NUMERICAL ALGORITHMS FOR SOLVING THE ELASTO-PLASTIC PROBLEM WITH MIXED HARDENING

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Abstract: The paper deals with elasto-plastic models for mixed hardening material, described by rate-type constitutive equations. In the first part of the paper update algorithms for solving rate type elastic-plastic models with mixed hardening mixed for small deformations have been emphasized, based on the radial return mapping algorithms proposed by Simo and Hughes [28]. We exemplify the elastic prediction and plastic corrector steps of the algorithms as well as the calculus of the appropriate algorithmic tangent moduli. We emphasize the role played by the algorithmic tangent moduli in solving the quasi-static, initial and boundary value problem, based on the methodology proposed and developed in the paper by Cleja-Tigoiu and Stoicuta [8].

Key words: small strain elasto-plastic model mixed hardening, rate type constitutive models, Radial Return Algorithm, finite element method, numerical application.

1. INTRODUCTION

The paper deals with the mathematical description of the elasto-plastic models with mixed hardening, within the constitutive framework of small elasto-plastic strains. The rate independent type constitutive models were described and discussed by Cleja-Tigoiu and Cristescu [6], Kachanov [15], Khan and Huang [16], Paraschiv-Munteanu et al. [24], Simo and Hughes [28].

The variational formulation of the elasto-plastic problem is proposed and described by Han and Reddy [10], Johnson [14] and Hughes [12]. The general finite element method (FEM) is developed by Fish and Belytschko in [9], Belytschko et al. in [3], and it is applied to solve the elastic type problems, can see

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also Bath [2]. The assemble matrix of the internal forces and external forces vectors is given by Hughes [13].

In Section 2 we briefly introduce the rate-type constitutive model.

In the Section 3 of the paper we presented the update algorithm applied to the rate type independent elasto-plastic model with mixed hardening, which allow us to incrementally describe the elasto-plastic response of the model when the history of the total strain is given. We will focused mainly on the methodology proposed by Simo and Taylor [27], Simo and Hughes [28], which is based on the Radial Return Mapping Algorithm. The elastic prediction and plastic corrector are the steps of the algorithm. The algorithm has been widely studied by Maenchen and Sack [20], Nagtegaal [21] and Simo and Taylor [27]. Other generalized forms of Radial Return Method and their application in solving of some elastic-plastic or viscoplastic problems can be found in the following works: Schreye [26], Loret and Prevost [18] or Ortiz and Popov [23]. The classical Radial Return Method is proposed for the first time by Wilkins [31], Krieg and Key [17] and the development proposed by Simo and Hughes [28] is based on the implicit backward Euler method. The Radial Return Mapping Algorithm proposed by Simo and Hughes [28] provided as well as the calculus of the appropriate algorithmic tangent moduli.

In Section 3 we discuss the algorithm proposed by Simo and Hughes [28] for the Prager hardening law, the modified algorithm for a in-plane stress state given by Cleja-Tigoiu and Stoicuta [8], and the algorithmic procedure applied by Lubarda and Benson [19] for Amstrong-Frederick hardening model. We emphasize the role played by the elastic trial stress state and the procedure to calculate the appropriate algorithmic tangent moduli.

The update algorithm applied to the rate-type model is coupled with the equilibrium equation to be satisfy by the Cauchy stress at any time. The initial and boundary value problem for the rate-type elasto-plastic material consists of the equilibrium equation to be satisfies by the Cauchy stress, the elasto-plastic constitutive equations, which involves the Cauchy stress, at which the boundary and initial conditions have to be considered. In order to numerically solve the problem the discretized weak form of the equilibrium equation is coupled with the update algorithm. The quasi-static problem with the initial and boundary values associated with the rate-type elasto-plastic model with mixed hardening is formulated in the Section 4.1, following the numerical procedure developed by Cleja-Tigoiu and Stoicuta [8]. The discrete variational formulation of the problem and application of the finite element method is performed in Section 4.2. The non-linear equation for the discrete displacement vector is solved by Newton-Raphson method, which requests the algorithmic tangent moduli, already calculated.

Further we shall use the following notations:

$\mathbb{R}$ the set of real numbers and $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} | x \leq 0\}$; Lin the second order tensors; $Sym \subset Lin$ the space of symmetric tensors; Sym_2 the set of symmetric plane tensors; (e_1, e_2, e_3) a Cartesian basis ; $\boldsymbol{\sigma}' = dev\boldsymbol{\sigma} = \boldsymbol{\sigma} - \frac{1}{3}(tr\boldsymbol{\sigma})I_3$ the deviatoric part of the stress tensor; I_3 the identity tensor in $\mathbb{R}^3$; $I_2 = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ the second-order identity tensor in $\mathbb{R}^2$; $\dot{\boldsymbol{\sigma}} = \partial\boldsymbol{\sigma}/\partial t$ the time derivative of $\boldsymbol{\sigma}$; $\partial_C \mathcal{F}$ the partial derivative with respect to C of the function $\mathcal{F}$; $\mathcal{E}$ the fourth order tensor; E the Young modulus; ν the Poisson ratio; $\|\mathbf{x}\|$ the norm of $\mathbf{x}$; $\mathcal{H}(\mathcal{F}) = \begin{cases} 0, \text{ if } \mathcal{F} < 0 \\ 1, \text{ if } \mathcal{F} \geq 0 \end{cases}$ the Heaviside function; $\langle z \rangle = \frac{1}{2}(z + |z|)$, $\forall z \in \mathbb{R}$ the positive part of the real number z; $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{A} \cdot \mathbf{B}$ vector and tensor scalar products, respectively; $\nabla\mathbf{u}(x,t) = u_{i,j}\mathbf{e}_i \otimes \mathbf{e}_j$ the displacement gradient; $H^m(\Omega)$ the Sobolev space; $H_0^m(\Omega)$ closed space $C_0^\infty(\Omega)$ in $H^m(\Omega)$.

2. THE CONSTITUTIVE FRAMEWORK OF THE ELASTO-PLASTIC MODEL WITH MIXED HARDENING

We assume that the elasto-plastic body occupies a domain $\Omega \subset \mathbb{R}^3$ which is an open and bounded set with smooth boundary $\partial\Omega$ and closure $\bar{\Omega}$.

Let us denote by $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ a material point. The time interval is denoted by $I = [0, T] \subset \mathbb{R}_+$. We introduce the following notation:

- $\mathbf{u} : \Omega \times I \to \mathbb{R}^3$, $\mathbf{u}(x,t) \equiv u_i(x,t)\mathbf{e}_i$ the displacement vector;
- $\boldsymbol{\sigma} : \Omega \times I \to Sym$, $\boldsymbol{\sigma}(x,t) \equiv \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ the Cauchy stress tensor;
- $\boldsymbol{\varepsilon} : \Omega \times I \to Sym$, $\boldsymbol{\varepsilon}(x,t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\mathbf{e}_i \otimes \mathbf{e}_j$ the infinitesimal strain tensor;
- $\boldsymbol{\alpha} : \Omega \times I \to Sym$, $\boldsymbol{\alpha}(x,t) \equiv \boldsymbol{\alpha}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ the kinematic hardening variable;
- $k : \Omega \times I \to \mathbb{R}$ the isotropic hardening variable.

The set of the symmetric tensors is defined by *Sym*.
We define the elastic domain:

$$E = \left\{ (\boldsymbol{\sigma}, \boldsymbol{\alpha}, k) \in Sym \times Sym \times \mathbb{R} \,\middle|\, \mathcal{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}, k) \leq 0 \right\}. \tag{1}$$

The set of the symmetric tensors is defined by:

$$Sym := \left\{ \boldsymbol{\tau} \in \mathbb{R}^{3\times3} \middle| \boldsymbol{\tau}^T = \boldsymbol{\tau},\ \tau_{ij} \in L^2(\Omega) \right\}, \tag{2}$$

where $L^2(\Omega)$ is the space of the square integrable functions defined on Ω.

The yield function $\mathcal{F} : \Omega \subset Sym \times Sym \times \mathbb{R} \to \mathbb{R}_{\leqslant 0}$ is dependent on the stress and hardening variables and is written in the form

$$\mathcal{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}, k) = \|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\| - \sqrt{\frac{2}{3}} F(k). \tag{3}$$

The variable $\boldsymbol{\alpha}$ is called back-stress and characterizes the translational motion of the yield surface in the space of symmetrical tensor, *Sym*, while *F* gives the shape of the yield surface.

We listed here some representations for the isotropic hardening function *F*, which is generally an increasing function of k :

- the linear isotropic hardening

$$F(k) = Hk + \sigma_Y, \tag{4}$$

where $H > 0$ is the hardening constant and σ_Y represents the initial yield constant,

- the form proposed by Chaboche [5]

$$F(k) = Q(1 - e^{-bk}) + \sigma_Y, \tag{5}$$

where $Q,\ b > 0,\ \sigma_Y$ are the material constants.

- the form introduced by Voce in [29]

$$F(k) = Hk + (K_\infty - K_0)(1 - e^{-bk}) + \sigma_Y, \tag{6}$$

where $K_\infty > K_0 > 0,\ b > 0,\ H,\ \sigma_Y$ are the material constants.

The rate of the plastic strain tensor is described by the associated flow rule through:

$$\dot{\boldsymbol{\varepsilon}}^p = \lambda \partial_{\boldsymbol{\sigma}} \mathcal{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}, k), \tag{7}$$

where the plastic factor λ is a function of the material state of material and is defined by the Kuhn-Tucker conditions

$$\lambda \geq 0, \quad \mathcal{F} \leq 0, \quad \lambda \mathcal{F} = 0 \tag{8}$$

and the consistency condition

$$\lambda \dot{\mathcal{F}} = 0. \tag{9}$$

The elastic type constitutive equation is given by:

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p)\,, \tag{10}$$

with $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$ the elastic strain tensor and $\mathcal{E} : Sym \to Sym$ is linear mapping and is called the stiffness elastic tensor.

The time variation of the isotropic hardening variable is given by the relation:

$$\dot{k} = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^p \cdot \dot{\boldsymbol{\varepsilon}}^p}\ . \tag{11}$$

The time variation of the kinematic hardening variable is given by Prager type evolution equation [25], see also Ziegler [30]:

$$\dot{\boldsymbol{\alpha}} = C\,\dot{\boldsymbol{\varepsilon}}^p \tag{12}$$

or by the Armstrong-Frederick type [1]

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}C\dot{\boldsymbol{\varepsilon}}^p - \gamma\dot{k}\,, \tag{13}$$

with C and γ material constant, generally considered to be constant. As a peculiar case we mention the so-called linear hardening law with $C = H'(k)$ which has been introduced by Simo and Hughes [28] and which will be considered together with $\gamma = 0$ in the Section 4.1.

PROPOSITION 1. *The plastic factor* $\lambda : \Omega \times I \to \mathbb{R}$ *is calculated on the yield surface from the consistency condition (9) and the Kuhn Tucker conditions (8) by*

$$\lambda = \frac{\langle\beta\rangle}{h}\mathcal{H}(\mathcal{F})\,, \tag{14}$$

where β *the complementary plastic factor is given by the expression*

$$\beta = \mathcal{E}\,\partial_{\boldsymbol{\sigma}}\mathcal{F} \cdot \dot{\boldsymbol{\varepsilon}} \tag{15}$$

and the hardening parameter h *is given by*

$$h = \partial_{\boldsymbol{\sigma}}\mathcal{F} \cdot \mathcal{E}\,\partial_{\boldsymbol{\sigma}}\mathcal{F} + C - \gamma\partial_{\boldsymbol{\sigma}}\mathcal{F} \cdot \boldsymbol{\alpha} + \sqrt{\frac{2}{3}}F'(k)\,, \tag{16}$$

under the hypothesis that $h > 0$. *Here* C, γ, σ_Y *are material constants.*

PROPOSITION 2. *The three-dimensional elasto-plastic model with hardening by the given Armstrong-Frederick law is described by the following rate-type constitutive equation*

$$\begin{cases} \dot{\boldsymbol{\sigma}} = \mathcal{E}\dot{\boldsymbol{\varepsilon}} - \dfrac{\langle\beta\rangle}{h}\mathcal{E}\,\partial_{\boldsymbol{\sigma}}\mathcal{F}\,\mathcal{H}(\mathcal{F}) \\ \dot{\boldsymbol{\varepsilon}}^{p} = \dfrac{\langle\beta\rangle}{h}\partial_{\boldsymbol{\sigma}}\mathcal{F}\,\mathcal{H}(\mathcal{F}) \\ \dot{\boldsymbol{\alpha}} = \dfrac{\langle\beta\rangle}{h}\left(C\partial_{\boldsymbol{\sigma}}\mathcal{F} - \gamma\boldsymbol{\alpha}\right)\mathcal{H}(\mathcal{F}) \\ \dot{k} = \dfrac{\langle\beta\rangle}{h}\mathcal{H}(\mathcal{F}) \end{cases}, \tag{17}$$

with $Q, b, C, \gamma, \sigma_Y$ material constants and associated with the yield function given by (3). The expressions of the complementary plastic factor and the hardening parameter are given by relations (15) and (16).

