STOCHASTIC LINEARIZATION OF SYSTEMS WITH HYSTERETIC CHARACTERISTICS

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Abstract. In this paper is presented a linearization method for a stochastic differential system with hysteretic characteristics and random excitation. The Gaussian equivalent linearization technique is applied to a system with hysteretic characteristic modeled by a Bouc-Wen equation. In order to verify the efficiency of the linearization method, the standard deviations of the equivalent linear system response are compared with those of Bouc-Wen nonlinear system for two case studies: Gaussian white noise and seismic inputs.

Key words: stochastic linearization, hysteretic characteristic, random excitation, Bouc-Wen model.

1. INTRODUCTION

The Bouc-Wen model, widely used in structural and mechanical engineering, gives an analytical description of a smooth hysteretic behavior. It was introduced by Bouc [1] and extended by Wen [2], who demonstrated its versatility by producing a variety of hysteretic characteristics. The hysteretic behavior of materials, structural elements or vibration isolators is treated in a unified manner by a single strongly nonlinear differential equation.

One of the most efficient techniques for approximating non-linear models within the system operating domain is the linearization method, both for deterministic and stochastic inputs. An important advantage of this approach is, unlike other methods, it can be applied to complex systems having many degrees of freedom and different types of excitations. There are many studies regarding statistical linearization, [3–5] that have proved the efficiency of this approach.

The method of stochastic equivalent linearization has its roots in the work of Krylov and Bogoliubov [6] on deterministic linearization and it first appeared in papers by Caughey [7], Iwan [8] and others in a probabilistic framework. A great impulse to practical applications of the method was given by a paper of Atalik and

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Utku [9], who demonstrated that the assumption of a Gaussian behavior of all the state variables greatly simplifies the computation of the linearization coefficients. Later it was proved that such simplification is valid only for the Gaussian density [10]. Mathematical issues such as the existence and uniqueness of the solutions had already been discussed [11]. A monograph by Roberts and Spanos [5] on computational techniques in nonlinear stochastic dynamics shows that this method is suitable to practical analysis of large structures. On the other hand the introduction of a smooth and versatile model of hysteresis by Bouc and Wen [12–13] and later developed by others [14–16], opened the way to the application of the method in many structural dynamics fields.

2. LINEARIZATION METHOD

The linearization method is presented and applied for a nonlinear SDOF oscillating system with hysteretic characteristic shown in Fig. 1.

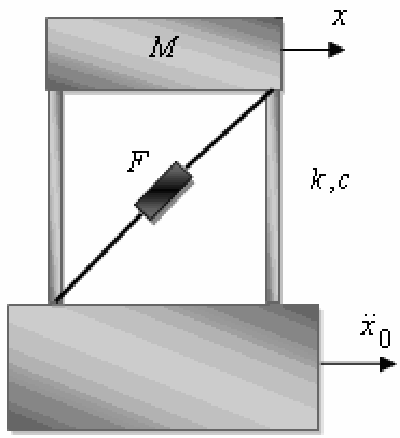

Fig. 1 – Schematic of mechanical system.

This system is often used in the analysis of the seismic response of share frame structures for the first vibration mode.

The equation of motion is:

$$M\ddot{x} + c\dot{x} + kx + F(x, \dot{x}) = -M\ddot{x}_0, \tag{1}$$

where $F(x,\dot{x})$ is the symmetric restoring force developed by the bracing hysteretic device within the operating range $-x_m \le x(t) \le x_m$, $-F_m \le F(x) \le F_m$. The system input $\ddot{x}_0$ is a Gaussian stationary white noise process.

Introducing the notations:

$$\xi(t) = x(t)/x_\mathrm{m}\,, \quad z(t) = F(x_\mathrm{m}\xi)/Mx_\mathrm{m}\,,$$
$$\xi_\mathrm{m} = \max\left|\,\xi(t)\right|, \;\; z_\mathrm{m} = \max\left|z(t)\right| = F_\mathrm{m}/Mx_\mathrm{m}\,, \tag{2}$$

a generic plot of the symmetric hysteresis loop $z(\xi)$ is represented in Fig. 2. For simplicity, the output $\xi(t)$ and input $\eta(t)$ will be given the same name as its physical counterparts $x(t)$ and $\ddot{x}_0(t)$.

