INTERNAL REACTION FORCES IN DYNAMICS OF A PLANAR PARALLEL MANIPULATOR

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Abstract. Three identical legs connecting to the moving platform of a planar parallel manipulator are located in the same vertical plane. Knowing the motion of the platform, we determine the positions, velocities and accelerations of the robot. Using an approach based on the principle of virtual work, compact matrix dynamics equations and graphs of simulation for the input torques of three actuators and internal joint reaction forces are obtained.

Key words: dynamics, planar parallel manipulator, internal reaction forces, principle of virtual work, matricial formulation.

1. INTRODUCTION

Compared with serial manipulators, the followings are the potential advantages of parallel architectures: higher kinematical precision, lighter weight and better stiffness, greater load bearing, stabile capacity and suitable position of arrangement of actuators. Equipped with revolute or prismatic actuators, parallel manipulators have a robust construction and can move bodies of large dimensions with high velocities and accelerations. That is reason why the devices, which produce translation or spherical motion to a platform, technologically are based on the concept of parallel manipulators [1].

Over the past three decades, considerable efforts have been devoted to the kinematics and dynamic analysis of parallel manipulators. Among these, the class of manipulators known as Stewart-Gough platform have received great attention (Stewart [2], Merlet [3], Parenti Castelli and Di Gregorio [4]). They are used in flight simulators and more recently for Parallel Kinematics Machines. The prototype of Delta parallel robot (Clavel [5], Tsai and Stamper [6]), developed by Clavel at the Federal Polytechnic Institute of Lausanne and by Tsai and Stamper at the University of Maryland as well as the Star parallel manipulator (Hervé and

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Sparacino [7]), both are equipped with three motors, which train on the mobile platform in a three-degrees-of-freedom general translation motion.

A mechanism is said to be a *planar robot* if all the moving links in the mechanism perform the planar motions. For a planar mechanism, the loci of all points in all links can be drawn conveniently on a plane. In a planar linkage, the axes of all revolute joints must be normal to the plane of motion, while the direction of translation of a prismatic joint must be parallel to the plane of motion.

Aradyfio and Qiao [8] examined the inverse kinematics solution for the three different 3-DOF planar parallel robots. Gosselin and Angeles [9] and Pennock and Kassner [10] each present a kinematical study of a planar parallel robot, where a moving platform is connected to a fixed base by three links, each leg consisting of two binary links connected through three parallel revolute joints. Sefrioui and Gosselin [11] give an interesting numerical solution in the inverse and direct kinematics of this kind of planar robot. Daniali *et al.* [12] present a study of velocity relationships and singular conditions for general planar parallel robots. Williams and Reinholtz [13] analysed the dynamics and the control of a planar three-degrees-of-freedom parallel manipulator at Ohio University, while Yang et al. [14] concentrate on the singularity analysis of a class of 3-<u>R</u>RR planar parallel robots developed in its laboratory. Bonev, Zlatanov and Gosselin [15] describe several types of singular configurations by studying the direct kinematics model of a 3-<u>R</u>PR planar parallel robot with actuated base joints.

A recursive method is introduced in the present paper, to reduce significantly the number of relations and computation operations by using a set of matrices for kinematics and dynamics of the 3-RRR planar parallel robot. Based on the principle of virtual work, compact equations establish a direct determination of the time-history evolution for the input torques of three actuators and the internal reaction forces in joints.

2. KINEMATICS MODELLING

Having a closed-loop structure, the planar parallel robot 3-RRR is a special symmetrical mechanism composed of three planar kinematical chains with identical topology, all connecting the fixed base to the moving platform. The centres A_1, B_1, C_1 of three fixed pivots define the position of a fixed base and the three moving revolute joints A_3, B_3, C_3 define the geometry of the planar moving platform. Each leg of the mechanism is represented by two binary links with three parallel revolute joints. Together, the manipulator consists of seven moving links, nine revolute joints and three revolute actuators installed on the fixed base. Grübler-Kutzbach mobility criterion predicts certainly three degrees of freedom of the robot's moving platform (Fig. 1).

For the purpose of the analysis, we attach a Cartesian frame $O\boldsymbol{x}_0\boldsymbol{y}_0\boldsymbol{z}_0(T_0)$ to the fixed base with its origin located at the triangle centre, the $\boldsymbol{z}_0$ axis perpendicular to the base and the $\boldsymbol{x}_0$ axis pointing along the direction C_1A_1. Another mobile reference frame $D_3x_3^Dy_3^Dz_3^D$ is attached to the moving platform. The origin of this coordinate central system is located just at the centre D_3 of the moving triangle.

In what follows we consider that the moving platform is initially located at a *central configuration*, where the platform is not rotated with respect to the fixed base and the mass centre D_3 is at the origin O of the fixed frame. It is noted that the relative rotation of anybody T_k with $\varphi_{k,k-1}$ angle must be always pointing about the direction of $\boldsymbol{z}_k$ axis.