Remark 1. The set of constitutive equations allows us to define a differential system for the unknowns $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\alpha}$ and k, when the deformation tensor $\boldsymbol{\varepsilon}$ is a given time dependent function.

3. UPDATE ALGORITHM APPLIED TO THE RATE TYPE INDEPENDENT ELASTO-PLASTIC MODEL WITH MIXED HARDENING

In this section we describe some algorithms for solving the differential system which describes the rate-type elasto-plastic constitutive model when the history of the strain tensor $\boldsymbol{\varepsilon}$ is given. We discuss the algorithm proposed by Simo and Hughes [28] for the Prager hardening law, the modified algorithm for a in-plane stress state given by Cleja-Tigoiu and Stoicuta [8], and the algorithmic procedure applied by Lubarda and Benson [19] for Amstrong-Frederick hardening model.

3.1. The radial return algorithm proposed by Simo and Hughes [28]

The numerical algorithm for solving the rate-type elastic-plastic model with hardening proposed and developed by Simo and Hughes [28] is a *predictor-corrector method,* which applies the return mapping algorithm.

Elastic Predictor step specifies *for a given strain increment* on the interval $[t_n, t_{n+1}]$ the trial elastic state, which is defined for the irreversible freezing state. If the trial elastic state remains inside the appropriate yield condition, then an elastic state is realized. If not, namely if the trial state is outside the yield surface, then the state corresponds to plastic behavior.

Plastic Corrector step and return mapping algorithm has the purpose to restore the consistency of the plastic state, which means that the corresponding plastic stress has to be on the appropriate yield surface.

Finally, the algorithmic elasto-plastic tangent modulus is built.

Now we describe the Radial Return Algorithm proposed by Simo and Hughes in [28], for solving the elasto-plastic model with mixed hardening characterized by Prager type law. This elasto-plastic model is described by the following differential type system:

$$\begin{cases} \dot{\boldsymbol{\sigma}} = \mathcal{E}\left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p\right) \\ \dot{\boldsymbol{\varepsilon}}^p = \lambda \dfrac{\mathbf{s} - \boldsymbol{\alpha}}{F(k)} \\ \dot{\boldsymbol{\alpha}} = C\lambda \dfrac{\mathbf{s} - \boldsymbol{\alpha}}{F(k)} \\ \dot{k} = \lambda \end{cases}, \tag{18}$$

where $\mathbf{s} = \boldsymbol{\sigma}'$ is the deviatoric part of the stess tensor.

If the time interval $[0, T]$ is discretyzed by the time sequence $0 = t_0 < t_1 < \ldots < t_{\mathrm{N}} = T$ and the notation $X(t_{n+1}) = X_{n+1}$ is used to denote the the value of the field X at time t_{n+1}, the elastic type constitutive relation at time t_{n+1} becomes:

$$\boldsymbol{\sigma}_{n+1} = \mathcal{E}(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^p_{n+1}). \tag{19}$$

We apply the backward Euler method for the equation from (18), and the following algorithmic relationships are provided

$$\begin{aligned} \boldsymbol{\varepsilon}^p_{n+1} &= \boldsymbol{\varepsilon}^p_n + \Delta\lambda \frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|} \\ k_{n+1} &= k_n + \Delta\lambda \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + C\Delta\lambda \frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|} \end{aligned}, \tag{20}$$

where $\Delta\lambda = \lambda\Delta t = \lambda(t_{n+1} - t_n)$ satisfy the discrete Kuhn-Tucker conditions

$$\Delta\lambda \geq 0, \quad \mathcal{F}(\mathbf{s}_{n+1}, \boldsymbol{\alpha}_{n+1}, k_{n+1}) \leq 0, \quad \Delta\lambda\mathcal{F}(\mathbf{s}_{n+1}, \boldsymbol{\alpha}_{n+1}, k_{n+1}) = 0. \tag{21}$$

The yield function is defined by:

$$\mathcal{F}\left(\mathbf{s}_{n+1}, \boldsymbol{\alpha}_{n+1}, k_{n+1}\right) = \left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\| - \sqrt{\frac{2}{3}} F\left(k_{n+1}\right). \tag{22}$$

Predictor elastic. The trial elastic stress tensor and its deviatoric part, respectively, are defined by freezing the plastic strain at the previous time

$$\boldsymbol{\sigma}_{n+1}^{\mathrm{tr}} = k_a \mathrm{tr}\boldsymbol{\varepsilon}_{n+1} 1 + 2\mu(\mathrm{dev}\,\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p); \quad \mathbf{s}_{n+1}^{tr} = \mathbf{s}_n + 2\mu\left(\mathrm{dev}\,\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p\right). \quad (23)$$

As $\boldsymbol{\varepsilon}_{n+1}^p, \boldsymbol{\alpha}_{n+1}$ and k_{n+1} are frozen at their previous values it follows that

$$\boldsymbol{\varepsilon}_{n+1}^p \equiv \boldsymbol{\varepsilon}_n^p, \ \boldsymbol{\alpha}_{n+1} \equiv \boldsymbol{\alpha}_n, \ k_{n+1} \equiv k_n \quad (24)$$

and consequently, the test function $\mathcal{F}_{n+1}^{\mathrm{tr}}$ is defined by

$$\mathcal{F}_{n+1}^{\mathrm{tr}} := \left\| \mathbf{s}_{n+1}^{\mathrm{tr}} - \boldsymbol{\alpha}_n \right\| - \sqrt{\frac{2}{3}} F\left(k_n\right). \quad (25)$$

Corrector plastic. Return algorithm. If the elastic trial state is outside the yield surface, then $\Delta\lambda$ cannot be zero and the following relations take place

$$\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{tr} - 2\mu\Delta\lambda \frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\| \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} \right\|}; \quad \boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n + C\Delta\lambda \frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\| \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} \right\|}. \quad (26)$$

The unit normal $\mathbf{n}_{n+1}^{tr}$ is determined exclusively in terms of the trial elastic stress:

$$\mathbf{n}_{n+1}^{tr} = \frac{\mathbf{s}_{n+1}^{tr} - \boldsymbol{\alpha}_n}{\left\| \mathbf{s}_{n+1}^{tr} - \boldsymbol{\alpha}_n \right\|}. \quad (27)$$

If we denote with $\xi_{n+1} = \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}$ and $\xi_{n+1}^{tr} = \mathbf{s}_{n+1}^{tr} - \boldsymbol{\alpha}_n$. Then we have

$$\xi_{n+1} = \mathbf{s}_{n+1}^{tr} - \boldsymbol{\alpha}_n - 2\mu\Delta\lambda\left(2\mu + C\right)\mathbf{n}_{n+1}^{tr}. \quad (28)$$

PROPOSITION 3 (The update algorithm). *The following loading/unloading conditions hold if* $\mathcal{F}_{n+1}^{\mathrm{tr}} \leq 0$ then $\Delta\lambda = 0$ *and the solution is elastic*

$$\begin{aligned} &\boldsymbol{\sigma}_{n+1}^{\mathrm{tr}} = k_a \mathrm{tr}\boldsymbol{\varepsilon}_{n+1} 1 + 2\mu(\mathrm{dev}\,\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p); \quad \mathbf{s}_{n+1}^{\mathrm{tr}} = \mathbf{s}_n + 2\mu\left(\mathrm{dev}\,\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p\right) \\ &\boldsymbol{\varepsilon}_{n+1}^p \equiv \boldsymbol{\varepsilon}_n^p, \quad \boldsymbol{\alpha}_{n+1} \equiv \boldsymbol{\alpha}_n, \quad k_{n+1} \equiv k_n. \end{aligned} \quad (29)$$

Precisely these relationships are taken to be the initial conditions for the solution of the plastic corrector problem. The consistency of the plastic condition is restored by the return-mapping algorithm.

If $\mathcal{F}_{n+1}^{\mathrm{tr}} > 0$ then $\Delta\lambda \neq 0$ and the algorithmic expression of the plastic factor λ_{n+1} is calculated from the consistency condition characterized by Kuhn-Tucker conditions. Hence, the plastic factor results

$$\Delta\lambda = \frac{\left\| \xi_{n+1}^{\mathrm{tr}} \right\| - \sqrt{\frac{2}{3}} F(k_n)}{\sqrt{\frac{2}{3}} F'(k_n) + 2\mu + \mathrm{C}}, \tag{30}$$

and the update values of the state at time t_{n+1}

$$\begin{aligned} &\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{\mathrm{tr}} - 2\mu\Delta\lambda \frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\| \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} \right\|}, \quad \boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n + C\Delta\lambda \frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\| \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} \right\|}, \\ &\boldsymbol{\varepsilon}_{n+1}^{p} = \boldsymbol{\varepsilon}_n^{p} + \Delta\lambda \frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\| \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} \right\|}, \qquad k_{n+1} = k_n + \Delta\lambda. \end{aligned} \tag{31}$$

The radial return mapping algorithm is completely determined, if the algorithmic elastic-plastic tangent modulus is calculated.

PROPOSITION 4 (Simo and Hughes [28]). *The algorithmic elasto-plastic tangent modulus can be approximated by the following expression*:

$$\mathcal{E}_{n+1}^{\mathrm{ep}} = \begin{cases} \mathcal{E} & , \ \mathcal{F}^{\mathrm{tr}} \leqslant 0 \\ k_a 1 \otimes 1 + 2\mu_a \left[\theta_1\right]_{n+1} I_{dev} - 2\mu_a \left[\theta_2\right]_{n+1} \mathbf{n}_{n+1} \otimes \mathbf{n}_{n+1}, & \mathcal{F}^{\mathrm{tr}} > 0 \end{cases}. \tag{32}$$

$$\left[\theta_1\right]_{n+1} = 1 - \Delta\lambda \frac{2\mu_a}{\left\| \xi_{n+1}^{\mathrm{tr}} \right\|}; \quad \left[\theta_2\right]_{n+1} = \frac{2\mu_a}{\sqrt{\frac{2}{3}} F'(k_n) + 2\mu_a + C} - \Delta\lambda \frac{2\mu_a}{\left\| \xi_{n+1}^{\mathrm{tr}} \right\|}. \tag{33}$$

3.2. The algorithm proposed by Cleja-Tigoiu and Stoicuta [8]

In [8] a new numerical approach of the elasto-plastic problem with mixed hardening in the *plane stress state*, within the classical constitutive framework of small deformation, has been proposed. The numerical algorithm (which is an incremental one) consists of the Radial Return Mapping Algorithm, originally proposed by Simo and Hughes [28], adapted to the stress plane problem and coupled with a pseudo-elastic problem. The proposed algorithm has been called the revisited Simo-Hughes algorithm and it is applied to exemplify the mode of integration of the bidimensional problem.