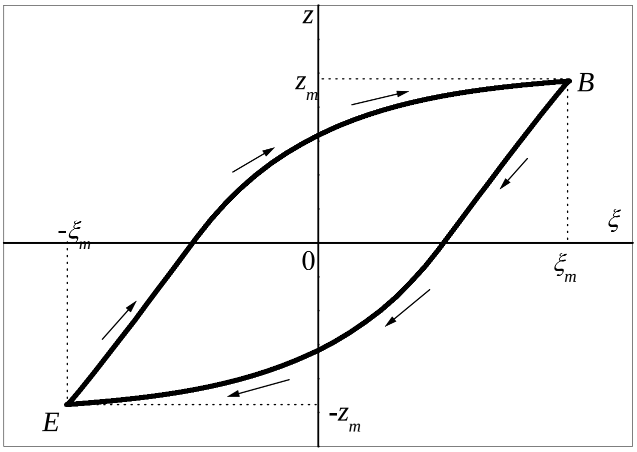

Fig. 2 – Generic experimental hysteretic loop.

Equation (1) can be written as:

$$\ddot{\xi} + 2\omega\zeta\dot{\xi} + \omega^2\xi + z = \eta(t)$$
$$\omega = \sqrt{\frac{k}{M}}, \;\; \zeta = \frac{c}{2\sqrt{kM}}, \;\; \eta(t) = -\frac{\ddot{x}_0}{x_\mathrm{m}}, \tag{3}$$

where $E[\eta(t')\,\eta(t'+t)] = 2\pi S_0\,\delta(t),\; S_\eta(\omega) = S_0$.

The Bouc-Wen model, chosen to fit the hysteresis loop shown in Figure 2, is described by the following non-linear differential equation:

$$\dot{z} = h(\dot{\xi}, z) = [A - |z|^n(\beta + \gamma\,\mathrm{sgn}(\dot{\xi}z))]\,\dot{\xi}, \tag{4}$$

where A, β, γ, n are loop parameters controlling the shape and magnitude of the hysteresis loop $z(\xi)$. For $n = 1$, the dimensions of model parameters are $[A] = T^{-2}$, $[\beta] = T^{-1}$, $[\gamma] = T^{-1}$.

Analytical relations derived from the Bouc-Wen differential equation to determine the parameters of the model such as the predicted hysteresis curves and the experimental loops to have same absolute values of the maximum force, same coordinates of the loop-axes crossing points, and same slopes of the tangent at these key points of hysteretic loops have been developed in [17]. As the measured data have a certain degree of imprecision, one of the most suitable methods for approximating the model parameters is the genetic algorithms (GA) approach. An analytical method for the identification of a differential equivalent linear model to approximate experimental hysteretic loops is presented in [18].

The parameters of the linear equivalent model

$$\dot{z} = az + b\dot{\xi} + c\xi, \tag{5}$$

with $a < 0$ (derived from the stability condition) and $[a] = T^{-1}$, $[b] = T^{-2}$, $[c] = T^{-3}$ are calculated by minimizing the expected value of the square error:

$$\varepsilon^2 = [h(\dot{\xi}, z) - (az + b\dot{\xi} + c\xi)]^2. \tag{6}$$

The nonlinear system modeled by equations (3) and (4)

$$\begin{cases} \ddot{\xi} + 2\omega\zeta\dot{\xi} + \omega^2\xi + z = \eta(t) \\ \dot{z} = [A - |z|^n (\beta + \gamma\,\mathrm{sgn}(\dot{\xi}z))]\dot{\xi} \end{cases} \tag{7}$$

is replaced by the following linear equivalent system:

$$\begin{cases} \ddot{\xi} + 2\omega\zeta\dot{\xi} + \omega^2\xi + z = \eta(t) \\ \dot{z} = az + b\dot{\xi} + c\xi \end{cases}, \tag{8}$$

where the parameters of linear model of hysteretic loop are determined by a stochastic equivalent linearization method, which will be described in this paper. In this case, it is necessary to determine the second order statistical moments of stabilized solution of stochastic system (8) by solving the attached Fokker-Planck equation.

