AN APPLICATION OF THE OPTIMAL AUXILIARY FUNCTIONS TO BLASIUS PROBLEM

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Abstract. In this work, a new procedure namely the Optimal Auxiliary Functions Method (OAFM) is proposed to obtain an explicit analytical solution of the Blasius problem. Our solutions are compared with those obtained by numerical solution, revealing that our procedure is highly accurate. This proves the validity and great potential of the proposed method as a new kind of powerful analytical tool for nonlinear problems.

Key words: Optimal Auxiliary Functions Method, Blasius equation, viscous flow, optimal parameters.

1. INTRODUCTION

The flow of the non-Newtonian fluids is very important for engineers, because of its several applications in various fields of science and engineering. In fluid mechanics a Blasius boundary layer (named after Paul Richard Heinrich Blasius), describes the steady two dimensional laminar boundary layer that forms a semi-infinite plate which is held parallel to a constant unidirectional flow.

In the last few decades, these fluids have attracted considerable attention from researchers in many branches of nonlinear dynamical systems. Many powerful methods have been presented. Perturbation methods have been widely applied to solve nonlinear problems [1] but unfortunately they are based on such assumption that a small parameter exist. Until now, lots of other analytical procedures were proposed to solve Blasius equation. Belhachmi *et al.* [2] studied in detail the concave solutions of initial value problems involving the Blasius equation, Fazio [3] introduced a numerical parameter and require to an extended scaling group involving this parameter, Ganji *et al*. [4] and Towsyfyan *et al*. [5] employed the homotopy perturbation method to solve Blasius nonlinear differential equation, Aminikhah [6] applied Laplace transform and a new homotopy perturbation method. Variational iteration method is used by Liu and Kurra [7], optimized artificial neural networks approximation with sequential quadratic programing algorithm and hybrid AST-

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INP techniques is employed by Ahmad and Bilal [8] and an integrated neural network and gravitational search algorithm (HNN GSA) is applied by Biglari *et al.* [9]. Peker *et al.* [10] studied the Blasius problem by means of Padé approximation with differential transformation method, Adanhounme and Codo [11] applied the Adomian Decomposition Method to solve the Blasius problem. Lal and Neeraj [12] employed the Taylor series with higher radius of convergence and parameters of asymptotic variation and Robin [13] developed a new uniform approximation from existing rational approximations. The optimal homotopy asymptotic method is used by Marinca and Herisanu [14–16] to obtain an explicit analytic solution and initial slope with high accuracy. Other important works on Blasius problem were done by Boyd [17, 18] and Costin and Tanveer [19].

In the present work, we apply the Optimal Auxiliary Functions Method (OAFM) to solve the nonlinear differential equation of Blasius problem. Our procedure is a powerful tool for solving strongly nonlinear problems, ensuring the conditions of convergence of the solutions after only one iteration.

2. THE GOVERNING EQUATION

For the steady, incompressible, two-dimensional boundary layer equations for continuity and momentum are, respectively:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where ν is kinematic viscosity. If δ is the boundary-layer thickness and L is natural length scale and U is a constant velocity of the mainstream at infinity, then balancing between viscosity and convective inertia it results the scaling argument

$$\frac{U^2}{L}\approx\nu\frac{U}{\delta^2}. \tag{3}$$

From the scaling argument it is apparent that the boundary layer grows with the downstream coordinate x, e.g.:

$$\delta(x)\approx\left(\frac{\nu x}{U}\right)^{1/2} \tag{4}$$

such that we introduce the following similarity variable

$$\eta=\frac{y}{\delta(x)}=y\left(\frac{U}{\nu x}\right)^{1/2} \tag{5}$$

the stream function

$$\Psi = \left(\nu x U\right)^{1/2} f(\eta) \tag{6}$$

and the function

$$u = U f'(\eta)\,. \tag{7}$$

Now, differentiating to find the velocities, after simple manipulations we obtain Blasius equation [20]:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0 \tag{8}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1\,, \tag{9}$$

where prime denotes the derivative with respect to η.