- We restrict ourselves to a *plane stress state*

$$\boldsymbol{\sigma} = \sum_{i,j=1}^{2} \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \boldsymbol{\sigma} \in Sym_2, \quad \text{i.e.} \quad \boldsymbol{\sigma}' \equiv \boldsymbol{\sigma}_{(2)}, \tag{34}$$

since $\sigma_{13}=\sigma_{23}=\sigma_{33}=0$. Here is avoid the representation such as $(\sigma_{11},\sigma_{22},2\sigma_{12})$ is avoided, and for a plane symmetric tensor $\boldsymbol{\sigma}$ we have the representation

$$\boldsymbol{\sigma}=\sigma_{11}e_1\otimes e_1+\sigma_{22}e_2\otimes e_2+\sigma_{12}e_1\otimes e_2+\sigma_{21}e_2\otimes e_1, \quad \text{with} \quad \sigma_{12}=\sigma_{21}. \tag{35}$$

- The *kinematic hardening variable* $\boldsymbol{\alpha}$ is defined:

$$\boldsymbol{\alpha}=\sum_{i,j=1}^{3}\alpha_{ij}\mathbf{e}_i\otimes\mathbf{e}_j, \quad \boldsymbol{\alpha}\in Sym \tag{36}$$

  and $\alpha_{13}=\alpha_{23}=0$, by trace null $\operatorname{tr}\alpha=0$.
- The *generalized strain state* is characterized by a strain tensor $\boldsymbol{\varepsilon}$ that has a non-zero component in the direction perpendicular to the shell, denoted by ε_{33}, which is added to the in plane strain $\varepsilon_{(2)}$:

$$\boldsymbol{\varepsilon}=\varepsilon_{(2)}+\varepsilon_{33}e_3\otimes e_3, \quad \varepsilon_{(2)}=\sum_{i,j=1}^{2}\varepsilon_{ij}\mathbf{e}_i\otimes\mathbf{e}_j, \quad \varepsilon_{(2)}\in Sym_2. \tag{37}$$

- The tensorial representations for the plain stress state and for the plane kinematic variable:

$$\begin{aligned}&\boldsymbol{\sigma}'=\sigma'_{(2)}+\sigma'_{33}e_3\otimes e_3; \quad \boldsymbol{\alpha}=\alpha_{(2)}+\alpha_{33}e_3\otimes e_3; \quad \mathbf{s}=\boldsymbol{\sigma}'_{(2)}; \quad \hat{\boldsymbol{\alpha}}=\boldsymbol{\alpha}_{(2)};\\ &\sigma'_{33}=-\operatorname{tr}s; \quad \operatorname{tr}s=s_{11}+s_{22}; \quad \alpha_{33}=-\operatorname{tr}\hat{\alpha}; \quad \operatorname{tr}\hat{\alpha}=\hat{\alpha}_{11}+\hat{\alpha}_{22}.\end{aligned} \tag{38}$$

- The passage from $\boldsymbol{\sigma}\equiv\boldsymbol{\sigma}_{(2)}\subset Sym_2$ to $\mathbf{s}=\boldsymbol{\sigma}'_{(2)}$ can be characterized by an invertible mapping P given by the matrix

$$P=\begin{pmatrix}2/3 & -1/3 & 0 & 0\\ -1/3 & 2/3 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad \mathbf{s}=P\boldsymbol{\sigma}, \quad \boldsymbol{\sigma}=\boldsymbol{\sigma}_{(2)}. \tag{39}$$

The plastic factor $\lambda:\Omega\times I\to\mathbb{R}$ is calculated on the yield surface $\mathcal{F}(s,\hat{\alpha},k)=0$ by:

$$\lambda=\frac{\langle\beta\rangle}{h}\mathcal{H}(\mathcal{F}), \tag{40}$$

where

$$\beta=\frac{E}{1+\nu}\left(\partial_{s_{11}}\mathcal{F}+\frac{\nu}{1-\nu}\partial_{trs}\mathcal{F}\right)\dot{\varepsilon}_{11}+\frac{E}{1+\nu}\left(\partial_{s_{22}}\mathcal{F}+\frac{\nu}{1-\nu}\partial_{trs}\mathcal{F}\right)\dot{\varepsilon}_{22}+\frac{2E}{1+\nu}\partial_{s_{12}}\mathcal{F}\dot{\varepsilon}_{12}$$
$$h=\left(\frac{E}{1+\nu}+C\right)\left\|\partial_s\mathcal{F}\right\|^2+\left(\frac{E}{1+\nu}+C+\frac{E(1-2\nu)}{1-\nu^2}\right)\left\|\partial_{trs}\mathcal{F}\right\|^2+\sqrt{\frac{2}{3}}H. \tag{41}$$

Here C is the kinematic hardening modulus.

The variation in time of the component of the strain ε_{33} is defined by

$$\dot{\varepsilon}_{33}=-\frac{\nu}{1-\nu}\left(\dot{\varepsilon}_{11}+\dot{\varepsilon}_{22}\right)-\frac{1-2\nu}{1-\nu}\lambda\,\partial_{trs}\mathcal{F}\,. \tag{42}$$

Remark 2. The evolution equation for the strain component ε_{33} is derived from the consistency requirement to have a plane stress state, applying the procedure developed by Cleja-Tigoiu [7].

In the case of a plane stress state associated with the generalized plane strain, the elasto-plastic tangent modulus can be expressed in the following modified form

$$\mathcal{E}^{\mathrm{ep}}=\begin{cases}\mathcal{E}_{(2)} & \text{if } \mathcal{F}^{\mathrm{tr}}\le 0\\ E_{(2)}[\Theta_1]-[\Theta_2]\,\mathbf{n}\otimes\left(E_{(2)}P\,\mathbf{n}+n^{\perp}E_{(2)}P\,I_{(2)}\right) & \text{if } \mathcal{F}^{\mathrm{tr}}>0\end{cases}, \tag{43}$$

where

$$[\Theta_1]=I_4-\frac{[\lambda^*]_{n+1}}{\sqrt{\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|^2+(\mathrm{tr}(\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|))^2}}PE_{(2)}$$
$$[\Theta_2]=\mathcal{E}_{(2)}\left(\frac{1}{\sqrt{\frac{2}{3}}H+\frac{E}{3(1-\nu)}+C}-\frac{[\lambda^*]_{n+1}}{\sqrt{\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|^2+(\mathrm{tr}(\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|))^2}}\right). \tag{44}$$

In the relation above, the tensor $E_{(2)}:Sym_2\times Sym_2$ is the elastic tensor and the normals $\mathbf{n}$ and $n^{\perp}$ have the following form

$$\mathbf{n}\equiv\frac{\mathbf{s}-\hat{\boldsymbol{\alpha}}}{\sqrt{\|\mathbf{s}-\hat{\boldsymbol{\alpha}}\|^2+(\mathrm{tr}(\mathbf{s}-\hat{\boldsymbol{\alpha}}))^2}}=\frac{\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}}{\sqrt{\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|^2+(\mathrm{tr}(\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|))^2}},$$
$$n^{\perp}\equiv\frac{\mathrm{tr}(\mathbf{s}-\hat{\boldsymbol{\alpha}})}{\sqrt{\|\mathbf{s}-\hat{\boldsymbol{\alpha}}\|^2+(\mathrm{tr}(\mathbf{s}-\hat{\boldsymbol{\alpha}}))^2}}=\frac{\mathrm{tr}(\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}})}{\sqrt{\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|^2+(\mathrm{tr}(\|\mathbf{s}^{\mathrm{tr}}-\hat{\boldsymbol{\alpha}}^{\mathrm{tr}}\|))^2}}. \tag{45}$$

Only when the plane stress state is considered to be associated with the plane strain, namely when the strain in the direction perpendicular to the plane is neglected, the elasto-plastic tangent moduli provided here can be reduced to the formulae that can be found in Simo and Hughes [28], namely the appropriate fourth order tensors are replaced by two scalars, written here in the formulae (32).

3.3. Algorithmic consistency condition proposed by Ludarda and Benson [19]

We exemplify the update algorithm proposed in [19] for the rate-type model, which is written in Proposition 2.

The elasto-plastic model with mixed hardening characterized by the Armstrong-Frederick type rule is described by the following differential system:

$$\begin{cases} \dot{\mathbf{s}} = 2\mu\dot{\mathbf{e}} - 2\mu\dot{\boldsymbol{\varepsilon}}^p \\ \dot{\boldsymbol{\varepsilon}}^p = \lambda\partial_{\mathbf{s}}\mathcal{F} \\ \dot{\boldsymbol{\alpha}} = C\dot{\boldsymbol{\varepsilon}}^p - \gamma\boldsymbol{\alpha}\dot{k} \\ \dot{k} = \sqrt{\|\dot{\boldsymbol{\varepsilon}}^p\|} \end{cases}, \tag{46}$$

where

$$\begin{cases} \lambda = \dfrac{\beta}{h_c} \\ \beta = \partial_{\mathbf{s}}\mathcal{F}\cdot\dot{\mathbf{s}} \\ h_c = 2F'(k) + C - \gamma\dfrac{(\mathbf{s}-\boldsymbol{\alpha})\cdot\boldsymbol{\alpha}}{\sqrt{2}F(k)}, \end{cases} \tag{47}$$

and which is associated with the yield surface given

$$\mathcal{F}(s,\alpha,k) = 1/\sqrt{2}\,\|s-\alpha\| - F(k)\,. \tag{48}$$

To determine the algorithmic consistency condition, the authors applied the Euler forward method

$$\begin{cases} \mathbf{s}_{n+1} = \mathbf{s}_n + 2\mu\mathrm{d}\mathbf{e} - 2\mu\lambda^*_{n+1}\mathbf{n}_{n+1} \\ k_{n+1} = k_n + \lambda^*_{n+1} \\ \boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n + (\mathrm{d}\boldsymbol{\alpha})_{n+1} \\ \boldsymbol{\varepsilon}^p_{n+1} = \boldsymbol{\varepsilon}^p_n + \lambda^*_{n+1}\mathbf{n}_{n+1} \end{cases}, \tag{49}$$

where

$$\begin{cases} \mathbf{n}_{n+1} = \dfrac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\| \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} \right\|} \\ (\mathrm{d}\boldsymbol{\alpha})_{n+1} = \lambda^*_{n+1}(C\mathbf{n}_{n+1} - \gamma\boldsymbol{\alpha}_{n+1}) \,. \\ \lambda^*_{n+1} = \Delta t\, \lambda(t_{n+1}) \end{cases} \tag{50}$$

In the method proposed in [19] in the second member of the equality written in $(50)_2$ $\boldsymbol{\alpha}_{n+1}$ is replaced with $\theta\boldsymbol{\alpha}_n + (1-\theta)\boldsymbol{\alpha}_{n+1}$. Thus the values at time t_{n+1} has been evaluated from the equality

$$\boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}_n = \lambda^*_{n+1}(Cn_{n+1} - \gamma(\theta\boldsymbol{\alpha}_n + (1-\theta)\boldsymbol{\alpha}_{n+1}))\,, \tag{51}$$

where $\theta \in [0,1]$. In the case $\theta = 0$ the forward Euler rule takes place, $\theta = 1$ corresponds to the backward Euler rule and $\theta = \dfrac{1}{2}$ is used in the generalized midpoint rule. We remark that a hybrid method has been employed to approximate the rate type equations which describe the model.