One of three active legs (for example leg A) consists of a fixed revolute joint A_1, a moving crank **1** of length l_1, mass m_1 and tensor of inertia $\hat{\mathbf{J}}_1$, which has a rotation about $\boldsymbol{z}_1^A$ axis with the angle φ_{10}^A, the angular velocity $\omega_{10}^A = \dot{\varphi}_{10}^A$ and the angular acceleration $\varepsilon_{10}^A = \ddot{\varphi}_{10}^A$. A new element of the leg is a rigid rod **2** linked at the $A_2\boldsymbol{x}_2^A\boldsymbol{y}_2^A\boldsymbol{z}_2^A$ frame, having a relative rotation with the angle φ_{21}^A, velocity $\omega_{21}^A = \dot{\varphi}_{21}^A$ and acceleration $\varepsilon_{21}^A = \ddot{\varphi}_{21}^A$. It has the length l_2, mass m_2 and tensor of inertia $\hat{\mathbf{J}}_2$. Finally, a revolute joint is introduced at the moving platform, which is schematised as an equilateral triangle with the edge $l = r\sqrt{3}$, mass m_3 and a symmetrical tensor of inertia $\hat{\mathbf{J}}_3$ with respect to central frame $D_3\boldsymbol{x}_3^D\boldsymbol{y}_3^D\boldsymbol{z}_3^D$.

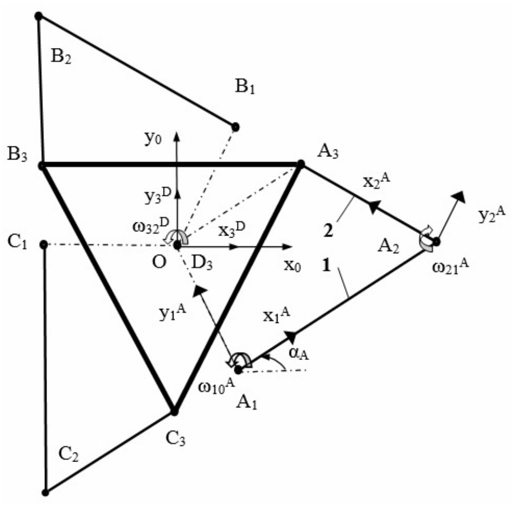

Fig. 1 – Planar 3-RRR parallel robot.

In the study of the kinematics of robot manipulators, we are interested in deriving a matrix equation relating the location of an arbitrary T_k body to the joint variables. We call the matrix $\mathbf{q}^{\varphi}_{k,k-1}$, for example, the orthogonal transformation 3×3 matrix of relative rotation with the angle $\varphi^i_{k,k-1}$ of link T^i_k around $\boldsymbol{z}^i_k$ axis. Starting from the fixed reference origin O and pursuing the independent legs $OA_1A_2A_3$, $OB_1B_2B_3$, $OC_1C_2C_3$, we obtain the following transformation matrices:

$$\mathbf{q}_{10} = \mathbf{q}^{\varphi}_{10}\boldsymbol{\theta}^i_{\alpha},\ \mathbf{q}_{21} = \mathbf{q}^{\varphi}_{21}\boldsymbol{\theta}_2\boldsymbol{\theta}_1,\ \mathbf{q}_{20} = \mathbf{q}_{21}\mathbf{q}_{10}\ \ (\mathbf{q} = \mathbf{a}, \mathbf{b}, \mathbf{c})\ \ (i = A, B, C)\,, \tag{1}$$

where we denote [16]:

$$\mathbf{q}^{\varphi}_{k,k-1} = \mathrm{rot}(\boldsymbol{z},\varphi^i_{k,k-1}) = \begin{bmatrix} \cos\varphi^i_{k,k-1} & \sin\varphi^i_{k,k-1} & 0 \\ -\sin\varphi^i_{k,k-1} & \cos\varphi^i_{k,k-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} (k = 1, 2)\,,\quad \boldsymbol{\theta}^i_{\alpha} = \mathrm{rot}(\mathbf{z},\alpha_i)$$

$$\boldsymbol{\theta}_1 = \mathrm{rot}(\boldsymbol{y},\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\quad \boldsymbol{\theta}_2 = \mathrm{rot}(\boldsymbol{z},\frac{\pi}{3}) = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{2}$$

In the inverse geometric problem, it can be considered that the position of the planar mechanism is completely given through the coordinates $\lambda^D_{10} = x^{D_3}_0$, $\lambda^D_{21} = y^{D_3}_0$ of the mass centre D_3 of the moving platform and the orientation angle $\varphi^D_{32} = \phi$ of the central frame $D_3\boldsymbol{x}^D_3\boldsymbol{y}^D_3\boldsymbol{z}^D_3$. The known orthogonal rotation matrix $\mathbf{R} = \mathbf{d}_{30} = \mathbf{d}_{32}\mathbf{d}_{21}\mathbf{d}_{10}$ of the platform from $O\boldsymbol{x}_0\boldsymbol{y}_0\boldsymbol{z}_0$ to $D_3\boldsymbol{x}^D_3\boldsymbol{y}^D_3\boldsymbol{z}^D_3$ are obtained by multiplying some relative basic matrices:

$$\mathbf{d}_{10} = \boldsymbol{\theta}_3\,,\ \ \mathbf{d}_{21} = \boldsymbol{\theta}_3\boldsymbol{\theta}_4\,,\ \ \mathbf{d}_{32} = \mathbf{d}^{\phi}_{32}\boldsymbol{\theta}_4\boldsymbol{\theta}_3\boldsymbol{\theta}_4\,,\ \ \mathbf{d}_{20} = \mathbf{d}_{21}\mathbf{d}_{10}\,, \tag{3}$$

with the notations:

$$\boldsymbol{\theta}_3 = \mathrm{rot}(\boldsymbol{y},\frac{\pi}{2}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ \ \boldsymbol{\theta}_4 = \mathrm{rot}(\boldsymbol{z},\frac{\pi}{2}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \ \mathbf{d}^{\phi}_{32} = \mathrm{rot}(\boldsymbol{z},\phi)\,. \tag{4}$$

Further, we suppose that the position vector of D_3 centre $\vec{r}^{\,D_3}_0 = [x^{D_3}_0\ \ y^{D_3}_0\ \ 0]^{\mathrm{T}}$ and the orientation angle ϕ, which are expressed by following analytical functions:

$$x_0^{D_3} = x_0^{D_3 *}(1-\cos\frac{\pi}{3}t), \quad y_0^{D_3} = y_0^{D_3 *}(1-\cos\frac{\pi}{3}t), \quad \phi = \phi^{*}(1-\cos\frac{\pi}{3}t) \tag{5}$$

can describe the general absolute motion of the moving platform in its *vertical plane*.