3. BASIC IDEAS OF OAFM

In general, Eq. (8) with the boundary conditions (9) can be written as

$$L[f(\eta)] + g(\eta) + N[f(\eta)] = 0\,, \tag{10}$$

where L is a linear operator, g is a known function and N is a nonlinear operator, with the boundary conditions:

$$B\left(f(\eta), \frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta} \right) = 0\,. \tag{11}$$

We assume that Eqs. (10) and (11) have the approximate solution in the form with only two components

$$\bar{f}(\eta) = f_0(\eta) + f_1(\eta, C_i), \quad i = 1,2,...,s\,, \tag{12}$$

where the initial approximation $f_0(\eta)$ and the first approximation $f_1(\eta, C_i)$ will be determined as follows. Substituting Eq. (12) into Eq. (10), it results in

$$L[f_0(\eta)] + L[f_1(\eta, C_i)] + g(\eta) + N[f_0(\eta) + f_1(\eta, C_i)] = 0\,. \tag{13}$$

The initial approximation $f_0(\eta)$ can be determined from the linear equation

$$L[f_0(\eta)] + g(\eta) = 0, \quad B\left[f_0(\eta), \frac{\mathrm{d}f_0(\eta)}{\mathrm{d}\eta} \right] = 0 \tag{14}$$

and the first approximation $f_1(\eta, C_i)$ from the equation

$$L[f_1(\eta, C_i)] + N[f_0(\eta) + f_1(\eta, C_i)] = 0, \quad B\left[f_1(\eta, C_i), \frac{\mathrm{d}f_1(\eta, C_i)}{\mathrm{d}\eta}\right] = 0. \quad (15)$$

Now, the nonlinear term from Eq. (15) is expanded in the form

$$N[f_0(\eta) + f_1(\eta, C_i)] = N[f_0(\eta)] + \sum_{k=1}^{\infty} \frac{f_1^k(\eta)}{k!} N^{(k)}[f_0(\eta)]. \quad (16)$$

In order to avoid the difficulties that appears in solving the nonlinear differential equation (15) and to accelerate the rapid convergence of the first approximation $f_1(\eta, C_i)$ and implicit of the solution $\bar{f}(\eta)$, instead of the last term arising into Eq. (15), we propose another expression, such that Eq. (15) can be written as

$$\begin{gathered} L[f_1(\eta, C_i)] + A_1[f_0(\eta), C_i] N[f_0(\eta)] + A_2[f_0(\eta), C_j] = 0, \\ B\left(f_1(\eta, C_i), \frac{\mathrm{d}f_1(\eta, C_i)}{\mathrm{d}\eta}\right) = 0, \end{gathered} \quad (17)$$

where A_1 and A_2 are two arbitrary auxiliary functions depending on the initial approximation $f_0(\eta)$ and several unknown parameters C_i and C_j, $i = 1, 2, \ldots, s$, $j = s+1, s+2, \ldots, p$. The auxiliary functions A_1 and A_2 (called optimal auxiliary functions) are not unique, and are of the same form like $f_0(\eta)$ or of the form of $N[f_0(\eta)]$ or combinations of the forms of $f_0(\eta)$ and $N[f_0(\eta)]$.

For example, if $f_0(\eta)$ or $N[f_0(\eta)]$ are polynomial functions, then $A_1[f_0(\eta, C_i)]$ and $A_2[f_0(\eta, C_j)]$ are sums of polynomial functions; if $f_0(\eta)$ or $N[f_0(\eta)]$ contain exponential functions, then A_1 and A_2 would be sums of exponential functions; if $f_0(\eta)$ or $N[f_0(\eta)]$ are trigonometric functions, then A_1 and A_2 would be sums of trigonometric functions, and so on. If, in a special case, $N[f_0(\eta)] = 0$ then it is clear that $f_0(\eta)$ is an exact solution of Eq. (10). The unknown parameters C_i and C_j can be optimally identified via different method such as the Galekin method, the Ritz method, the least square method, the collocation method. The first option should be minimizing the square residual error using

$$J(C_i, C_j) = \int_a^b R^2(\eta, C_i, C_j)\mathrm{d}\eta \quad (18)$$

where

$$R(\eta, C_i, C_j) = L[\overline{f}(\eta, C_i, C_j)] + g(\eta) + N[\overline{f}(\eta, C_i, C_j)],$$
$$i = 1,2,...,s; \quad j = s+1, s+2, ..., p$$

Finally, by this novel approach, the approximate solution (12) is well determined.

Our procedure proves to be a powerful tool for solving nonlinear problems not depending on small or large parameters. It should be emphasized that our method contains the optimal auxiliary functions A_1 and A_2 which provides us with a simple way to adjust and control the convergence of the approximate solutions after only one iteration.

