5$ .

## 7. CONCLUSIONS

It may appear that the work presented here has little application to robotics. However, the main motivation of this work is to be able to solve problems in the kinematics of spatial mechanisms. The work presented here is part of a larger project to look at the geometry defined by mechanisms and robots. The first steps were to look at some simple mechanisms defined by a few joints connected in series. This work introduces what is hoped will be a key tool in this area—birational transformation of the group  $SE(3)$ . This tool is well known in Algebraic Geometry, birational maps of the plane to itself are often called Cremona transforms. A particular example, inversion in a circle, is particularly useful in solving problems concerning circles in the plane.

The original intention of this work was to use this representation to study the intersection of point-plane constraints. In [11] a start was made, the generic intersections of 2, 3, 4, 5 and 6 point-plane constraints were found. However, there are many special cases to consider and the methods used in that work are too cumbersome to produce the desired results in a simple manner. It is assumed that since the point-plane constraints transform to hyperplanes in this representation, the problem of understanding intersections will be reduced to understanding intersections of linear varieties and how these varieties can lie in relation to the variety  $\tilde{V}$  corresponding to the group.

There are many other representations of the group of rigid-body displacements and these will produce other birational transformations of the Study quadric. For example the adjoint representation. From the discussion in Section 5.4 it is clear that using the transformation based on this representation, the variety of group elements which preserve the incidence of a line with a given linear line complex would lie on a hyperplane.

A more interesting example however, would be provided by the representation given in Section 5.2 . Using this representation it is clear that the point-sphere constraints would be transformed to hyperplanes. This would have immediate applications to systems of S-S dyads for example Gough-Stewart platforms. The difficulty with carrying out such analysis is that the image variety of the transformation from the Study quadric may be difficult to determine.

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