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COUPLED TRANSVERSAL AND LONGITUDINAL VIBRATIONS OF A PLANE MECHANICAL SYSTEM WITH TWO IDENTICAL BEAMS

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Abstract. The paper aims to highlight some vibrations properties for a plane beams structure, where transverse vibrations are coupled with the longitudinal vibrations. In engineering practice it often happens that identical parts are used in projects for reasons which relate to design time, material costs and execution time. Considering only the statics approach, these types of systems have been studied in strength of materials. In the case of dynamics, some comments on the calculation of symmetrical systems have been made in the literature, but a systematic study does not yet exist; some particular cases were treated in the literature. In this paper we aim to study a symmetrical system of beams that presents vibrations in the system's plane. The determination of properties of such systems would decrease the computational time and effort and would automatically imply lower development and testing costs and would increase the accuracy of calculus.

Keywords: vibration, eigenvalue, eigenmode, beam, symmetry.

1. INTRODUCTION

Many of the machines and devices made for engineering applications exhibit symmetries that can lead, in some cases, to the simplification of the dynamic analysis made for these structures. One consequence would be to shorten design time and thus lowering costs of this stage. The repetitive information provided by the structure with symmetry or identical parts can help ease the computational effort. For the case of a static approach, solutions to this problem are presented in strength of materials and allow for easier calculus. However, from the perspective of dynamic vibrations for systems with elastic elements, although some properties have been observed more [1], a systematic study of the problem has not yet been developed. Particular cases were studied in [2–6]. In the following we will study the case of a mechanical system composed of 3 beams, considered in the plane, of which two are identical. The transverse vibrations are considered to take place in the plane defined by

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the three beams structure. Longitudinal vibrations will also take place. Some coupling occurs between the longitudinal and transverse vibration, this being determined by the rigid link between the beams.

2. BOUNDARY CONDITIONS

Let us consider a mechanical structure (Fig. 1) which consists of two identical beams, MP and NP, rigidly fixed at right angles (for example by welding or soldering in the case of an engineering system) on a third beam, PR. The beams may present vibrations in their defining plane and, at the same time, also longitudinal vibrations along their axes.

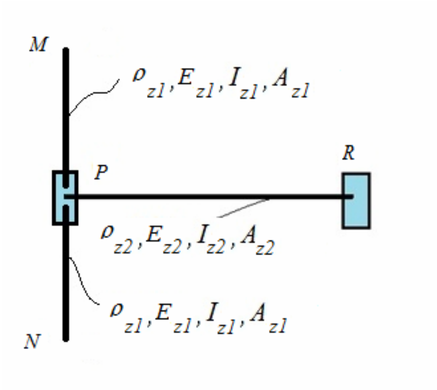

Fig. 1 – The mechanical system.

M and N are free ends, and from this results that the bending moment, shear force and longitudinal force are zero at these points (six Boundary Conditions). The endpoint R of the beam PR is clamped. As a result, the arrow (the deflection), the rotation (the tangential slope) at that point are zero (3 Boundary Conditions). The transverse displacement of the point P of the beam PR is equal to the axial displacement of the beams MP and NP (two Boundary Conditions). The longitudinal displacements of the point P of the beam PR are equal to the transverse displacement of the beams MP and NP (two Boundary Conditions). The tangential slope of the three beams in the point P are equal (two more Boundary Conditions). If we do the balance for the infinitesimal element around the point P, it will result that the sum of the shear forces that occur in MP and NP, in the point P, is equal to the axial force in P for the beam PR. The sum of the axial forces in P for MP and NP is equal to the shear force in P for the beam PR. The sum of the three bending moments, corresponding to the three beams, in P, will be zero. Finally the problem introduces 18 Boundary Conditions. They will allow the writing of the analytical conditions that will allow the determination of the 18 constants of integration.

3. EQUATION OF TRANSVERSE AND LONGITUDINAL VIBRATION

In the following we will consider a continuous beam with constant section. For this situation, the transverse vibration of the beams are described, if there are no forces distributed along the length of the beam, by the well-known equations [7, 12, 13, 14]

$$\frac{\partial^4 v}{\partial x} + \frac{\rho A}{E I_z} \frac{\partial^2 v}{\partial x^2} = 0. \tag{1}$$

In equation (1) we denoted with: v – deflection of the beam, A – the sectional area of the beam, ρ – the density of the material, E – Young's modulus, I_z – the second moment of the area with respect to the z axis in the center of mass and x – the distance from the left end of the beam to the point where v is the deflection.