Six variables $\varphi_{10}^A, \varphi_{21}^A, \varphi_{10}^B, \varphi_{21}^B, \varphi_{10}^C, \varphi_{21}^C$ will be determined by several vector-loop equations, as follows:

$$\vec{r}_{10}^{\,i} + \mathbf{q}_{10}^{\mathrm{T}} \vec{r}_{21}^{\,i} + \mathbf{q}_{10}^{\mathrm{T}} \mathbf{q}_{21}^{\mathrm{T}} \vec{r}_{32}^{\,i} = \vec{r}_0^{\,D_3} + \mathbf{R}^{\mathrm{T}} \vec{r}_{D_3}^{\,i_3} \quad (i = A, B, C) \ \ (\mathbf{q} = \mathbf{a}, \mathbf{b}, \mathbf{c}), \tag{6}$$

where we denote:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \tilde{\mathbf{u}}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\vec{r}_{10}^{\,A} = 0.25r[\sqrt{3} \quad -3 \quad 0]^{\mathrm{T}}, \quad \vec{r}_{10}^{\,B} = 0.25r[\sqrt{3} \quad 3 \quad 0]^{\mathrm{T}}, \quad \vec{r}_{10}^{\,C} = [-0.5r\sqrt{3} \quad 0 \quad 0]^{\mathrm{T}}$$

$$\begin{aligned} &\vec{r}_{D_3}^{\,A_3} = 0.5r[\sqrt{3} \quad 1 \quad 0]^{\mathrm{T}}, \ \vec{r}_{D_3}^{\,B_3} = 0.5r[-\sqrt{3} \quad 1 \quad 0]^{\mathrm{T}}, \ \vec{r}_{D_3}^{\,C_3} = [0 \quad -r \quad 0]^{\mathrm{T}} \\ &\vec{r}_{21}^{\,i} = 1.5r\vec{u}_1, \quad \vec{r}_{32}^{\,i} = r\vec{u}_1. \end{aligned}, \tag{7}$$

Actually, these vector equations mean that there is only one inverse geometric solution for the manipulator's position:

$$\begin{aligned} &1.5r\cos(\varphi_{10}^i + \alpha_i) - r\cos(\varphi_{10}^i + \alpha_i - \varphi_{21}^i - \frac{\pi}{3}) = x_0^{D_3} - x_{10}^i + \vec{u}_1^{\mathrm{T}} \mathbf{R}^{\mathrm{T}} \vec{r}_{D_3}^{\,i_3}, \\ &1.5r\sin(\varphi_{10}^i + \alpha_i) - r\sin(\varphi_{10}^i + \alpha_i - \varphi_{21}^i - \frac{\pi}{3}) = y_0^{D_3} - y_{10}^i + \vec{u}_2^{\mathrm{T}} \mathbf{R}^{\mathrm{T}} \vec{r}_{D_3}^{\,i_3}. \end{aligned} \tag{8}$$

We develop the *inverse kinematics problem* and determine the velocities and accelerations, supposing that the planar motion of the moving platform is known. First, we compute the linear and angular velocities of each leg in terms of the angular velocity $\vec{\omega}_{30}^{D} = \dot{\varphi}_{32}^{D} \vec{u}_3$ and the centre's velocity $\dot{\vec{r}}_0^{\,D_3} = [\dot{\lambda}_{10}^{D} \quad \dot{\lambda}_{21}^{D} \quad 0]^{\mathrm{T}}$ of the moving platform.

The rotations of compounding elements of each leg i are characterized by recursive relations of following skew-symmetric matrices:

$$\tilde{\boldsymbol{\omega}}_{k0}^{i} = \mathbf{q}_{k,k-1} \tilde{\boldsymbol{\omega}}_{k-1,0}^{i} \mathbf{q}_{k,k-1}^{\mathrm{T}} + \omega_{k,k-1}^{i} \tilde{\mathbf{u}}_3, \quad \omega_{k,k-1}^{i} = \dot{\varphi}_{k,k-1}^{i}, \tag{9}$$

which are *associated* to the absolute angular velocities:

$$\vec{\omega}_{10}^{i} = \dot{\varphi}_{10}^{i} \vec{u}_3, \quad \vec{\omega}_{20}^{i} = \mathbf{q}_{21} \vec{\omega}_{10}^{i} + \vec{\omega}_{21}^{i} = (\dot{\varphi}_{21}^{i} - \dot{\varphi}_{10}^{i}) \vec{u}_3. \tag{10}$$

Following expressions give the absolute velocities of the centres of all joints:

$$\vec{v}_{k0}^{i} = \mathbf{q}_{k,k-1}\vec{v}_{k-1,0}^{i} + \mathbf{q}_{k,k-1}\tilde{\boldsymbol{\omega}}_{k-1,0}^{i}\vec{r}_{k,k-1}^{i} \quad (k = 1, 2). \tag{11}$$