Therefore we derived that $\mathrm{d}\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}_n$, with

$$\mathrm{d}\boldsymbol{\alpha}_{n+1} = a_{n+1}\lambda^*_{n+1}\left(\mathbf{n}_{n+1} - \frac{\gamma}{C}\boldsymbol{\alpha}_n\right), \tag{52}$$

where

$$a_{n+1} = \frac{C}{1 + \gamma(1-\theta)\lambda^*_{n+1}}\,. \tag{53}$$

The following equality results

$$\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} = \mathbf{s}_n - \boldsymbol{\alpha}_n + 2\mu\mathrm{d}\mathbf{e} - 2\mu\lambda^*_{n+1}\mathbf{n}_{n+1} - a_{n+1}\lambda^*_{n+1}\left(\mathbf{n}_{n+1} - \frac{\gamma}{C}\boldsymbol{\alpha}_n\right). \tag{54}$$

This can be finally written in the form

$$\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1} + (2\mu + a_{n+1})\lambda^*_{n+1}\mathbf{n}_{n+1} = \mathbf{s}_n - \boldsymbol{\alpha}_n + 2\mu\mathrm{d}\mathbf{e} + b_{n+1}\lambda^*_{n+1}\boldsymbol{\alpha}_n\,, \tag{55}$$

with

$$b_{n+1} = \frac{\gamma}{1 + \gamma(1-\theta)\lambda^*_{n+1}}. \tag{56}$$

Remark 3. Let us denote by $\mathbf{s}^*_{n+1}$ the trial elastic stress $\mathbf{s}^*_{n+1} = \mathbf{s}_n + 2\mu\mathrm{d}\mathbf{e}$ that corresponds to the elastic strain increment d**e**. First the frozen plastic behavior has to be considered. Only if the trial elastic state is outside the appropriate trial yield condition at time t_{n+1} the approximate solution has to be characterized by the non-zero plastic factor. Following the method proposed by Simo and Hughes [28] the procedure to restore the consistency condition has to be pursued.

If we take a trace product for the equation given above taking into account the fact that the plasticity function $F(\boldsymbol{\sigma},\boldsymbol{\alpha},k)=0$, namely $\|\boldsymbol{\sigma}'-\boldsymbol{\alpha}\|=\sqrt{2}F(k)$, then it becomes:

$$\sqrt{2}F(k_n+\sqrt{2}\lambda^*_{n+1})+(2\mu+a_{n+1})\lambda^*_{n+1}=$$
$$=\left[2F^2(k_n)+\left\|2\mu\mathrm{d}\mathbf{e}+b_{n+1}\lambda^*_{n+1}\boldsymbol{\alpha}_n\right\|^2+2(\mathbf{s}_n-\boldsymbol{\alpha}_n)\cdot(2\mu\mathrm{d}\mathbf{e}+b_{n+1}\lambda^*_{n+1}\boldsymbol{\alpha}_n)\right]^{1/2}. \quad (57)$$

That is the key equation of the numerical method proposed by Lubarda and Benson [19]. There is the algorithmic consistency condition for the considered hardening model with the Armstrong-Frederick evolution of the back stress. The nonlinear equation for the loading parameter λ^*_{n+1} is proposed to be resolved using Newton's iterative method.

$$G=\sqrt{2}F(k_n+\sqrt{2}\lambda^*_{n+1})+(2\mu+a_{n+1})\lambda^*_{n+1}-$$
$$-\left[2F^2(k_n)+\left\|2\mu\mathrm{d}\mathbf{e}+b_{n+1}\lambda^*_{n+1}\boldsymbol{\alpha}_n\right\|^2+2(\mathbf{s}_n-\boldsymbol{\alpha}_n)\cdot(2\mu\mathrm{d}\mathbf{e}+b_{n+1}\lambda^*_{n+1}\boldsymbol{\alpha}_n)\right]^{1/2}. \quad (58)$$

The iterative procedure to find the zero value of G is then based on the approximate formula

$$\lambda^{*,j+1}_{n+1}=\lambda^{*,j}_{n+1}-\frac{G(\lambda^{*,j}_{n+1})}{G'(\lambda^{*,j}_{n+1})}, \quad (59)$$

where with $G'(\lambda^{*,j}_{n+1})$ was denoted the derivative of the function $G(\lambda^{*,j}_{n+1})$. The iterations are ended when a desired accuracy has been reached, i.e., when $\left|G(\lambda^{*,j}_{n+1})\right|$ falls into a prescribed error tolerance.

Remark 4. The numerical procedure proposed in [19] has not been applied to solve any elasto-plastic non-homogeneous problem. The homogeneous simple shear deformation process has been considered and compared to the other models.

4. NUMERICAL ALGORITHM OF SOLVING THE ELASTO-PLASTIC PROBLEM WITH MIXED HARDENING

The quasi-static elasto-plastic problem with the initial and boundary values is formulated within constitutive framework of mixed hardening material with small strains.

The discrete variational formulation of the problem is provided and the finite element method is performed in Section 4.2 in order to solve the problem, following the procedure applied by Cleja-Tigoiu and Stoicuta [8]. The rate type constitutive model considered here is the one introduced in Simo and Hughes [28].

The model has been called non-linear hardening, due to the presence of the scalar hardening variable k in the evolution equation of the Prager-type for the back stress., i.e. the constant C is replaced by $H'(k)$.

The Radial Return Algorithm is written to determine the algorithmic expression of the plastic factor λ from the consistency condition when the history $t \to \varepsilon(u(t))$ is given in a fixed material particle and the tangent elasto-plastic moduli have also been calculated. The initial and boundary value problem will be formulated as in Cleja-Tigoiu and Stoicuta [8].

4.1. Mathematical formulation of the elasto-plastic problem with mixed hardening

Problem P. Let be given the function $f, g : \Omega \to \mathbb{R}^3$. Find the functions $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\alpha}, k$ which are defined in $\Omega \times [0,T)$, which satisfy the differential type constitutive equations:

$$\begin{cases} \dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{u}) - \mathcal{E}\lambda\,\partial_{\boldsymbol{\sigma}}\mathcal{F} \\ \dot{\boldsymbol{\varepsilon}}^p = \lambda\,\partial_{\boldsymbol{\sigma}}\mathcal{F} \\ \dot{\boldsymbol{\alpha}} = \lambda\dfrac{2}{3}H'(k)\partial_{\boldsymbol{\sigma}}\mathcal{F} \\ \dot{k} = \sqrt{\dfrac{2}{3}}\lambda \end{cases} \tag{60}$$

where

$$\begin{cases} \mathcal{F} = \|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\| - \sqrt{\dfrac{2}{3}}F(k) \\ \partial_{\boldsymbol{\sigma}}\mathcal{F} = \dfrac{\boldsymbol{\sigma}' - \boldsymbol{\alpha}}{\|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\|} \\ \lambda = \dfrac{\langle\beta\rangle}{h_c}H(F) \\ \beta = \partial_{\boldsymbol{\sigma}}\mathcal{F}\cdot\mathcal{E}\boldsymbol{\varepsilon}(\dot{u}) \\ h_c = \partial_{\boldsymbol{\sigma}}\mathcal{F}\cdot\mathcal{E}\partial_{\boldsymbol{\sigma}}\mathcal{F} + \partial_{\boldsymbol{\sigma}}\mathcal{F}\cdot\dfrac{2}{3}H'(k)\partial_{\boldsymbol{\sigma}}\mathcal{F} - \sqrt{\dfrac{2}{3}}F'(k). \end{cases} \tag{61}$$

The Cauchy stress $\boldsymbol{\sigma}$ satisfies quasi-static boundary value problem

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = 0 & \text{in} \quad \Omega \times I \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{f} & \text{in} \quad \Gamma_\sigma \times I \\ \mathbf{u} = \mathbf{g} & \text{in} \quad \Gamma_u \times I \end{cases} \tag{62}$$

at which the initial conditions have to be added

$$\begin{cases} \mathbf{u}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\varepsilon}^p(0) = \boldsymbol{\varepsilon}_0^p, \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0, \quad k(0) = k_0 \,. \end{cases} \tag{63}$$

To solve numerically the Problem P, we propose the following strategy:

- we introduce the weak formulation associated with the equations in order to find the stress $\boldsymbol{\sigma}_{n+1}$ and the displacement vector $\mathbf{u}_{n+1}$ at the time t_{n+1};
- for a given incremental strain history, an update algorithm for $\mathbf{s}, \boldsymbol{\varepsilon}^p, \boldsymbol{\alpha}, k$ is derived by using the constitutive equation;
- at every time t_{n+1} and for any material point in $\mathbf{x} \in \Omega(t_{n+1})$, we use an iterative solution of the rate type given by the update algorithm, to constitutive equation boundary and initial value problem for the discretized equilibrium balance equation and coupled with the update algorithm.

4.2. Discretized variational formulation and application of the finite element method

We proceed to discretize the weak formulation of the problem to solve an associate non-linear elastic-like problem. We supposed that the boundary of the domain denoted with $\partial\Omega$ is decomposed into two parts Γ_u and Γ_σ, with $\overline{\Gamma_u \cup \Gamma_\sigma} = \partial\Omega$ and $\Gamma_u \cap \Gamma_\sigma = \varnothing$.

We define the set of kinematically admissible velocity field at every time t, is denoted by:

$$\mathcal{V}_{ad} = \left\{ w_i \in L^2(\Omega);\ \frac{\partial w_i}{\partial x_j} \in L^2(\Omega),\ w_i\big|_{\Gamma_u} = g_i, \quad i, j = 1,2 \right\} \subset \left(H^1(\Omega)\right)^2 \tag{64}$$

where $H^1(\Omega)$ is the Sobolev space.

We replace the continuous time interval $[0,T]$ by the sequence of discrete times $t_0, t_1, ..., t_N$ with $t_{n+1} = t_n + \Delta t$ and we formulate at the moment t_{n+1} the pseudo-elastic problem P_{n+1}^e.

Problem P^{e}_{n+1}. Find the displacement field $\mathbf{u}_{n+1}$, solution of the discretized weak formulation:

$$\int_{\Omega} \mathcal{E}(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^{p}_{n+1}) \cdot \boldsymbol{\varepsilon}(\mathbf{w}_{n+1})\,\mathrm{d}x = L(\mathbf{w}_{n+1}), \tag{65}$$

with $L\left(\mathbf{w}_{n+1}\right)$ given by

$$L(\mathbf{w}_{n+1}) = \int_{\Gamma_\sigma} \mathbf{f} \cdot \mathbf{w}_{n+1}\mathrm{d}\Gamma + \int_{\Gamma_u} \boldsymbol{\sigma}_{n+1}\mathbf{n} \cdot \mathbf{g}\,\mathrm{d}\Gamma + \int_{\Omega} \mathbf{b} \cdot \mathbf{w}_{n+1}\mathrm{d}x, \tag{66}$$

for all $\mathbf{w}_{n+1} \in \mathcal{V}^{h}_{ad}$, which is the finite-dimensional approximation to $\mathcal{V}_{ad}$.

At each iteration $n+1$ one solves a so called pseudo-elastic problem P^{e}_{n+1}, where $\mathbf{w}_{n+1}$ is the test function associated with the $n+1$ iteration. The finite element method is utilized to solve the discrete variational formulation of the problem.