To solve the differential equation above we search for a solution like:

$$v(x,t) = \Phi(x)\sin(pt + \theta). \tag{2}$$

If we set the condition that (2) verifies (1) at any moment in time, we obtain:

$$\frac{\partial^4 \Phi}{\partial x^2} - p^2 \frac{\rho A}{E I_z} \Phi = 0. \tag{3}$$

We denote:

$$\lambda^4 = \frac{\rho A}{E I_z}. \tag{4}$$

In this situation, (1) becomes:

$$\frac{\partial^4 \Phi}{\partial x^2} - p^2 \lambda^4 \Phi = 0, \tag{5}$$

where Φ is the function which gives the deformed beam (the eigenmode) which will vibrate with the pulsation p. In the following we will write the equations (5) for the part of the mechanical system represented by the three beams. In this way, three differential equations, which will correspond to the portions MP, NP and PR, will be obtained.

– for the first segment, MP: $$\frac{\partial^4 \Phi_{MP}}{\partial x^2} - p^2 \frac{\rho_1 A_1}{E_1 I_{z1}} \Phi_{MP} = 0 \tag{5.1}$$

– for the second segment, NP: $$\frac{\partial^4 \Phi_{NP}}{\partial x^2} - p^2 \frac{\rho_1 A_1}{E_1 I_{z1}} \Phi_{NP} = 0 \tag{5.2}$$

– for the segment PR: $$\frac{\partial^4 \Phi_{PR}}{\partial x^2} - p^2 \frac{\rho_2 A_2}{E_2 I_{z2}} \Phi_{PR} = 0 \tag{5.3}$$

If we note:

$$\lambda_1^4 = \frac{\rho_1 A_1}{E_1 I_{z1}}; \quad \lambda_2^4 = \frac{\rho_2 A_2}{E_2 I_{z2}}, \tag{6}$$

the solution to the system of equations of 4th degree (5.1), (5.2) and (5.3) will be given by [8–12]:

$$\Phi_{MP}(x) = \alpha_1 \sin\lambda_1\sqrt{p}x + \alpha_2 \cos\lambda_1\sqrt{p}x + \alpha_3 \mathrm{sh}\lambda_1\sqrt{p}x + \alpha_4 \mathrm{ch}\lambda_1\sqrt{p}x \tag{6.1}$$

$$\Phi_{NP}(x) = \beta_1 \sin\lambda_1\sqrt{p}x + \beta_2 \cos\lambda_1\sqrt{p}x + \beta_3 \mathrm{sh}\lambda_1\sqrt{p}x + \beta_4 \mathrm{ch}\lambda_1\sqrt{p}x \tag{6.2}$$

$$\Phi_{PR}(x) = \gamma_1 \sin\lambda_1\sqrt{p}x + \gamma_2 \cos\lambda_1\sqrt{p}x + \gamma_3 \mathrm{sh}\lambda_1\sqrt{p}x + \gamma_4 \mathrm{ch}\lambda_1\sqrt{p}x. \tag{6.3}$$

The longitudinal vibrations of the beams are given by:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} = 0, \tag{7}$$

where u is axial displacement of the beam, x is the ordinate of the point with the displacement u. In order to solve the equation, we will consider the solution u in the form of:

$$u(x,t) = \Psi(x)\sin(pt + \theta). \tag{8}$$

Bringing it into (7), we obtain:

$$\frac{\partial^2 \Psi}{\partial x^2} + p^2 \delta^2 \Psi = 0 \tag{9}$$

where:

$$\delta^2 = \frac{\rho}{E}. \tag{10}$$

If we make these notations, the solution of the system of equations obtained by applying (9) for the beams MP, NP and PR looks like:

$$\Psi_{MP}(x) = \alpha_5 \sin\delta_1 px + \alpha_6 \cos\delta_1 px \tag{11.1}$$

$$\Psi_{NP}(x) = \beta_5 \sin\delta_1 px + \beta_6 \cos\delta_1 px \tag{11.2}$$

$$\Psi_{PR}(x) = \gamma_5 \sin\delta_1 px + \gamma_6 \cos\delta_1 px. \tag{11.3}$$

The equations (6.1), (6.2), (6.3), (11.1), (11.2), (11.3) describe the free vibrations of the studied mechanical system. We observe that in these systems we have 18 constants of integration: $\alpha_i, \beta_i, \gamma_i \ (i=\overline{1,6})$, which will be determined considering the Boundary Conditions.

If we note M^b the bending moment of a beam in the section *x*, *T* the shear force and *S* the axial force, the Boundary Conditions can be written as:

a) For the MP beam, the M end is free, so we will have: $M^b_{MP}(0,t)=0$;

   $T_{MP}(0,t)=0$; $S_{MP}(0,t)=0$;

b) For the NP beam, the N is free, so we will have: $M^b_{NP}(0,t)=0$;

   $T_{NP}(0,t)=0$; $S_{NP}(0,t)=0$;

c) For the PR beam, the R end is fixed, so we will have: $v_{PR}(l_2,t)=0$;