Equations of geometrical constraints (6) can be derived with respect to time to obtain the following *matrix conditions of connectivity* [17]:

$$\omega_{10}^{i}\vec{u}_{j}^{\mathrm{T}}\mathbf{q}_{10}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\left(\vec{r}_{21}^{i} + \mathbf{q}_{21}^{\mathrm{T}}\vec{r}_{32}^{i}\right) + \omega_{21}^{i}\vec{u}_{j}^{\mathrm{T}}\mathbf{q}_{20}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\vec{r}_{32}^{i} = \vec{u}_{j}^{\mathrm{T}}\dot{\vec{r}}_{0}^{D_3} + \dot{\varphi}_{32}^{D}\vec{u}_{j}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\mathbf{R}^{\mathrm{T}}\vec{r}_{D_3}^{i_3}$$
$$(i = A, B, C) \quad (j = 1, 2). \tag{12}$$

From these equations, we obtain immediately the *complete* Jacobian matrix of the manipulator and the relative angular velocities $\omega_{10}^{i}, \omega_{21}^{i}$ as functions of angular velocity of the platform and velocity of its mass centre D_3.

As for the relative angular accelerations $\varepsilon_{10}^{i}, \varepsilon_{21}^{i}$, the derivatives with respect to time of the equations (12) give other following conditions of connectivity

$$\varepsilon_{10}^{i}\vec{u}_{j}^{\mathrm{T}}\mathbf{q}_{10}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\left(\vec{r}_{21}^{i} + \mathbf{q}_{21}^{\mathrm{T}}\vec{r}_{32}^{i}\right) + \varepsilon_{21}^{i}\vec{u}_{j}^{\mathrm{T}}\mathbf{q}_{20}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\vec{r}_{32}^{i} = \vec{u}_{j}^{\mathrm{T}}\ddot{\vec{r}}_{0}^{D_3} + \vec{u}_{j}^{\mathrm{T}}\left(\dot{\varphi}_{32}^{D}\dot{\varphi}_{32}^{D}\tilde{\mathbf{u}}_{3}\tilde{\mathbf{u}}_{3} + \ddot{\varphi}_{32}^{D}\tilde{\mathbf{u}}_{3}\right)\mathbf{R}^{\mathrm{T}}\vec{r}_{D_3}^{i_3} -$$
$$-\omega_{10}^{i}\omega_{10}^{i}\vec{u}_{j}^{\mathrm{T}}\mathbf{q}_{10}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\tilde{\mathbf{u}}_{3}\{\vec{r}_{21}^{i} + \mathbf{q}_{21}^{\mathrm{T}}\vec{r}_{32}^{i}\} - \omega_{21}^{i}\omega_{21}^{i}\vec{u}_{j}^{\mathrm{T}}\mathbf{q}_{20}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\tilde{\mathbf{u}}_{3}\vec{r}_{32}^{i} - 2\omega_{10}^{i}\omega_{21}^{i}\vec{u}_{j}^{\mathrm{T}}\mathbf{q}_{10}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\mathbf{q}_{21}^{\mathrm{T}}\tilde{\mathbf{u}}_{3}\vec{r}_{32}^{i},$$
$$(i = A, B, C) \quad (j = 1, 2). \tag{13}$$

The following recursive relations give the angular accelerations $\vec{\varepsilon}_{k0}^{i}$ and the accelerations $\vec{\gamma}_{k0}^{i}$ of the joints:

$$\vec{\varepsilon}_{k0}^{i} = \mathbf{q}_{k,k-1}\vec{\varepsilon}_{k-1,0}^{i} + \varepsilon_{k,k-1}^{i}\vec{u}_{3} + \omega_{k,k-1}^{i}\mathbf{q}_{k,k-1}\tilde{\boldsymbol{\omega}}_{k-1,0}^{i}\mathbf{q}_{k,k-1}^{\mathrm{T}}\vec{u}_{3}, \quad \varepsilon_{k,k-1}^{i} = \ddot{\varphi}_{k,k-1}^{i},$$

$$\tilde{\boldsymbol{\omega}}_{k0}^{i}\tilde{\boldsymbol{\omega}}_{k0}^{i} + \tilde{\boldsymbol{\varepsilon}}_{k0}^{i} = \mathbf{q}_{k,k-1}\left(\tilde{\boldsymbol{\omega}}_{k-1,0}^{i}\tilde{\boldsymbol{\omega}}_{k-1,0}^{i} + \tilde{\boldsymbol{\varepsilon}}_{k-1,0}^{i}\right)\mathbf{q}_{k,k-1}^{\mathrm{T}} + \omega_{k,k-1}^{i}\omega_{k,k-1}^{i}\tilde{\mathbf{u}}_{3}\tilde{\mathbf{u}}_{3} +$$
$$+\varepsilon_{k,k-1}^{i}\tilde{\mathbf{u}}_{3} + 2\omega_{k,k-1}^{i}\mathbf{q}_{k,k-1}\tilde{\boldsymbol{\omega}}_{k-1,0}^{i}\mathbf{q}_{k,k-1}^{\mathrm{T}}\tilde{\mathbf{u}}_{3},$$

$$\vec{\gamma}_{k0}^{i} = \mathbf{q}_{k,k-1}\vec{\gamma}_{k-1,0}^{i} + \mathbf{q}_{k,k-1}\left(\tilde{\boldsymbol{\omega}}_{k-1,0}^{i}\tilde{\boldsymbol{\omega}}_{k-1,0}^{i} + \tilde{\boldsymbol{\varepsilon}}_{k-1,0}^{i}\right)\vec{r}_{k,k-1}^{i} \quad (k = 1, 2). \tag{14}$$

The matrix relations (13) and (14) will be further used for the computation of wrench of the inertia forces for every rigid of the robot.