4. APPROXIMATE SOLUTION OF THE BLASIUS PROBLEM BY MEANS OF OAFM

In the following, we apply our procedure to obtain an approximate solution of Eqs. (8) and (9). The initial approximation $f_0(\eta)$ which verify the boundary conditions (9) can be chosen as

$$f_0(\eta) = \eta + \frac{e^{-k\eta} - 1}{k}, \tag{19}$$

where k is an unknown parameter at this moment.

Taking into consideration Eq. (19), we define the linear operator and the function g in the form (the linear operator would not be unique):

$$L[f(\eta)] = f'''(\eta) + kf''(\eta), \quad g(\eta) = 0 \tag{20}$$

or

$$L[f(\eta)] = f''(\eta) + kf'(\eta), \quad g(\eta) = -k \tag{21}$$

or

$$L[f(\eta)] = f'''(\eta) - k^2 f'(\eta), \quad g(\eta) = k^2. \tag{22}$$

If we consider only the linear operator given through Eq. (20), we can obtain the nonlinear operator as

$$N[f(\eta)] = \frac{1}{2} f(\eta) f''(\eta) - kf''(\eta). \tag{23}$$

It is clear that the Eq. (14) is defined by

$$L[f_0(\eta)] = 0 \tag{24}$$

whose solution is given by Eq. (19).

Now, substituting Eq. (19) into Eq. (23), it holds that

$$N[f_0(\eta)]=\left(\frac{1}{2}k\eta-\frac{1}{2}-k^2\right)e^{-k\eta}+\frac{1}{2}e^{-2k\eta}. \tag{25}$$

We have the freedom to choose the auxiliary functions A_1 and A_2 in the following forms:

$$A_1[f_0(\eta),C_i]=-\frac{1}{N[f_0(\eta)]}(C_1\eta^3+C_2\eta^2+C_3\eta+C_4)e^{-k\eta} \tag{26}$$

$$A_2[f_0(\eta),C_i]=-(C_5\eta+C_6)e^{-2k\eta} \tag{27}$$

or

$$A_1[f_0(\eta),C_i]=C_1\eta^2+C_2\eta+C_3 \tag{28}$$

$$A_2[f_0(\eta),C_i]=(C_4\eta^3+C_5\eta^2+C_6\eta+C_7)e^{-k\eta} \tag{29}$$

or

$$A_1[f_0(\eta),C_i]=0 \tag{30}$$

$$A_2[f_0(\eta),C_i]=(C_1\eta+C_2)e^{-k\eta}+(C_3\eta+C_4)e^{-2k\eta}+(C_5\eta+C_6)e^{-3k\eta} \tag{31}$$

and so on. Hence, we consider the auxiliary functions A_1 and A_2 given by Eqs. (26) and (27). In this case, Eq. (17) may be written as

$$\begin{aligned}&f_1'''(\eta)+kf_1''(\eta)=(C_1\eta^3+C_2\eta^2+C_3\eta+C_4)e^{-k\eta}+(C_5\eta+C_6)e^{-2k\eta},\\&f_1(0)=f_1'(0)=f_1'(\infty)=0\end{aligned} \tag{32}$$

whose solution is

$$\begin{aligned}f_1(\eta)=&\left[\frac{C_1}{4k^2}\eta^4+\left(\frac{2C_1}{k^3}+\frac{C_2}{3k^2}\right)\eta^3+\left(\frac{9C_1}{k^4}+\frac{2C_2}{k^3}+\frac{C_3}{2k^2}\right)\eta^2+\right.\\&+\left(\frac{24C_1}{k^5}+\frac{6C_2}{k^4}+\frac{2C_3}{k^3}+\frac{C_4}{k^2}\right)\eta+\frac{24C_1}{k^6}+\frac{6C_2}{k^5}+\frac{2C_3}{k^4}+\frac{C_4}{k^3}+\\&\left.+\frac{3C_5}{4k^4}+\frac{C_6}{2k^3}\right]e^{-k\eta}-\left(\frac{C_5}{4k^3}\eta+\frac{C_5}{2k^4}+\frac{C_6}{4k^3}\right)e^{-2k\eta}-\frac{24C_1}{k^6}-\\&-\frac{6C_2}{k^5}-\frac{2C_3}{k^4}-\frac{C_4}{k^3}-\frac{C_5}{4k^4}-\frac{C_6}{4k^3}.\end{aligned} \tag{33}$$