PROPOSITION 5. *The discrete version of the weak formulation, namely of the problem* P^{e}_{n+1} *reads*:

$$\hat{G}\left(\hat{\mathbf{u}}_{n+1}\right) = F^{\text{int}}\left(\hat{\mathbf{u}}_{n+1}\right) - F^{ext}_{n+1} = 0. \tag{67}$$

Proof. We proceed to discretize the weak form to give the associate non-linear pseudo-elastic-like problem. Thus, the displacement field $\hat{\mathbf{u}}_{n+1}$ has be solution of the following equations

$$\begin{aligned}
&\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}^{e}_{n+1}\right)^{\mathrm{T}}\left(\int_{-1}^{1}\int_{-1}^{1}\left[\left(B^{e}(\xi,\eta)\right)^{\mathrm{T}}\boldsymbol{\sigma}\left(\hat{\mathbf{u}}^{e}_{n+1}\right)(\xi,\eta)\left|J^{e}(\xi,\eta)\right|\right]\mathrm{d}\xi\mathrm{d}\eta\right)\right\} = \\
&=\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}^{e}_{n+1}\right)^{\mathrm{T}}\int_{-1}^{1}\int_{-1}^{1}\left[\left(N(\xi,\eta)\right)^{\mathrm{T}}N(\xi,\eta)\hat{\mathbf{b}}^{e}_{n+1}\left|J^{e}(\xi,\eta)\right|\right]\mathrm{d}\xi\mathrm{d}\eta\right\} + \\
&+\sum_{i=1}^{2}\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}^{e}_{n+1}\right)^{\mathrm{T}}\int_{-1}^{1}\left[\left(N(\xi,-2\left(i-1\right)+1)\right)^{\mathrm{T}}N(\xi,-2\left(i-1\right)+1)\hat{\mathbf{f}}^{e}_{n+1}\left(J_1\right)^{e}_{i}\right]\mathrm{d}\xi\right\} + \\
&+\sum_{i=1}^{2}\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}^{e}_{n+1}\right)^{\mathrm{T}}\int_{-1}^{1}\left[\left(N(-2\left(i-1\right)+1,\eta)\right)^{\mathrm{T}}N(-2\left(i-1\right)+1,\eta)\hat{\mathbf{f}}^{e}_{n+1}\left(J_2\right)^{e}_{i}\right]\mathrm{d}\eta\right\},
\end{aligned} \tag{68}$$

where $(J_1)^e = \left[(J_1)_1^e \quad (J_1)_2^e \right]^{\mathrm{T}}$; $(J_2)^e = \left[(J_2)_1^e \quad (J_2)_2^e \right]^{\mathrm{T}}$, with

$$(J_1)_1^e = \frac{\sqrt{\left((x_1)_3^e-(x_1)_4^e\right)^2+\left((x_2)_3^e-(x_2)_4^e\right)^2}}{2}; \quad (J_1)_2^e = \frac{\sqrt{\left((x_1)_1^e-(x_1)_2^e\right)^2+\left((x_2)_1^e-(x_2)_2^e\right)^2}}{2}; \tag{69}$$
$$(J_2)_1^e = \frac{\sqrt{\left((x_1)_2^e-(x_1)_3^e\right)^2+\left((x_2)_2^e-(x_2)_3^e\right)^2}}{2}; \quad (J_2)_2^e = \frac{\sqrt{\left((x_1)_1^e-(x_1)_4^e\right)^2+\left((x_2)_1^e-(x_2)_4^e\right)^2}}{2},$$

for any $\hat{\mathbf{w}} \in \mathcal{V}_{ad}^h$.

In equality ,(68) the shape function matrix $N(\xi,\eta)$ is by the form:

$$N(\xi,\eta) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}, \tag{70}$$

with

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta); \quad N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta),$$
$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta); \quad N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta), \tag{71}$$

and the strain-displacement matrix $B^e(\xi,\eta)$

$$B^e(\xi,\eta) = \begin{bmatrix} B_{11}^e & 0 & B_{12}^e & 0 & B_{13}^e & 0 & B_{14}^e & 0 \\ 0 & B_{21}^e & 0 & B_{22}^e & 0 & B_{23}^e & 0 & B_{24}^e \\ B_{21}^e & B_{11}^e & B_{22}^e & B_{12}^e & B_{23}^e & B_{13}^e & B_{24}^e & B_{14}^e \\ B_{12}^e & B_{11}^e & B_{22}^e & B_{21}^e & B_{32}^e & B_{31}^e & B_{42}^e & B_{41}^e \end{bmatrix}. \tag{72}$$

In the discrete form ,(68), for simplicity, we introduce the following notations:

$$\left[f^{\mathrm{int}}\left(\hat{\mathbf{u}}_{n+1}^e\right)\right] = \int_{-1}^{1}\int_{-1}^{1}\left[\left(B^e(\xi,\eta)\right)^{\mathrm{T}} \boldsymbol{\sigma}\left(\hat{\mathbf{u}}_{n+1}^e\right)(\xi,\eta)\left|J^e(\xi,\eta)\right|\right]\mathrm{d}\xi\mathrm{d}\eta$$
$$\left[f_1^{ext}\right]_{n+1}^e = \int_{-1}^{1}\int_{-1}^{1}\left[\left(N(\xi,\eta)\right)^{\mathrm{T}} N(\xi,\eta)\hat{\mathbf{b}}_{n+1}^e\left|J^e(\xi,\eta)\right|\right]\mathrm{d}\xi\mathrm{d}\eta$$
$$\left[\left(f_2^{ext}\right)_i\right]_{n+1}^e = \int_{-1}^{1}\left[\left(N(\xi,-2(i-1)+1)\right)^{\mathrm{T}} N(\xi,-2(i-1)+1)\hat{\mathbf{f}}_{n+1}^e (J_1)_i^e\right]\mathrm{d}\xi \quad . \tag{73}$$
$$\left[\left(f_3^{ext}\right)_i\right]_{n+1}^e = \int_{-1}^{1}\left[\left(N(-2(i-1)+1,\eta)\right)^{\mathrm{T}} N(-2(i-1)+1,\eta)\hat{\mathbf{f}}_{n+1}^e (J_2)_i^e\right]\mathrm{d}\eta$$

The integrals of the form . (73) are approximated by the *Gauss method*. Given the above, the equality ,(68) can be written as

$$\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}_{n+1}^{e}\right)^{\mathrm{T}}\left[f^{\mathrm{int}}\left(\hat{\mathbf{u}}_{n+1}^{e}\right)\right]\right\}=\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}_{n+1}^{e}\right)^{\mathrm{T}}\left[f_{3}^{ext}\right]_{n+1}^{e}\right\}+ \\ +\sum_{i=1}^{2}\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}_{n+1}^{e}\right)^{\mathrm{T}}\left[\left(f_{1}^{ext}\right)_{i}\right]_{n+1}^{e}\right\}+\sum_{i=1}^{2}\sum_{e=1}^{nel}\left\{\left(\hat{\mathbf{w}}_{n+1}^{e}\right)^{\mathrm{T}}\left[\left(f_{2}^{ext}\right)_{i}\right]_{n+1}^{e}\right\} . \tag{74}$$

In the following, the assembled matrix of internal forces and vectors of the external forces are denoted by

$$F^{\mathrm{int}}\left(\hat{\mathbf{u}}_{n+1}\right)=\mathop{A}_{e=1}^{nel}\left[f^{\mathrm{int}}\left(\hat{\mathbf{u}}_{n+1}^{e}\right)\right];\quad \left(F_{1}^{ext}\right)_{n+1}=\mathop{A}_{e=1}^{nel}\left[f_{1}^{ext}\right]_{n+1}^{e};$$
$$\left(F_{2}^{ext}\right)_{n+1}=\mathop{A}_{e=1}^{nel}\left[\left[\left(f_{2}^{ext}\right)_{1}\right]_{n+1}^{e}+\left[\left(f_{2}^{ext}\right)_{2}\right]_{n+1}^{e}\right]; \tag{75}$$
$$\left(F_{3}^{ext}\right)_{n+1}=\mathop{A}_{e=1}^{nel}\left[\left[\left(f_{3}^{ext}\right)_{1}\right]_{n+1}^{e}+\left[\left(f_{3}^{ext}\right)_{2}\right]_{n+1}^{e}\right].$$

We also use the following notation:

$$F_{n+1}^{ext}=\sum_{i=1}^{3}\left(F_{i}^{ext}\right)_{n+1}. \tag{76}$$

Then, discrete variational formulation . (74) yields

$$\tilde{G}\left(\hat{\mathbf{u}}_{n+1},\hat{\mathbf{w}}_{n+1}\right)=\hat{\mathbf{w}}_{n+1}^{T}\left[F^{\mathrm{int}}\left(\hat{\mathbf{u}}_{n+1}\right)-F_{n+1}^{ext}\right]=0,\quad \forall\hat{\mathbf{w}}\in\mathcal{V}_{ad}^{h} . \tag{77}$$

Consequently, the resulting system of non-linear equations (77) can be written as (2).

4.3. Radial Return Algorithm for the model with non-linear hardening

We use the Radial Return Mapping Algorithm which has been presented in the previous section 3.1, this time for the evolution equations written in (1). At the moment t_{n+1} we have the elastic type constitutive equation (19) together with

$$\boldsymbol{\varepsilon}_{n+1}^{p}=\boldsymbol{\varepsilon}_{n}^{p}+\lambda_{n+1}^{*}\frac{\mathbf{s}_{n+1}-\boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1}-\boldsymbol{\alpha}_{n+1}\right\|};\quad k_{n+1}=k_{n}+\lambda_{n+1}^{*};$$
$$\boldsymbol{\alpha}_{n+1}=\boldsymbol{\alpha}_{n}+\sqrt{\frac{2}{3}}(H(k_{n+1})\text{ - }H(k_{n}))\frac{\mathbf{s}_{n+1}-\boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1}-\boldsymbol{\alpha}_{n+1}\right\|}, \tag{78}$$

where $\lambda^*_{n+1} = \lambda\Delta t = \lambda(t_{n+1} - t_n)$ satisfy the discrete Kuhn-Tucker conditions

$$\mathcal{F}(\mathbf{s}_{n+1}, \boldsymbol{\alpha}_{n+1}, k_{n+1}) \leq 0, \quad \lambda^*_{n+1} \geq 0, \quad \lambda^*_{n+1}\mathcal{F}(\mathbf{s}_{n+1}, \boldsymbol{\alpha}_{n+1}, k_{n+1}) = 0 . \tag{79}$$

The yield function is defined by:

$$\mathcal{F}\left(\mathbf{s}_{n+1}, \boldsymbol{\alpha}_{n+1}, k_{n+1}\right) = \left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\| - \sqrt{\frac{2}{3}}F(k_{n+1}) . \tag{80}$$

Elastic Predictor. We define trial elastic stress tensor like in (23) and in this stage $\boldsymbol{\varepsilon}^p_{n+1}, \boldsymbol{\alpha}_{n+1}$ and k_{n+1} are frozen as previous values like in (24). The test function $\mathcal{F}^{tr}_{n+1}$ is defined as in (25).

The second step of the algorithm, *Plastic corrector and return mapping algorithm* is further applied. If the elastic trial state is outside the yield surface, then the plastic factor can not be zero and the procedure to restore the consistency condition follows:

$$\begin{aligned} \mathbf{s}_{n+1} &= \mathbf{s}^{tr}_{n+1} - 2\mu\lambda^*_{n+1}\frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|}, \\ \boldsymbol{\alpha}_{n+1} &= \boldsymbol{\alpha}_n + \sqrt{\frac{2}{3}}\left(H(k_{n+1}) - H(k_n)\right)\frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|}. \end{aligned} \tag{81}$$

We introduce the notation $\overline{\boldsymbol{\beta}}_{n+1} = \mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}$ and $\overline{\boldsymbol{\beta}}^{tr}_{n+1} = \mathbf{s}^{tr}_{n+1} - \boldsymbol{\alpha}_n$ and as a direct consequence of the collinearity of $\overline{\boldsymbol{\beta}}_{n+1}$ and $\overline{\boldsymbol{\beta}}^{tr}_{n+1}$, i.e.

$$\mathbf{n}_{n+1} = \frac{\overline{\boldsymbol{\beta}}_{n+1}}{\left\|\overline{\boldsymbol{\beta}}_{n+1}\right\|} = \frac{\overline{\boldsymbol{\beta}}^{tr}_{n+1}}{\left\|\overline{\boldsymbol{\beta}}^{tr}_{n+1}\right\|} . \tag{82}$$

In this conditions, from (81) we have:

$$\overline{\boldsymbol{\beta}}_{n+1} = \overline{\boldsymbol{\beta}}^{tr}_{n+1} - \left(2\mu\lambda^*_{n+1} + \sqrt{\frac{2}{3}}\left(H(k_{n+1}) - H(k_n)\right)\right)n_{n+1} . \tag{83}$$

PROPOSITION 6. *The following loading/unloading conditions take place if* $\mathcal{F}^{tr}_{n+1} \leq 0$ then $\lambda^*_{n+1} = 0$ *and the solution is an elastic one*

$$\begin{aligned} &\boldsymbol{\sigma}^{tr}_{n+1} = k_a \mathrm{tr}\boldsymbol{\varepsilon}_{n+1}1 + 2\mu(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^p_n); \quad \mathbf{s}^{tr}_{n+1} = \mathbf{s}_n + 2\mu\left(\mathrm{dev}\,\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^p_n\right) \\ &\boldsymbol{\varepsilon}^p_{n+1} \equiv \boldsymbol{\varepsilon}^p_n, \quad \boldsymbol{\alpha}_{n+1} \equiv \boldsymbol{\alpha}_n, \quad k_{n+1} \equiv k_n . \end{aligned} \tag{84}$$