3. DYNAMICS EQUATIONS

In the context of the real-time control, neglecting the frictions forces and considering the gravitational effects, the relevant objective of the complete dynamics is first to determine the input torques or forces, which must be exerted by the actuators in order to produce a given trajectory of the end-effector, but also to calculate the internal joint forces or torques.

A lot of works have focused on the dynamics of Stewart platform. Dasgupta and Mruthyunjaya [18] used the Newton-Euler approach to develop closed-form dynamic equations of Stewart platform, considering all dynamic and gravity effects as well as viscous friction at joints. Tsai [1] presented an algorithm to solve the inverse dynamics for a Stewart platform-type using also Newton-Euler equations. This commonly known approach requires computation of all constraint forces and moments between the links.

In the present paper we apply the principle of virtual work for the *inverse dynamic problem* in order to establish some definitive recursive matrix relations for the calculus of input torques of the actuators and internal forces in the joints.

Three electric motors A_1, B_1, C_1 that generate the moments $\vec{m}_{10}^A = m_{10}^A \vec{u}_3$, $\vec{m}_{10}^B = m_{10}^B \vec{u}_3$, $\vec{m}_{10}^C = m_{10}^C \vec{u}_3$ oriented about fixed parallel axes control the motion of the platform. The parallel robot can artificially be transformed in a set of three open chains C_i $(i = A, B, C)$ subject to the constraints. This is possible by cutting each joint for moving platform, and takes its effect into account by introducing the corresponding constraint conditions.

The wrench of two vectors $\vec{F}_k^*$ and $\vec{M}_k^*$ evaluates the influence of the action of the weight $m_k \vec{g}$ and of other eventually external and internal forces applied to the same element T_k of the mechanism:

$$\vec{F}_k^* = 9.81 m_k \mathbf{q}_{k0} \vec{u}_2 , \quad \vec{M}_k^* = 9.81 m_k \tilde{\mathbf{r}}_k^C \mathbf{q}_{k0} \vec{u}_2 , \quad (q = a, b, c) . \tag{15}$$

Now, we compute the force of inertia $\vec{F}_k^{\text{in}}$ and the resulting moment of inertia forces $\vec{M}_k^{\text{in}}$ of an arbitrary rigid body T_k of mass m_k with respect to the centre of its first joint:

$$\vec{F}_k^{\text{in}} = -m_k \left[\vec{\gamma}_{k0} + (\tilde{\boldsymbol{\omega}}_{k0}^2 + \tilde{\boldsymbol{\varepsilon}}_{k0}) \vec{r}_k^C \right], \; \vec{M}_k^{\mathbf{in}} = -\left(m_k \tilde{\mathbf{r}}_k^C \vec{\gamma}_{k0} + \hat{\mathbf{J}}_k \bar{\varepsilon}_{k0} + \tilde{\boldsymbol{\omega}}_{k0} \hat{\mathbf{J}}_k \vec{\omega}_{k0} \right). \tag{16}$$

Pursuing the leg i, for example, two *significant recursive relations* generate the vectors:

$$\vec{F}_k^i = \vec{F}_{k0}^i + \mathbf{q}_{k+1,k}^{\text{T}} \vec{F}_{k+1}^i , \quad \vec{M}_k^i = \vec{M}_{k0}^i + \mathbf{q}_{k+1,k}^{\text{T}} \vec{M}_{k+1}^i + \tilde{\mathbf{r}}_{k+1,k}^i \mathbf{q}_{k+1,k}^{\text{T}} \vec{F}_{k+1}^i , \tag{17}$$

where we denote:

$$\vec{F}_{k0}^i = -\vec{F}_k^{\text{in},i} - \vec{F}_k^{*i} , \quad \vec{M}_{k0}^i = -\vec{M}_k^{\text{in},i} - \vec{M}_k^{*i} . \tag{18}$$

As example, starting from (17), we develop a set of recursive relations for the leg i:

$$\vec{F}_2^{\,i} = \vec{F}_{20}^{\,i}\,, \quad \vec{F}_1^{\,i} = \vec{F}_{10}^{\,i} + q_{21}^{T}\vec{F}_2^{\,i}\,,$$

$$\vec{M}_2^{\,i} = \vec{M}_{20}^{\,i}\,, \quad \vec{M}_1^{\,i} = \vec{M}_{10}^{\,i} + \mathbf{q}_{21}^{\mathrm{T}}\vec{M}_2^{\,i} + \tilde{\mathbf{r}}_{21}^{\,i}\mathbf{q}_{21}^{\mathrm{T}}\vec{F}_2^{\,i} \quad (i = A, B, C) \tag{19}$$

and for the moving platform:

$$\vec{F}_3^{\,D} = \vec{F}_{30}^{\,D}\,, \quad \vec{F}_2^{\,D} = \mathbf{d}_{32}^{\mathrm{T}}\vec{F}_3^{\,D}\,, \quad \vec{F}_1^{\,D} = \mathbf{d}_{21}^{\mathrm{T}}\vec{F}_2^{\,D}\,,$$

$$\vec{M}_3^{\,D} = \vec{M}_{30}^{\,D}\,,\; \vec{M}_2^{\,D} = \mathbf{d}_{32}^{\mathrm{T}}\vec{M}_3^{\,D}\,,\; \vec{M}_1^{\,D} = \mathbf{d}_{21}^{\mathrm{T}}\vec{M}_2^{\,D} + \tilde{\mathbf{r}}_{21}^{\,D_3}\mathbf{d}_{21}^{\mathrm{T}}\vec{F}_2^{\,D}\,,\; \tilde{\mathbf{r}}_{21}^{\,D_3} = [0 \quad \lambda_{21}^{D} \quad 0]^{\mathrm{T}} \tag{20}$$