Using the notations:

$$\boldsymbol{\xi} = [\xi,\ \dot{\xi},\ z]^{\mathrm{T}},\ \ \xi_1 = \xi,\ \ \xi_2 = \dot{\xi},\ \ \xi_3 = z,\ \ \mathrm{d}w(t) = \eta(t)\mathrm{d}t, \tag{9}$$

where $\mathrm{d}w(t) = w(t + \mathrm{d}t) - w(t)$ is the infinitesimal Wiener increment, the system (8) is equivalent with

$$\begin{cases} \dfrac{\mathrm{d}\xi_1}{\mathrm{d}t} = 0\xi_1 + 1\xi_2 + 0\xi_3 \\ \dfrac{\mathrm{d}\xi_2}{dt} = -\omega^2\xi_1 - 2\omega\zeta\xi_2 - \xi_3 + \eta(t) \\ \dfrac{\mathrm{d}\xi_3}{\mathrm{d}t} = c\xi_1 + b\xi_2 + a\xi_3 \end{cases} \tag{10}$$

which have the matrix form

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}=\begin{bmatrix}0 & 1 & 0\\ -\omega^2 & -2\omega\zeta & -1\\ c & b & a\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}+\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\eta(t). \tag{11}$$

The linear stochastic differential equation is

$$\mathrm{d}\xi(t)=\mathbf{A}\xi(t)\mathrm{d}t+\mathbf{G}\mathrm{d}w(t), \tag{12}$$

where

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ -\omega^2 & -2\omega\zeta & -1\\ c & b & a\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}. \tag{13}$$

Using the Fokker-Planck equation, yields the following algebraic equation [18]:

$$\mathbf{A}\mathbf{C}_\xi+\mathbf{C}_\xi\mathbf{A}^{\mathrm{T}}+2\pi\mathbf{G}S_0\mathbf{G}^{\mathrm{T}}=0, \tag{14}$$

where $\mathbf{C}_\xi$ is the solution of (14) and represents the steady-state instantaneous covariance matrix which have the form:

$$\mathbf{C}_\xi=\begin{bmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{bmatrix},\quad c_{12}=c_{21}=0. \tag{15}$$

The equation (14) yields:

$$\begin{gathered}\sigma_\xi^2=c_{11}=\frac{\pi S_0}{\Delta}\left(\omega^2 a-2\omega\zeta a^2+a^3-c\right),\\ \sigma_{\dot\xi}^2=c_{22}=\frac{\pi S_0}{\Delta}\left[\omega^4 a-2\omega^3\zeta a^2+\omega^2(a^3+ab-c)+2\omega\zeta ac-c(a^2+b)\right],\\ \sigma_z^2=c_{33}=\frac{\pi S_0}{\Delta}\left(\omega^2 ab^2-2\omega\zeta c^2+ac^2-b^2c\right),\end{gathered} \tag{16}$$

and

$$\begin{aligned}&\sigma_{\xi z}=c_{13}=\frac{\pi S_0}{\Delta}\left[\omega^2 ab+2\omega\zeta ac-c(a^2+b)\right],\\ &\sigma_{\dot\xi z}=c_{23}=\frac{\pi S_0}{\Delta}\left[-\omega^2 a(ab+c)+c^2+abc\right],\\ &\rho_{\xi z}=\frac{\sigma_{\xi z}}{\sigma_\xi\sigma_z},\quad \rho_{\dot\xi z}=\frac{\sigma_{\dot\xi z}}{\sigma_{\dot\xi}\sigma_z},\end{aligned} \tag{17}$$

with

$$\begin{aligned}\Delta = 2\omega^5\zeta a - 4\omega^4\zeta^2 a^2 + 2\omega^3\zeta(a^3 + ab - c) + \\ + \omega^2 a(4\zeta^2 c - ab - c) - 2\omega\zeta c(a^2 + b) + c^2 + abc \neq 0.\end{aligned} \tag{18}$$

Under the hypothesis of joint Gaussian behavior of the state vector $\boldsymbol{\xi} = [\xi,\ \dot{\xi},\ z]^{\mathrm{T}}$, the values of the coefficients are [4, 14]:

$$\begin{cases} a = E\left[\dfrac{\partial h}{\partial z}\right] = -\beta F_3 - \gamma F_4 \\ b = E\left[\dfrac{\partial h}{\partial \dot{\xi}}\right] = A - \beta F_1 - \gamma F_2\,, \\ c = 0 \end{cases} \tag{19}$$

where the functions F_i, $i = 1,2,3,4$ are given by:

$$\begin{aligned} F_1 &= \frac{\sigma_z^n}{\pi}\Gamma\left(\frac{n+2}{2}\right)2^{n/2} I_s, \quad F_2 = \frac{\sigma_z^n}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)2^{n/2} \\ F_3 &= \frac{n\sigma_{\dot{\xi}}\sigma_z^{n-1}}{\pi}\Gamma\left(\frac{n+2}{2}\right)2^{n/2}\left[2(1-\rho_{\dot{\xi}z}^2)^{\frac{n+1}{2}} + \rho_{\dot{\xi}z} I_s\right] \\ F_4 &= \frac{n\rho_{\dot{\xi}z}\sigma_{\dot{\xi}}\sigma_z^{n-1}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)2^{n/2}, \end{aligned} \tag{20}$$

with

$$I_s = 2\int_l^{\pi/2} \sin^n\theta\, \mathrm{d}\theta\,, \qquad l = \tan^{-1}\left(\frac{\sqrt{1-\rho_{\dot{\xi}z}^2}}{\rho_{\dot{\xi}z}}\right). \tag{21}$$

For $n = 1$, using (16) and (17), the relations (20) become:

$$\begin{aligned} F_1 &= \sqrt{2S_0}\,\frac{b}{\sqrt{\Delta_1\Delta_2}}, \quad F_2 = \sqrt{2S_0}\,\frac{b}{\sqrt{\Delta_1}}, \\ F_3 &= \sqrt{2\pi S_0}\,\frac{\Delta_2 - 2}{\sqrt{\Delta_1\Delta_2}}, \quad F_4 = -\frac{\sqrt{2S_0}}{\sqrt{\Delta_1}}, \end{aligned} \tag{22}$$

where

$$\Delta_1 = 2\omega^3\zeta - 4\omega^2\zeta^2 a + 2\omega\zeta a^2 + 2\omega\zeta b - ab, \quad \Delta_2 = \omega^2 - 2\omega\zeta a + a^2 + b\,, \tag{23}$$

with $\Delta_1 > 0$ and $\Delta_2 > 0$.

Therefore, the system (19) becomes:

$$\begin{cases} a\sqrt{\Delta_1} + \beta\sqrt{2S_0}\sqrt{\Delta_2} - \gamma\sqrt{2S_0}\,a = 0 \\ A\sqrt{\Delta_1\Delta_2} + \beta ab\sqrt{2S_0} - \gamma b\sqrt{2S_0}\sqrt{\Delta_2} - b\sqrt{\Delta_1\Delta_2} = 0. \\ c = 0 \end{cases} \tag{24}$$

It can be shown that this system doesn't always have physically realizable solutions for certain values of hysteresis loop parameters.

3. APPLICATION OF THE LINEARIZATION METHOD

In the proposed case study, the hysteresis characteristics are modeled by Bouc-Wen model (4) with parameters $A = 5\,\mathrm{s}^{-2}$, $\beta = 0.1\,\mathrm{s}^{-1}$, $\gamma = 4\,\mathrm{s}^{-1}$, $n = 1$. Figure 3 shows the hysteresis loop generated by this model for imposed motion amplitude $\xi_0 = 1$. This hysteric characteristic is typical for bracing seismic devices used from control and limitation of inter-storey drifts.

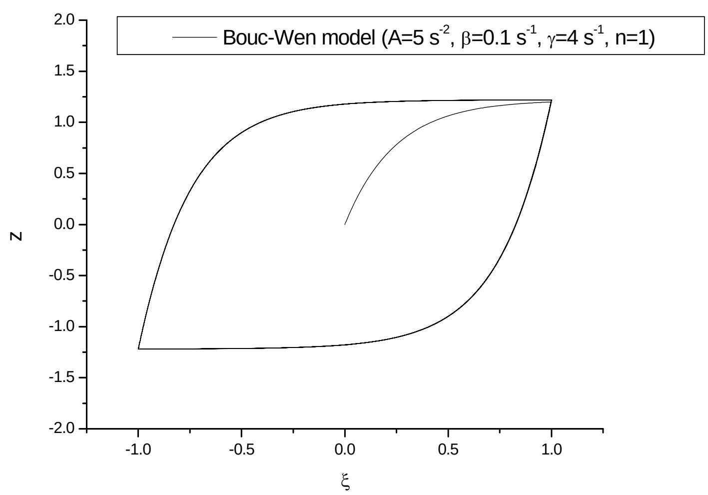

Fig. 3 – Hysteretic loops obtained by a virtual experiment.