If $\mathcal{F}^{tr}_{n+1} > 0$, then $\lambda_{n+1} \neq 0$ so the plastic factor λ^*_{n+1} is the solution of the nonlinear equation:

$$\left\|\overline{\boldsymbol{\beta}}^{tr}_{n+1}\right\| - \sqrt{\frac{2}{3}}F(k_{n+1}) - \left[2\mu\lambda^{*}_{n+1} + \sqrt{\frac{2}{3}}\left(H(k_n + \sqrt{\frac{2}{3}}\lambda^{*}_{n+1}) - H(k_n)\right)\right] = 0 . \quad (85)$$

Moreover, the plastic solution is described by the following relationships

$$\mathbf{s}_{n+1} = \mathbf{s}^{tr}_{n+1} - 2\mu\lambda^{*}_{n+1}\frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|},$$

$$\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n + \sqrt{\frac{2}{3}}\left(H(k_{n+1}) - H(k_n)\right)\frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|}, \quad (86)$$

$$\boldsymbol{\varepsilon}^{p}_{n+1} = \boldsymbol{\varepsilon}^{p}_{n} + \lambda^{*}_{n+1}\frac{\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}}{\left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|}, \quad k_{n+1} = k_n + \sqrt{\frac{2}{3}}\lambda^{*}_{n+1}.$$

Proof. The plastic factor λ^{*}_{n+1} is calculated from the discrete consistency condition. From the relation . (82) we have:

$$\overline{\boldsymbol{\beta}}_{n+1} = \left\|\overline{\boldsymbol{\beta}}_{n+1}\right\|\mathbf{n}_{n+1} \quad \text{and} \quad \overline{\boldsymbol{\beta}}^{tr}_{n+1} = \left\|\overline{\boldsymbol{\beta}}^{tr}_{n+1}\right\|\mathbf{n}_{n+1} \quad (87)$$

and the equality (83) becomes:

$$\left\|\overline{\boldsymbol{\beta}}_{n+1}\right\|\mathbf{n}_{n+1} = \left\|\overline{\boldsymbol{\beta}}^{tr}_{n+1}\right\|\mathbf{n}_{n+1} - \left(2\mu\lambda^{*}_{n+1} + \sqrt{\frac{2}{3}}\left(H(k_{n+1}) - H(k_n)\right)\right)\mathbf{n}_{n+1} \quad (88)$$

or

$$2\mu\lambda^{*}_{n+1} + \sqrt{\frac{2}{3}}\left(H(k_{n+1}) - H(k_n)\right) = \left\|\overline{\boldsymbol{\beta}}^{tr}_{n+1}\right\| - \left\|\overline{\boldsymbol{\beta}}_{n+1}\right\| . \quad (89)$$

On the yield surface $\tilde{\mathcal{F}}\left(\overline{\boldsymbol{\beta}}_{n+1}, k_{n+1}\right) = 0$ the equality $\left\|\overline{\boldsymbol{\beta}}_{n+1}\right\| = F(k_{n+1})$ holds, where $\left\|\overline{\boldsymbol{\beta}}_{n+1}\right\| = \left\|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\right\|$ and

$$F(k_{n+1}) = Q\left(1 - e^{-b\left(k_n + \sqrt{2/3}\lambda^{*}_{n+1}\right)}\right) + \sigma_Y . \quad (90)$$

Consequently, the plastic factor λ^{*}_{n+1} is determined from the following nonlinear equation

$$\left\|\overline{\beta}^{tr}_{n+1}\right\| - \sqrt{\frac{2}{3}}F\left(k_{n+1}\right) - \left[2\mu\lambda^{*}_{n+1} + \sqrt{\frac{2}{3}}\left(H\left(k_n + \sqrt{\frac{2}{3}}\lambda^{*}_{n+1}\right) - H\left(k_n\right)\right)\right] = 0 . \quad (91)$$

To solve the equation (91) we apply the Newton method, which consists in the following iterative procedure:

$$\lambda_{n+1}^{*,j+1} = \lambda_{n+1}^{*,j} - \frac{f(\lambda_{n+1}^{*,j})}{f'(\lambda_{n+1}^{*,j})}, \tag{92}$$

where the function $f(\lambda_{n+1}^{*,j})$ is given by

$$f\left(\lambda_{n+1}^{*,j}\right) = \left\|\bar{\boldsymbol{\beta}}_{n+1}^{tr}\right\| - \sqrt{\frac{2}{3}}F\left(k_{n+1}^{j}\right) - \left[2\mu\lambda_{n+1}^{*,j} + \sqrt{\frac{2}{3}}\left(H(k_n + \sqrt{\frac{2}{3}}\lambda_{n+1}^{*,j}) - H(k_n)\right)\right] \tag{93}$$

and $f'(\lambda_{n+1}^{*,j})$ is the derivative of the function $f(\lambda_{n+1}^{*,j})$:

$$f'(\lambda_{n+1}^{*}) = -2\mu - \frac{2}{3}\left(H'(k_{n+1}) + F'(k_{n+1})\right) = -2\mu\left(1 + \frac{H'(k_{n+1}) + F'(k_{n+1})}{3\mu}\right), \tag{94}$$

where $F'(k_{n+1}) = Qb\,\mathrm{e}^{-b(k_{n+1})}$ in numerical application.

The iterations are ended when a desired accuracy has been reached, i.e., when $\left|f(\lambda_{n+1}^{*,j})\right|$ falls into a prescribed error tolerance.

The Radial Return Algorithm is completely determined, if the algorithmic elastic-plastic tangent modulus $\mathcal{E}^{ep}$ is calculated.

PROPOSITION 7. *The algorithmic elasto-plastic tangent modulus can be approximated by the following expression*:

$$\left[\mathcal{E}^{ep}\right]_{n+1}^{e} = \begin{cases} \mathcal{E} & , \mathcal{F}^{tr} \leqslant 0 \\ k_a 1\otimes 1 + 2\mu_a\left[\theta_1\right]_{n+1}\mathrm{I}_{\mathrm{dev}} - 2\mu_a\left[\theta_2\right]_{n+1}\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}, & \mathcal{F}^{tr} > 0 \end{cases} \tag{95}$$

where

$$\left[\theta_1\right]_{n+1} = \frac{1 - 2\mu\lambda_{n+1}^{*}}{\left\|\bar{\boldsymbol{\beta}}_{n+1}^{tr}\right\|};$$
$$\left[\theta_2\right]_{n+1} = \left(1 + \frac{H'(k_{n+1}) + F'(k_{n+1})}{3\mu}\right)^{-1} - \left(1 - \left[\theta_1\right]_{n+1}\right). \tag{96}$$

Proof. The algorithmic elasto-plastic tangent modulus is calculated from the relation:

$$\mathcal{E}^{ep} = \frac{\partial\boldsymbol{\sigma}_{n+1}}{\partial\boldsymbol{\varepsilon}(\hat{u}_{n+1})}. \tag{97}$$

We differentiate the discrete constitutive equation (19) with respect to $\boldsymbol{\varepsilon}$:

$$\frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}(\hat{u}_{n+1})}=\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\varepsilon}(\hat{u}_{n+1})}\Big[k_a\left(\mathrm{tr}\boldsymbol{\varepsilon}(\hat{u}_{n+1})\right)1+2\mu(\mathrm{dev}\boldsymbol{\varepsilon}(\hat{u}_{n+1})-\boldsymbol{\varepsilon}_n^p)-2\mu\lambda_{n+1}^*\mathbf{n}_{n+1}\Big]=$$
$$=k_a 1\otimes 1+2\mu\mathrm{I}_{\mathrm{dev}}-2\mu\left(\lambda_{n+1}^*\frac{\partial n_{n+1}}{\partial\boldsymbol{\varepsilon}(\hat{u}_{n+1})}+\mathbf{n}_{n+1}\otimes\frac{\partial\lambda_{n+1}^*}{\partial\boldsymbol{\varepsilon}(\hat{u}_{n+1})}\right). \tag{98}$$

In the relation (98) we observe that the term $k_a 1\otimes 1+2\mu\mathrm{I}_{\mathrm{dev}}=\mathcal{E}$ is the tensor of elastic moduli, where $\mathrm{I}_{\mathrm{dev}}=\mathrm{I}-\frac{1}{3}1\otimes 1$ is the second order deviator tensor.

We calculate in the following, the expression of the derivatives witch appearing in (98). Thus

$$\frac{\partial\mathbf{n}_{n+1}}{\partial\boldsymbol{\varepsilon}(\hat{u}_{n+1})}=\frac{\partial\mathbf{n}_{n+1}}{\partial\overline{\boldsymbol{\beta}}_{n+1}^{tr}}\frac{\partial\overline{\boldsymbol{\beta}}_{n+1}^{tr}}{\partial\boldsymbol{\varepsilon}(\hat{u}_{n+1})}=\left(\mathrm{I}-\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}\right)\frac{2\mu}{\left\|\overline{\boldsymbol{\beta}}_{n+1}^{tr}\right\|}\mathrm{I}_{\mathrm{dev}}. \tag{99}$$

The derivative of the plastic factor with respect to the total strain at time t_{n+1} can be derived from the consistency condition written in (9)

$$\frac{\partial}{\partial\boldsymbol{\varepsilon}_{n+1}}\left(\left\|\overline{\boldsymbol{\beta}}_{n+1}^{tr}\right\|-\sqrt{\frac{2}{3}}F(k_{n+1})-\left[2\mu\lambda_{n+1}^*+\sqrt{\frac{2}{3}}\left(H(k_n+\sqrt{\frac{2}{3}}\lambda_{n+1}^*)-H(k_n)\right)\right]\right)=0$$
$$\frac{\partial\lambda_{n+1}^*}{\partial\boldsymbol{\varepsilon}(\hat{u}_{n+1})}\left[2\mu\left(1+\frac{H'(k_{n+1})+F'(k_{n+1})}{3\mu}\right)\right]=2\mu\mathbf{n}_{n+1}\mathrm{I}_{\mathrm{dev}} \tag{100}$$
$$\frac{\partial\lambda_{n+1}^*}{\partial\boldsymbol{\varepsilon}(\hat{u}_{n+1})}=\left(1+\frac{H'(k_{n+1})+F'(k_{n+1})}{3\mu}\right)^{-1}\mathbf{n}_{n+1}.$$

Consequently, the following formula can be proved

$$\frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}(\hat{u}_{n+1})}=k_a 1\otimes 1+2\mu_a\left[\theta_1\right]_{n+1}\mathrm{I}_{\mathrm{dev}}-2\mu_a\left[\theta_2\right]_{n+1}\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}, \tag{101}$$

where

$$\left[\theta_1\right]_{n+1}=\frac{1-2\mu\lambda_{n+1}^*}{\left\|\overline{\boldsymbol{\beta}}_{n+1}^{tr}\right\|};$$
$$\left[\theta_2\right]_{n+1}=\left(1+\frac{H'(k_{n+1})+F'(k_{n+1})}{3\mu}\right)^{-1}-\left(1-\left[\theta_1\right]_{n+1}\right). \tag{102}$$