The fundamental principle of the virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism.

The characteristic *virtual velocities* are expressed as functions of the pose of the mechanism by the general kinematical equations (12), where we add the contributions of successive virtual translations during some fictitious displacements along the directions $A_1x_1^A, A_1y_1^A, B_1x_1^B, B_1y_1^B, C_1x_1^C, C_1y_1^C$ of the middle revolute joints A_2, B_2, C_2:

$$\begin{aligned}\omega_{10}^{iv}\vec{u}_j^{\mathrm{T}}\mathbf{q}_{10}^{\mathrm{T}}\tilde{\mathbf{u}}_3\left(\vec{r}_{21}^{\,i} + \mathbf{q}_{21}^{\mathrm{T}}\vec{r}_{32}^{\,i}\right) + v_{21}^{ixv}\vec{u}_j^{\mathrm{T}}\mathbf{q}_{10}^{\mathrm{T}}\vec{u}_1 + v_{21}^{iyv}\vec{u}_j^{\mathrm{T}}\mathbf{q}_{10}^{\mathrm{T}}\vec{u}_2 + \omega_{21}^{iv}\vec{u}_j^{\mathrm{T}}\mathbf{q}_{20}^{\mathrm{T}}\tilde{\mathbf{u}}_3\vec{r}_{32}^{\,i} = \\ = v_{10}^{Dv}\vec{u}_j^{\mathrm{T}}\vec{u}_1 + v_{21}^{Dv}\vec{u}_j^{\mathrm{T}}\vec{u}_2 + \omega_{32}^{Dv}\vec{u}_j^{\mathrm{T}}\tilde{\mathbf{u}}_3\mathbf{R}^{\mathrm{T}}\vec{r}_{D_3}^{\,i_3} \quad (j = 1, 2).\end{aligned} \tag{21}$$

Considering some independent virtual motions of the planar mechanism, virtual displacements and velocities should be compatible with the virtual motions imposed by all kinematical constraints and joints at any instant in time. Let us assume that the robot has successively nine virtual motions determined by following sets of velocities:

$$\omega_{10a}^{Av} = 1\,, \quad \omega_{10a}^{iv} = 0 \;\; (i \neq A)\,, \quad v_{21a}^{ixv} = 0\,, \quad v_{21a}^{iyv} = 0\,;$$

$$\omega_{10b}^{Bv} = 1\,, \quad \omega_{10b}^{iv} = 0 \;\; (i \neq B)\,, \quad v_{21b}^{ixv} = 0\,, \quad v_{21b}^{iyv} = 0\,;$$

$$\omega_{10c}^{Bv} = 1\,, \quad \omega_{10c}^{iv} = 0 \;\; (i \neq C)\,, \quad v_{21c}^{ixv} = 0\,, \quad v_{21c}^{iyv} = 0\,;$$

$$\omega_{10a}^{iv} = 0\,, \quad v_{21a}^{Axv} = 1\,, \quad v_{21a}^{ixv} = 0 \;\; (i \neq A)\,, \quad v_{21a}^{iyv} = 0\,;$$

$$\omega_{10b}^{iv} = 0\,, \quad v_{21b}^{Axv} = 1\,, \quad v_{21b}^{ixv} = 0 \;\; (i \neq B)\,, \quad v_{21b}^{iyv} = 0\,;$$

$$\omega_{10c}^{iv} = 0\,,\;\; v_{21c}^{Axv} = 1\,,\;\; v_{21c}^{ixv} = 0\;\; (i \neq C)\,,\;\; v_{21c}^{iyv} = 0\,;$$
$$\omega_{10a}^{iv} = 0\,,\;\; v_{21a}^{ixv} = 0\,,\;\; v_{21a}^{Ayv} = 1\,,\;\; v_{21a}^{iyv} = 0\;\; (i \neq A)\,;$$
$$\omega_{10b}^{iv} = 0\,,\;\; v_{21b}^{ixv} = 0\,,\;\; v_{21b}^{Ayv} = 1\,,\;\; v_{21b}^{iyv} = 0\;\; (i \neq B)\,;$$
$$\omega_{10c}^{iv} = 0\,,\;\; v_{21c}^{ixv} = 0\,,\;\; v_{21c}^{Ayv} = 1\,,\;\; v_{21c}^{iyv} = 0\;\; (i \neq C)\,,\;\; (i = A,B,C)\,. \tag{22}$$

These virtual velocities are required into the computation of virtual power and virtual work of all forces applied to the compounding elements of the robot.