In order to study the efficiency of the method, the r.ms. response of the equivalent linear system (8), where $\zeta = 0.05$, $\omega = 2\pi f$, $f \in [0.35, 1]$ and the coefficients a, b, c are calculated from relation (24), is compared with the response obtained by numerical simulation of Bouc-Wen nonlinear system (7) excited by the same Gaussian white noise input with intensity $S_0 = 1\,\mathrm{s}^{-3}$. This range of frequency is typical for very flexible structures which can display large inter-storey drifts that must be kept within the imposed limits. The input intensity S_0 was chosen such that the maximum drift is less than x_{m}.

The r.m.s responses (inter-storey drift ξ and absolute acceleration of sprung mass $\ddot{\xi}_1 = \ddot{\xi} + \eta$) of the nonlinear and linear equivalent systems are comparatively plotted in Figs. 4 and 5.

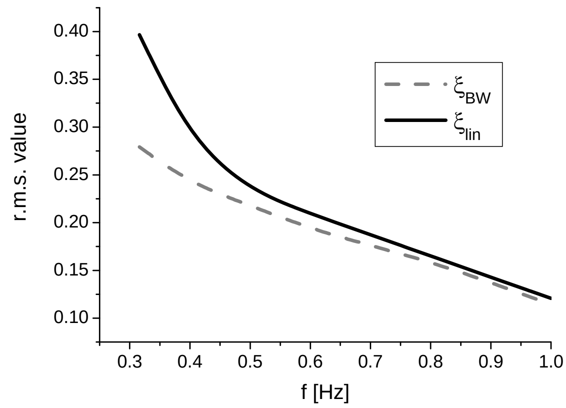

Fig. 4 – Inter-storey drift for nonlinear and linear equivalent systems.

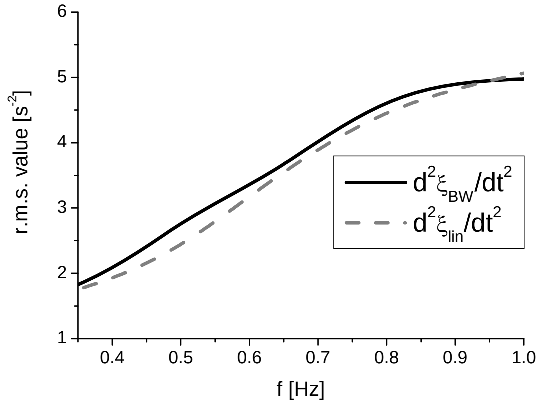

Fig. 5 – Absolute acceleration for nonlinear and linear equivalent systems.

Variation of hysteretic force for the nonlinear and the linear equivalent system response ($\zeta = 0.05$, $f = 0.5$ Hz, $S_0 = 1\,\mathrm{s}^{-3}$) is shown in Fig. 6.

From Figs. 4–6 one can see that the linearization method provides a sufficiently good approximation of the nonlinear system r.m.s. response.

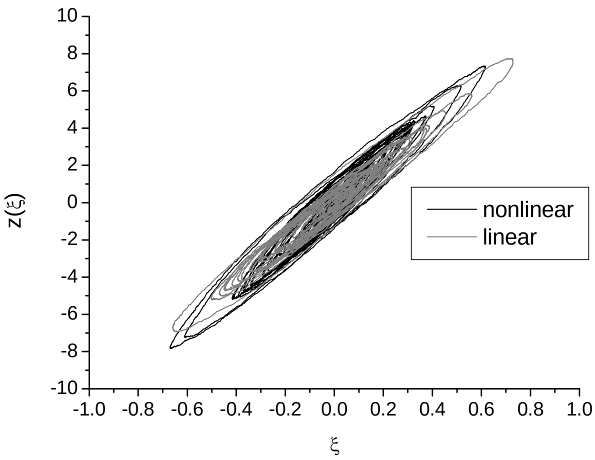

Fig. 6 – Hysteretic force for the nonlinear and the linear system response.

In order to illustrate the applicability of the method is considered a seismic input $\ddot{x}_0(t)$ with the acceleration response spectrum ($\zeta = 0.05$) shown in Fig. 7.