Newton Raphson algorithm The algorithmic elastic-plastic tangent modulus is involved in the Newton Raphson method which is applied for solving the nonlinear equation (2). The unknown $\hat{u}_{n+1}$ is obtained based on the following iterative relation:

$$\hat{\mathbf{u}}_{n+1}^{j+1} = \hat{\mathbf{u}}_{n+1}^{j} - \beta_j \left[K_{n+1}^{j} \right]^{-1} \hat{G}\left(\hat{\mathbf{u}}_{n+1}^{j}\right), \tag{103}$$

where $\beta_j \in (0,1]$ and the Jacobian of the function $\hat{G}$ is given by the following expression:

$$\begin{aligned} \left[K\left(\hat{\mathbf{u}}_{n+1}\right)\right]_{i,k} &= \frac{\partial \hat{G}_i\left(\hat{\mathbf{u}}_{n+1}\right)}{\partial \left(\hat{\mathbf{u}}_{n+1}\right)_k} = \mathop{A}_{e=1}^{nel} \left[\frac{\partial \left[\left(f^{\text{int}}\right)_i^e \right]_{n+1}}{\partial \left(\hat{\mathbf{u}}_{n+1}\right)_k} \right] = \\ &= \mathop{A}_{e=1}^{nel} \left[\int_{-1}^{1}\int_{-1}^{1} \left[\left(B^e(\xi,\eta)\right)^{\mathrm{T}} \left(\mathcal{E}^{ep}\right)^e B^e(\xi,\eta) \left| J^e(\xi,\eta) \right| \right] \mathrm{d}\xi \mathrm{d}\eta \right]. \end{aligned} \tag{104}$$

Finally the Radial Return Algorithm can be assembled

$$\begin{aligned} &\left\{ [\boldsymbol{\sigma}]_{n+1}^{e,j+1}, \left[\boldsymbol{\varepsilon}^p\right]_{n+1}^{e,j+1}, [\boldsymbol{\alpha}]_{n+1}^{e,j+1}, [k]_{n+1}^{e,j+1} \right\} = \\ &= \textbf{Radial Return Algorithm} \left\{ \mathbf{u}_{n+1}^{e,j}, [\boldsymbol{\sigma}]_{n+1}^{e}, \left[\boldsymbol{\varepsilon}^p\right]_{n}^{e}, [\boldsymbol{\alpha}]_{n+1}^{e}, [k]_{n}^{e} \right\}. \end{aligned}$$

1. compute the trial elastic stress for prescribed $[\boldsymbol{\varepsilon}]_{n+1}^{j+1} = [B]\hat{\mathbf{u}}_{n+1}^{j}$:
   $\boldsymbol{\sigma}_{n+1}^{tr} = k_a \mathrm{tr}\boldsymbol{\varepsilon}_{n+1} 1 + 2\mu(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p)$; $\mathbf{s}_{n+1}^{tr} = \mathbf{s}_n + 2\mu\left(\mathrm{dev}\,\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p\right)$;
2. check the yield condition $\tilde{\mathcal{F}}_{n+1}^{tr} := \left\| \overline{\boldsymbol{\beta}}_{n+1}^{tr} \right\| - \sqrt{\frac{2}{3}} \left[Q(1 - e^{-bk_n}) + \sigma_Y \right]$:

   If $\tilde{\mathcal{F}}_{n+1}^{tr} \le 0$, then:
   - elastic step:

   $\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{tr}$; $\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{tr}$; $\boldsymbol{\varepsilon}_{n+1}^p \equiv \boldsymbol{\varepsilon}_n^p$; $\boldsymbol{\alpha}_{n+1} \equiv \boldsymbol{\alpha}_n$; $k_{n+1} \equiv k_n$; $\mathcal{E}_{n+1}^{ep} = \mathcal{E}$

   else
   - plastic step: proceed to step 3.

   endif
3. Radial Return Algorithm:
- compute $\mathbf{n}_{n+1}$ and find λ_{n+1}^* with Newton method:

$$\mathbf{n}_{n+1} = \frac{\overline{\boldsymbol{\beta}}^{tr}_{n+1}}{\left\|\overline{\boldsymbol{\beta}}^{tr}_{n+1}\right\|}$$

Initialize: $\lambda^{*,0}_{n+1} = 0$ and $k^{0}_{n+1} = 0$

Do until $\left| f(\lambda^{*,j}_{n+1}) \right| < \mathrm{ERR}$,

$j \leftarrow j+1$

Compute iterate

$$f(\lambda^{*,j}_{n+1}) = \left\|\overline{\boldsymbol{\beta}}^{tr}_{n+1}\right\| - \sqrt{\frac{2}{3}} F(k^{j}_{n+1}) - \left[2\mu\lambda^{*,j}_{n+1} + \sqrt{\frac{2}{3}} \left(H(k_n + \sqrt{\frac{2}{3}}\lambda^{*,j}_{n+1}) - H(k_n) \right) \right]$$

$$f'(\lambda^{*}_{n+1}) = -2\mu - \frac{2}{3}\left(H'(k_{n+1}) + F'(k_{n+1}) \right) = -2\mu \left(1 + \frac{H'(k_{n+1}) + F'(k_{n+1})}{3\mu} \right)$$

$$\lambda^{*,j+1}_{n+1} = \lambda^{*,j}_{n+1} - \frac{f(\lambda^{*,j}_{n+1})}{f'(\lambda^{*,j}_{n+1})}$$

- update isotropic hardening variable:

$$k^{j+1}_{n+1} = k^{j}_{n} + \sqrt{\frac{2}{3}}\lambda^{*,j+1}_{n+1} .$$

- update stress, hardening variable and compute the algorithmic elasto-plastic tangent moduli:

$$\mathbf{s}_{n+1} = \mathbf{s}^{tr}_{n+1} - 2\mu\lambda^{*}_{n+1}\mathbf{n}_{n+1} \; ; \quad \boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n + \sqrt{\frac{2}{3}}\left(H(k_{n+1}) - H(k_n) \right)\mathbf{n}_{n+1} \; ;$$

$$\boldsymbol{\varepsilon}^{p}_{n+1} = \boldsymbol{\varepsilon}^{p}_{n} + \lambda^{*}_{n+1}\mathbf{n}_{n+1} \; ;$$

$$\mathcal{E}^{ep}_{n+1} = k_a 1 \otimes 1 + 2\mu_a \left[\theta_1\right]_{n+1} \mathrm{I}_{\mathrm{dev}} - 2\mu_a \left[\theta_2\right]_{n+1} \mathbf{n}_{n+1} \otimes \mathbf{n}_{n+1} \; ;$$

$$\left[\theta_1\right]_{n+1} = \frac{1 - 2\mu\lambda^{*}_{n+1}}{\left\|\xi^{tr}_{n+1}\right\|}; \quad \left[\theta_2\right]_{n+1} = \left(1 + \frac{H'(\mathrm{k}_{n+1}) + F'(\mathrm{k}_{n+1})}{3\mu} \right)^{-1} - \left(1 - \left[\theta_1\right]_{n+1} \right)$$

Return

Remark 5. The discrete functions $\left[\theta_1\right]_{n+1}$ and $\left[\theta_2\right]_{n+1}$ are dependent on the evolution equation for the back stress, which is involved into the model, as can be seen from (96) and (33).

5. NUMERICAL APPLICATION

In order to exemplify the mode of integration of the problem, we consider the following bi-dimensional example. Numerical application presented here, aims to highlight the numerical algorithm of solving the Problem P. For this, we chose the trapezoidal panel domain (Figure 1), for which the vertical left edge is fixed, and the bottom and the right vertical edges are traction free, i.e. $f = 0$. The traction $f = 50\sin t$ is applied on the top horizontal edge, while the internal forces are assumed to vanish $b = 0$. Inside of this plates is inserted a hole with a radius of 20 mm. The application is realized for a time interval $t \in [0,60]$. The sheet is made of steel DP600, with the material parameters given by Broggiato et al. [4]
$\{E = 182\,000 \text{ MPa};\ \sigma_Y = 394{,}4 \text{ MPa};\ \nu = 0{,}3;\ C = 17\,400 \text{ MPa};\ H = 1194 \text{ MPa}\}$.

The Ω domain for this example is by the form:

$$\Omega = \Omega_2 \setminus \Omega_1 \ \rightarrow \begin{cases} \Omega_2 = \left\{ (x_1, x_2) \subset \mathbb{R}^2 \,\middle|\, 0 \leqslant x_1 \leqslant 20\,, \dfrac{x_1}{4} \leqslant x_2 \leqslant 100 \right\} \\ \Omega_1 = \left\{ (x_1, x_2) \subset \mathbb{R}^2 \,\middle|\, (x_1 - 80)^2 + (x_2 - 60)^2 \leqslant 20 \right\} \end{cases}. \quad (105)$$

The numerical example put into evidence the behavior of the sheet, made of an elasto-plastic material with mixed hardening, when the Prager hardening law describes the evolution in time of the back stress.

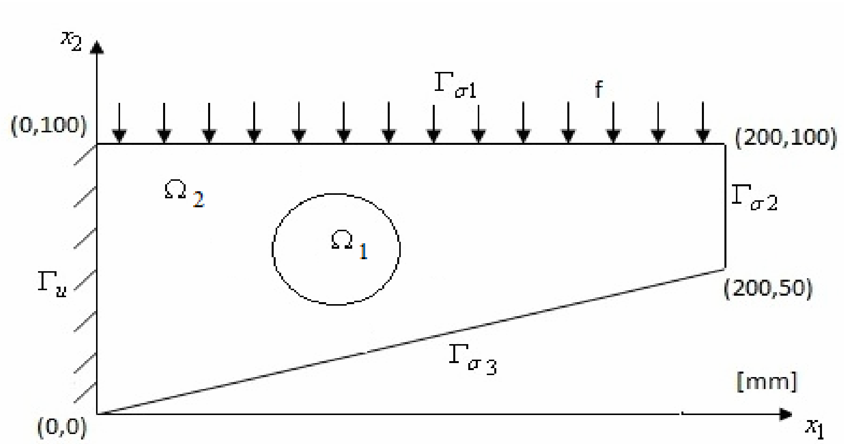

Fig. 1 – The geometry of the plate with the boundary σ_{ij} conditions.

In Cleja-Tigoiu and Stoicuta's paper [7] the numerical procedure to solve the boundary value problem has been illustrated for the deformation in plane stress of the trapezoidal sheet. We exemplify here the behavior in the sheet through the graphs of the stress components , versus the corresponding strain components ϵ_{ij}, produced in the four points of the trapezoidal plate with a hole. The cyclic behavior can be observed, as well as the peculiar behavior in compression and tension in the chosen material points. We specify also that for the meshing, the trapezoidal plate with hole was divided into 2 500 elements, each element having four nodes. The distribution of the stress component in the direction of the applied force σ_{11}, and the intensity of the Cauchy stress, i.e. the so-called Mises stress, which is denoted by σ_{Mises}, are plotted on the trapezoidal plate in Figs. 2 and 3.

The graphs are obtained for mixed hardening elasto-plastic materials, using the Cleja-Stoicuta solution for the in-plane stress and linear isotropic hardening given by (4) and the Simo-Hudges algorithm, with the nonlinear isotropic hardening described by (5). Figures 4–9 show the components of the stress versus the appropriate components of the strain. The components of the total strain ε_{33} and the plastic strain ε_{33}^{p} are plotted as function of time at the mentioned material point in Fig. 10.

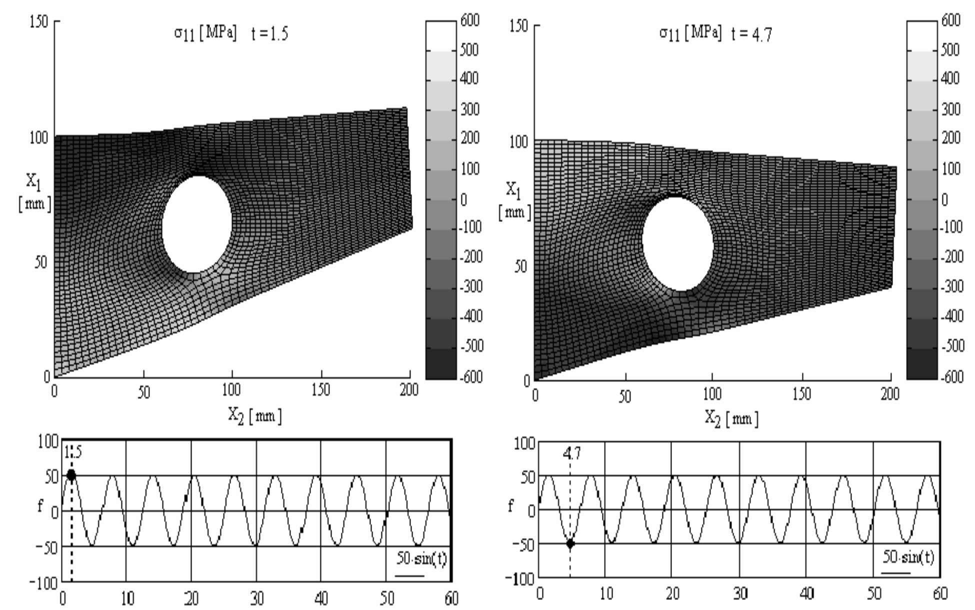

Fig. 2 – The distribution of the stress σ_{11} on the trapezoidal plate with hole at $t = 1.5$ and $t = 4.7$ which correspond to the minimum and maximum of the applied force *f*.