Total virtual work contributed by the inertia forces and moments of inertia forces $\vec{F}_k^{\text{in}}, \vec{M}_k^{\text{in}}$, by the wrench of known external forces $\vec{F}_k^*, \vec{M}_k^*$ and by the first actuator torque $\vec{m}_{10}^A$ or some eventually internal joint forces, for example, can be written in a compact form, *only* based on the relative virtual angular velocities. Applying the explicit form of the *equations of the parallel robots dynamics*, established by Stefan Staicu [19], following compact matrix relations results:

$$m_{10}^A = \vec{u}_3^{\mathrm{T}} \vec{M}_1^A + \vec{u}_3^{\mathrm{T}} \left(\omega_{21a}^{Av} \vec{M}_2^A + \omega_{21a}^{Bv} \vec{M}_2^B + \omega_{21a}^{Cv} \vec{M}_2^C + v_{10a}^{Dv} \vec{F}_1^D + v_{21a}^{Dv} \vec{F}_2^D + \omega_{32a}^{Dv} \vec{M}_3^D \right) \tag{23}$$

for the *torque* of first revolute actuator A_1,

$$f_{21}^{Ax} = \vec{u}_1^{\mathrm{T}} \mathbf{a}_{21}^{\mathrm{T}} \vec{F}_2^A + \vec{u}_3^{\mathrm{T}} \left(\omega_{21a}^{Av} \vec{M}_2^A + \omega_{21a}^{Bv} \vec{M}_2^B + \omega_{21a}^{Cv} \vec{M}_2^C + v_{10a}^{Dv} \vec{F}_1^D + v_{21a}^{Dv} \vec{F}_2^D + \omega_{32a}^{Dv} \vec{M}_3^D \right) \tag{24}$$

for the *first joint force* and

$$f_{21}^{Ay} = \vec{u}_2^{\mathrm{T}} \mathbf{a}_{21}^{\mathrm{T}} \vec{F}_2^A + \vec{u}_3^{\mathrm{T}} \left(\omega_{21a}^{Av} \vec{M}_2^A + \omega_{21a}^{Bv} \vec{M}_2^B + \omega_{21a}^{Cv} \vec{M}_2^C + v_{10a}^{Dv} \vec{F}_1^D + v_{21a}^{Dv} \vec{F}_2^D + \omega_{32a}^{Dv} \vec{M}_3^D \right) \tag{25}$$

for the *second joint force* acting in the middle internal joint A_2.

The relations (17–25) represent the *complete inverse dynamics model* of the 3-RRR planar parallel robot. The various dynamical effects, including the Coriolis and centrifugal forces coupling and the gravitational actions are considered in these explicit equations.

As application let us consider a 3-RRR planar robot, which has the following geometrical and architectural characteristics:

$$x_0^{G*} = 0.025\,\mathrm{m}\,,\;\; y_0^{G*} = 0.025\,\mathrm{m}\,,\;\; \phi^* = \frac{\pi}{12}\,,\;\; \Delta t = 3\,\mathrm{s}\,,\;\; r = 0.3\,\mathrm{m}\,,$$
$$l_1 = 1.5r\,,\;\; l_2 = r\,,\;\; l = r\sqrt{3}\,,\;\; m_1 = 2.25\,\mathrm{kg}\,,\;\; m_2 = 1.5\,\mathrm{kg}\,,\;\; m_3 = 5\,\mathrm{kg}\,.$$

Using the MATLAB software, a computer program was developed to solve the inverse dynamics of the planar parallel robot. To illustrate the algorithm, it is assumed that for a period of three seconds the platform starts at rest from a central configuration and rotates or translates along rectilinear directions.

Assuming that there is no external force and moment acting on the moving platform, a dynamic simulation is based on the computation of three input torques m_{10}^{i} required by each actuator during the platform's evolution and the internal joint forces f_{21}^{ix}, f_{21}^{iy}.

Following three examples are solved to illustrate the simulation.

For the first example we consider the *translation motion* of the moving platform along the *horizontal* axis $\boldsymbol{x}_0$ with variable acceleration while all the other positional parameters are held equal to zero. The input torques of three actuators and the internal joint forces are calculated by the program and plotted versus time as follows: Fig. 2, Fig. 3 and Fig. 4.

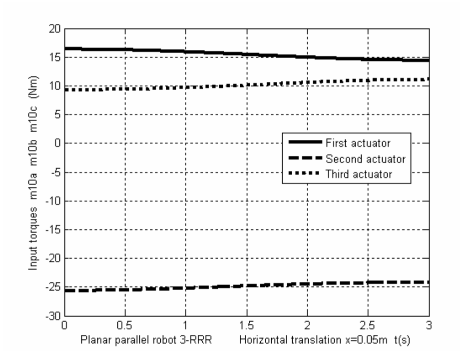

Fig. 2 – Input torques of three actuators.

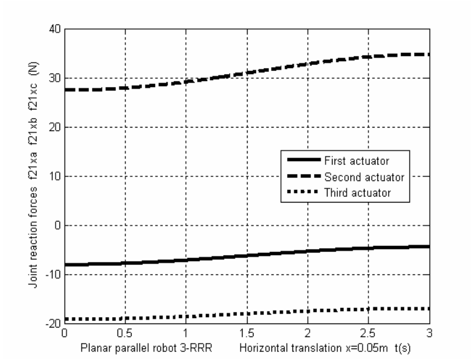

Fig. 3 – Joint forces $f_{21x}^{A}, f_{21x}^{B}, f_{21x}^{C}$.