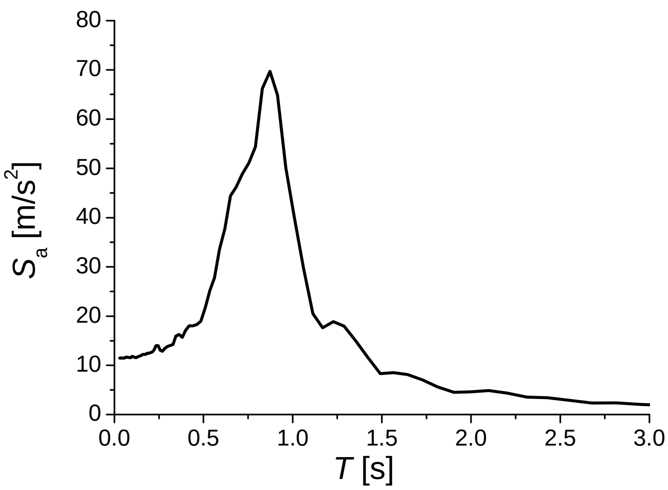

Fig. 7 – The response spectrum of seismic acceleration.

The period of the dominant frequency component is $T = 0.9\,\mathrm{s}$ being typical for a slow earthquake. The seismic response of the linear equivalent system obtained for $f = 1\,\mathrm{Hz}$ and $S_0 = 1\,\mathrm{s}^{-3}$ is compared with nonlinear hysteretic system with bracing seismic devices modeled by the Bouc-Wen equation (Fig. 8).

From Fig. 8 one can see the good agreement between the solutions of nonlinear and the linear system, obtained by statistical linearization method.

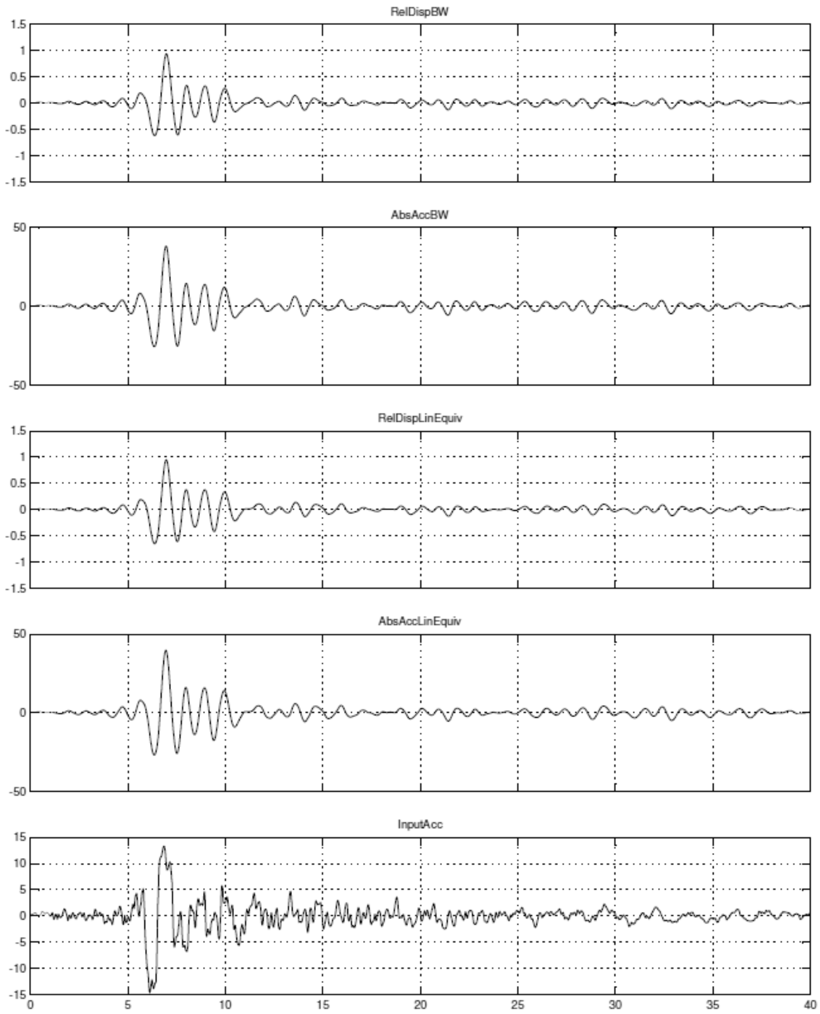

Fig. 8 – The seismic response of the linear equivalent system and nonlinear hysteretic system for protected structure.

4. CONCLUSIONS

The employed linearization method allows the determination of the equivalent linear system parameters by stochastic techniques.

The results obtained by statistical linearization of hysteretic system with white noise input show a good agreement between the r.m.s outputs. Moreover, the equivalent linear system can be used to approximate the behavior of the considered system with bracing devices, excited by a seismic motion with dominant frequency component within the range used for linearization.

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