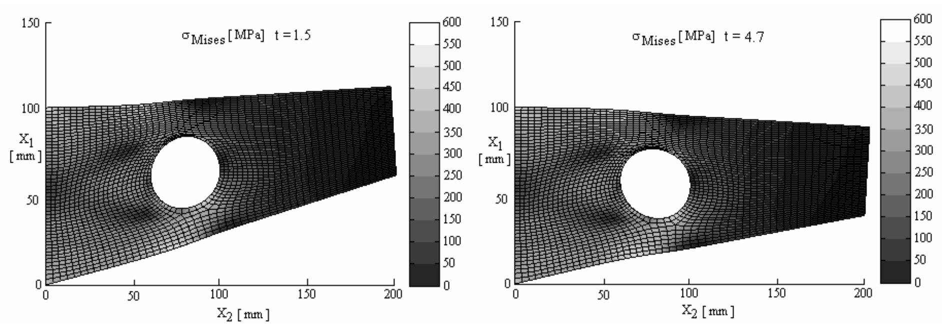

Fig. 3 – The distribution of the stress σ_{Mises} on the trapezoidal plate with hole at $t = 1.5$ and $t = 4.7$ which correspond to the minimum and maximum of the applied force *f* given as in Fig. 2.

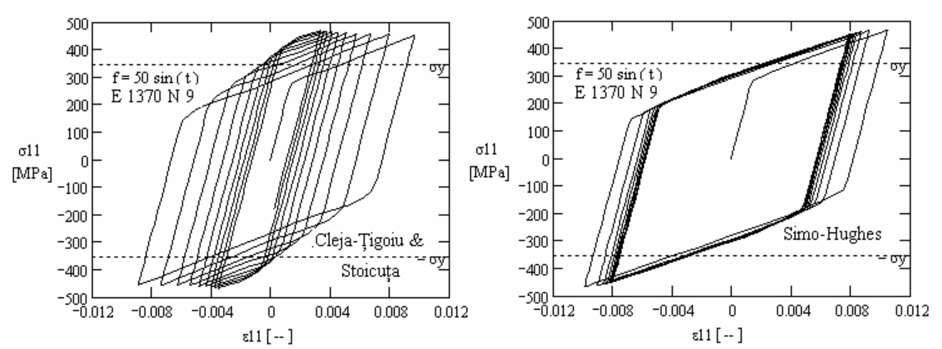

Fig. 4 – The component of the stress σ_{11} *versus* the strain ε_{11} at the material points given by node 9 obtained using the Cleja-Tigoiu & Stoicuta solution and the Simo-Hughes solution.

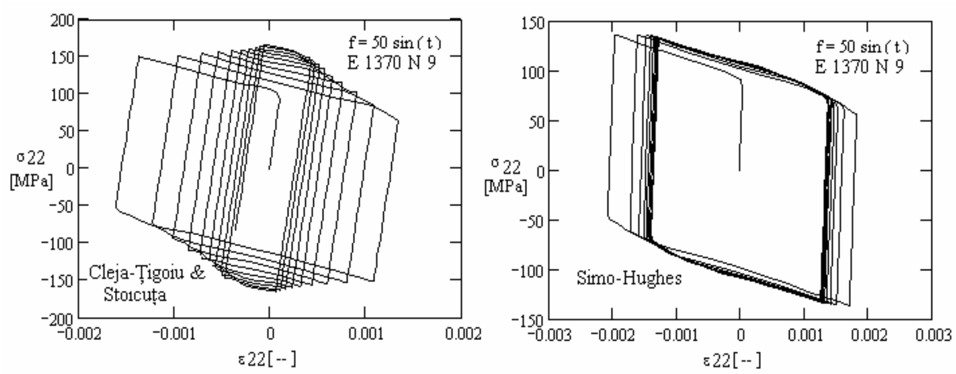

Fig. 5 – The component of the stress σ_{22} *versus* the strain ε_{22} at the material points given by node 9, obtained using the Cleja-Tigoiu & Stoicuta solution and the Simo-Hughes solution.

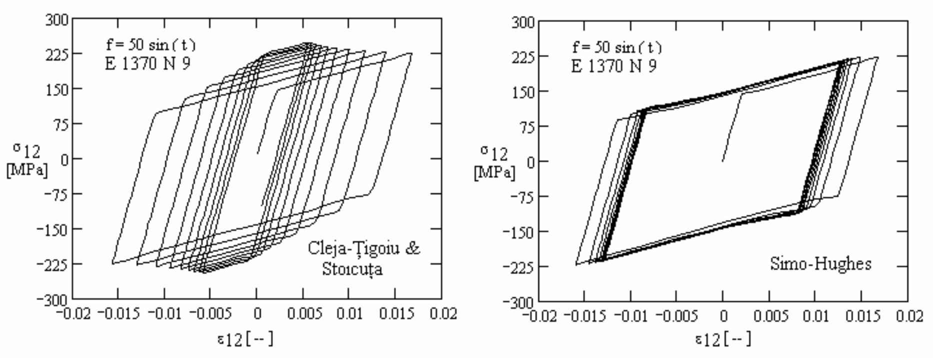

Fig. 6 – The component of the stress σ_{12} *versus* the strain ε_{12} at the material points given by node 9, obtained using the Cleja-Tigoiu & Stoicuta solution and the Simo-Hughes solution.

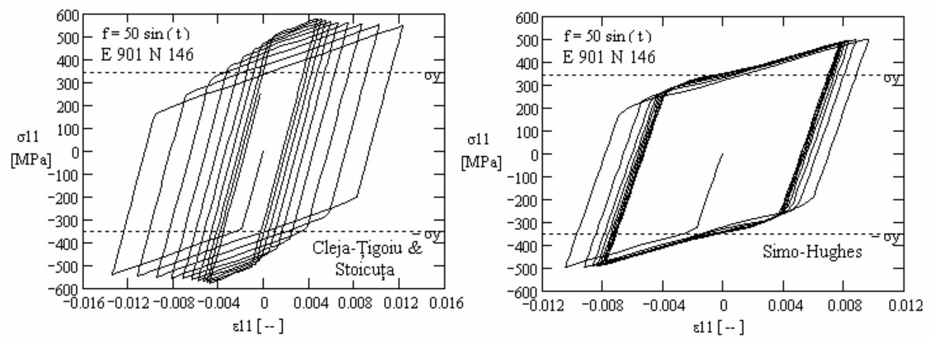

Fig. 7 – The component of the stress σ_{11} *versus* the strain ε_{11} at the material points given by node 146, obtained using the Cleja-Tigoiu & Stoicuta solution and the Simo-Hughes solution.

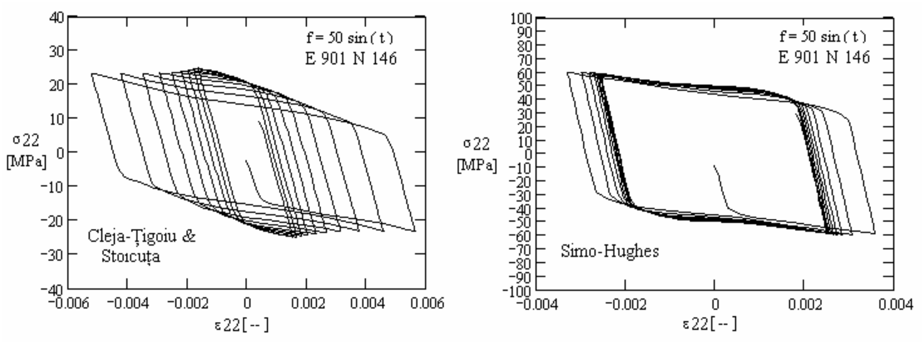

Fig. 8 – The component of the stress σ_{22} *versus* the strain ε_{22} at the material points given by node 146, obtained using the Cleja-Tigoiu & Stoicuta solution and the Simo-Hughes solution.

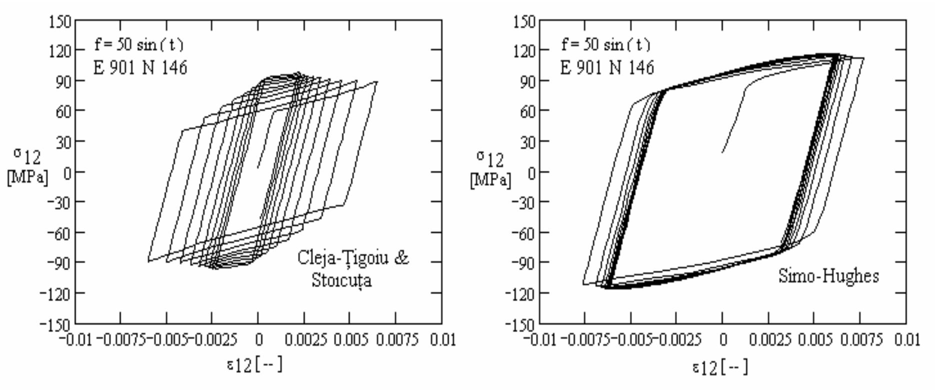

Fig. 9 – The component of the stress σ_{12} *versus* the strain ε_{12} at the material points given by node 146, obtained using the Cleja-Tigoiu & Stoicuta solution and the Simo-Hughes solution.

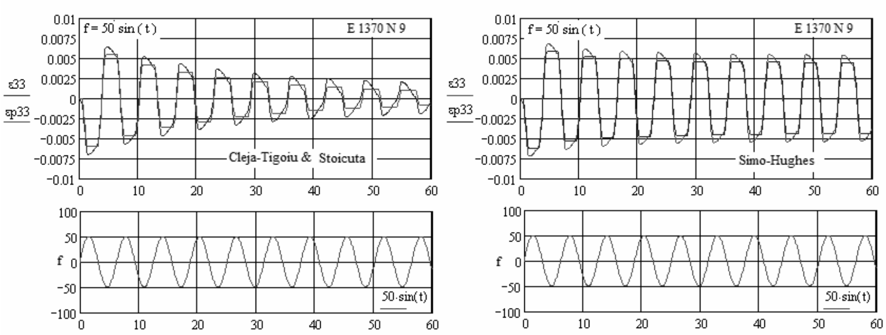

Fig. 10 – The graphs of the components of the strain ε_{33} and of the plastic strain ε_{33}^{p} plotted as functions of time at the material points given by node 9, obtained using the Cleja-Tigoiu & Stoicuta solution and the Simo-Hughes solution.

6. CONCLUSIONS

Concerning the comparison of the numerical results using the Simo-Hughes algorithm and the in-plane algorithm, we note that the material responses are completely different on one side for the considered methods, and the other hand for the points located in the vicinity of the loading boundary, in Figs. 4–6, and points close to the free boundary, Figs. 7–9.

Figures 4–6 show that the stresses take comparable values, while the strains are larger for the Simo-Hughes algorithm than in plane algorithm. Figures 7–9 show large differences between the components of the stress, especially for the shear and normal component in the perpendicular direction of the applied force.

Finally we remark that the Chaboche isotropic hardening low, (5), leads to the existence of *a limit cycle*, which characterizes the evolution of the curves stress versus strain after a large number of periods of the applied force. The linear isotropic hardening low, (4) leads to a *stress threshold* and *limit strain* after a large numbers of period of the applied force.

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