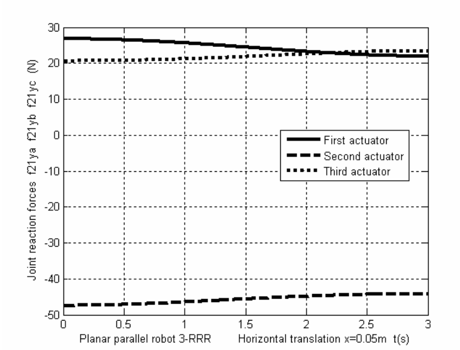

Fig. 4 – Joint forces $f_{21y}^{A}, f_{21y}^{B}, f_{21y}^{C}$.

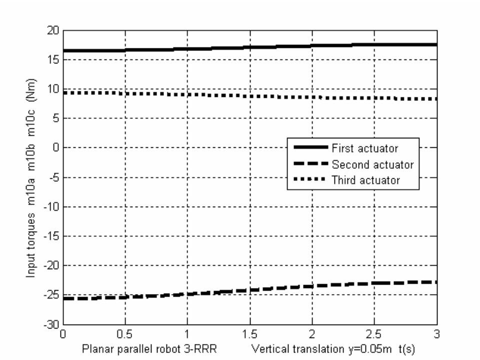

Fig. 5 – Input torques of three actuators.

If the platform's centre *D* moves along the *vertical* axis $\boldsymbol{y}_0$ without rotation of platform, the graphs are sketched in Fig. 5, Fig. 6 and Fig. 7.

For the third example we consider the *rotation motion* of the moving platform around the *horizontal* axis $\mathbf{z}_0$ with variable angular acceleration: Figs. 8, 9, 10.

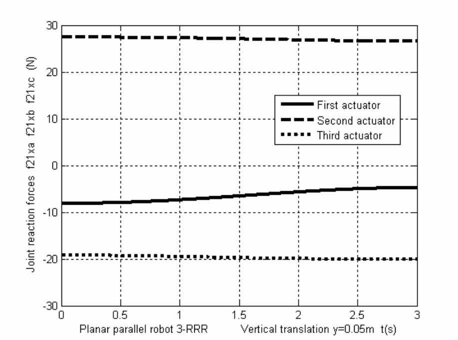

Fig. 6 – Joint forces f^A_{21x} , f^B_{21x} , f^C_{21x} .

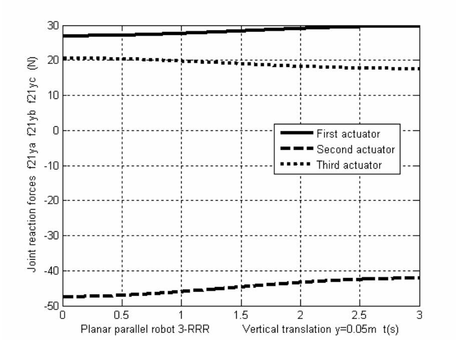

Fig. 7 – Joint forces f^A_{21y} , f^B_{21y} , f^C_{21y} .

The simulation through the program certifies that the current matrix recursive formulation can easily be transformed in a model which is successfully expected to be deployed for automatic robotic control of the parallel hybrid robot and that one of its major advantages is the effectiveness and accuracy of numerical computation.

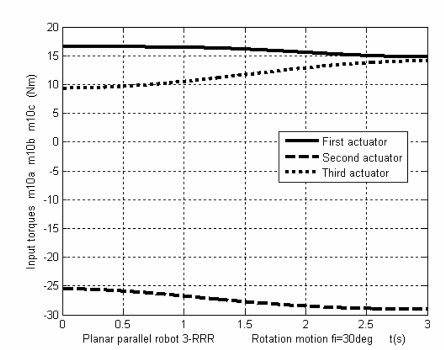

Fig. 8 – Input torques of three actuators.

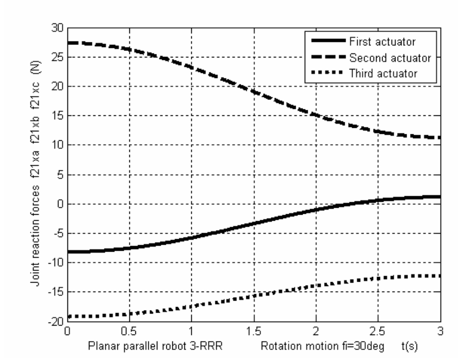

Fig. 9 – Joint forces f^A_{21x} , f^B_{21x} , f^C_{21x} .

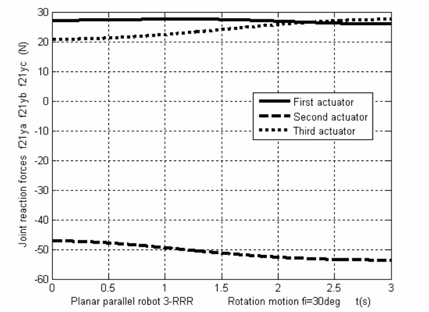

Fig. 10 – Joint forces f^A_{21y} , f^B_{21y} , f^C_{21y} .

4. CONCLUSIONS

Within the inverse kinematics analysis some exact relations that give in real-time the position, velocity and acceleration of each element of the parallel robot have been established in the present paper. The dynamics model takes into consideration the mass, the tensor of inertia and the action of weight and inertia force introduced by all compounding elements of the parallel mechanism.

Based on the principle of virtual work, this approach can formally eliminate the forces of the external and internal joints and establishes a direct determination of the time-history evolution for the torques required by the actuators and the external and internal forces or torques in joints.

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