The approximate solution (12) is obtained from Eqs. (19) and (33). The parameters k, C_1, $C_{2,\ ...,}$ C_6 are determined simply by collocation method, making collocation in 7 points, as follows:

$$k = 0.7213456124, \quad C_1 = -0.0200901684, \quad C_2 = 0.6332041491,$$
$$C_3 = -5.7019089005, \quad C_4 = 15.6635426754, \quad C_5 = -5.4440609736,$$
$$C_6 = -15.4081506377.$$

The approximate solution (12) for Eqs. (8) and (9) can be written as follows:

$$\begin{aligned}\bar{f}(\eta) &= (-0.00965243302\eta^4 + 0.2985860774\eta^3 - 2.7728487521\eta^2 + \\ &+ 11.2835892145\eta - 18.5769471597)e^{-0.7213456124\eta} + \\ &+ (3.62604179605\eta + 20.3162186281)e^{-1.4426912248\eta} + \eta - 1.73927146.\end{aligned} \tag{34}$$

5. DISCUSSION OF RESULTS

To solve Eqs. (8) and (9), Blasius in 1908 [20] provides a power series solution

$$f(\eta) = \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \delta^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{35}$$

where

$$A_0 = A_1 = 1, \quad A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1} \quad k \geq 2,$$

with $\binom{m}{n} = \dfrac{m!}{n!(m-n)!}$. But the expression (35) is not closed, because $\sigma = f''(0)$ is unknown.

Using two different approximations, Blasius obtained the numerical result σ=0.332. In 1938, Howarth [21] gained a more accurate value σ=0.33206, the expression of $f(\eta)$ given by Eq. (35) is valid in a small region $0 \leq \eta \leq 5.69$. Blasius' power series (35) is fundamentally an analytic-numerical solution because the value of σ is obtained by numerical techniques.

Asaithambi [22] found this number correct to nine decimal precision as σ=0.332057336. Using our procedure, from Eq. (34) we obtain

$$\sigma = \bar{f}''(0) = 0.332054298, \tag{36}$$

Obviously, our first-order approximate result (34) ensures an error ε=0.0009%, which is remarkable good.

From Table 1 we can conclude that the first-order approximate solution (34) obtained by means of OAFM is highly accurate.

Table 1

Comparison of analytical and numerical results

η	$\overline{f}$	f_{num} [22]	$\overline{f}'$	f'_{num} [22]
0	0	0	0	0
0.4	0.02658	0.0265	0.13276	0.1327
0.8	0.10601	0.1061	0.26415	0.2647
1	0.16535	0.1655	0.32918	0.3297
1.4	0.32262	0.322	0.45626	0.4562
2	0.64995	0.6560	0.63043	0.6297
2.4	0.92238	0.9222	0.72898	0.7289
2.8	1.23084	1.2309	0.81037	0.8115
3	1.3964	1.3961	0.84445	0.8460
3.6	1.92807	1.9295	0.92187	0.9233
4	2.30397	2.3057	0.95551	0.9555
4.4	2.69098	2.6923	0.97786	0.9758
5	3.2839	3.2832	0.99600	0.9915
5.4	3.68348	3.6803	1.0012	0.9974
6	4.28503	4.2796	1.00311	0.9989
6.4	4.68614	4.6793	1.00228	0.9996
7	5.28684	5.2792	0.99992	0.9999
7.4	5.68647	5.6792	0.99827	1
8	6.2848	6.2792	0.99628	1
8.4	6.68313	6.6792	0.99539	1
8.8	7.08116	7.0792	0.99484	1
9	7.28012	7.2797	0.99469	1

6. CONCLUSIONS

In the present work, a new technique is proposed to obtain an analytical solution of the Blasius problem. We obtained an effective analytic solution of the 2D laminar viscous flow over a semi-infinite flat plate. Our solution is valid in the whole region $0 \le \eta \le \infty$.

Comparison with numerical results reveals that the proposed method is very accurate. This method is valid even if the nonlinear equation does not contain any small or large parameters.

Our procedure provides us with a simple way to optimally control and adjusts the convergence of the solution and can give good approximations in a few terms after only one iteration. The convergence of the approximate solution given by

OAFM is determined by the optimal auxiliary functions A_1 and A_2 whose parameters C_i are optimally determined.

The obtained approximate analytical solution is in very good agreement with the numerical simulation results, which proves the validity of our procedure.

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