   $v'_{PR}(l_2,t)=0$; $u_{PR}(l_2,t)=0$,

so, in the end, we will have **9 conditions**. In the following, we will write these conditions in analytical form. The bending moment, shear force and axial force can be written using:

$$M^b(x)=-EI_z\frac{\partial^2 v(x)}{\partial x^2};\quad T(x)=-EI_z\frac{\partial^3 v(x)}{\partial x^3};\quad S(x)=EA\frac{\partial u(x)}{\partial x}. \tag{12}$$

By successive differentiations it obtains:

$$\frac{\partial^2 v(x)}{\partial x^2}=\Phi''_{MP}(x)\sin(pt+\theta)=$$
$$=\left(\lambda_1\sqrt{p}\right)^2\left(-\alpha_1\sin\lambda_1\sqrt{p}x-\alpha_2\cos\lambda_1\sqrt{p}x+\alpha_3\mathrm{sh}\lambda_1\sqrt{p}x+\alpha_4\mathrm{ch}\lambda_1\sqrt{p}x\right)\sin(pt+\theta) \tag{13}$$

$$\frac{\partial^3 v(x)}{\partial x^3}=\Phi'''_{MP}(x)\sin(pt+\theta)=$$
$$=\left(\lambda_1\sqrt{p}\right)^3\left(-\alpha_1\cos\lambda_1\sqrt{p}x+\alpha_2\sin\lambda_1\sqrt{p}x+\alpha_3\mathrm{ch}\lambda_1\sqrt{p}x+\alpha_4\mathrm{sh}\lambda_1\sqrt{p}x\right)\sin(pt+\theta) \tag{14}$$

$$\frac{\partial u(x)}{\partial x}=\Psi'_{MP}(x)\sin(pt+\theta)=\delta_1 p\left(\alpha_5\cos\delta_1 px-\alpha_6\sin\delta_a px\right)\sin(pt+\theta), \tag{15}$$

for the MP beam, and similar equations for the NP and PR beams.

Replacing (13), (14) and (15) in (12) and taking into consideration the 9 conditions a), b) and c) which we previously described, we obtain:

$$-\alpha_2 + \alpha_4 = 0 \tag{7.1}$$

$$-\alpha_1 + \alpha_3 = 0 \tag{7.2}$$

$$\alpha_5 = 0 \tag{7.3}$$

$$-\beta_2 + \beta_4 = 0 \tag{7.4}$$

$$-\beta_1 + \beta_3 = 0 \tag{7.5}$$

$$\alpha_5 = 0 \tag{7.6}$$

$$\gamma_1 \sin\lambda_2\sqrt{p}l_2 + \gamma_2 \cos\lambda_2\sqrt{p}l_2 + \gamma_3 \mathrm{sh}\lambda_2\sqrt{p}l_2 + \gamma_4 \mathrm{ch}\lambda_2\sqrt{p}l_2 = 0 \tag{7.7}$$

$$\gamma_1 \cos\lambda_2\sqrt{p}l_2 - \gamma_2 \sin\lambda_2\sqrt{p}l_2 + \gamma_3 \mathrm{ch}\lambda_2\sqrt{p}l_2 + \gamma_4 \mathrm{sh}\lambda_2\sqrt{p}l_2 = 0 \tag{7.8}$$

$$\gamma_5 \sin\delta_2 pl_2 + \gamma_6 \cos\delta_2 pl_2 = 0. \tag{7.9}$$

Let us consider now the continuity of the system in P. his leads to the following conditions: we know that the axial displacements of the MP and NP beams are equal and opposite, and equal to the displacement of the PR beam in P: $u_{MP}(l_1,t)$ = $u_{NP}(l_1,t)$ = $v_{CD}(0,t)$ (**two conditions**), the rotations in P for the beams MP, NP and PR are equal: $v'_{MP}(l_1,t)$ = $v'_{NP}(l_1,t)$ = $v'_{PR}(0,t)$ (**two conditions**), the displacements of the beams MP și NP in P are equal to the axial displacement of the PR beam: $v_{MP}(l_1,t)$ = $v_{NP}(l_1,t)$ = $u_{CD}(0,t)$ (**two conditions**). These six conditions lead to:

$$\alpha_5 \sin\delta_1 pl_1 + \alpha_6 \cos\delta_1 pl_1 = \gamma_2 + \gamma_4 \tag{7.10}$$

$$\beta_5 \sin\delta_1 pl_1 + \beta_6 \cos\delta_1 pl_1 = \gamma_2 + \gamma_4 \tag{7.11}$$

$$\lambda_1\left(\alpha_1 \cos\lambda_1\sqrt{p}l_1 - \alpha_2 \sin\lambda_1\sqrt{p}l_1 + \alpha_3 \mathrm{ch}\lambda_1\sqrt{p}l_1 + \alpha_4 \mathrm{sh}\lambda_1\sqrt{p}l_1\right) = \lambda_2(\gamma_1 + \gamma_3) \tag{7.12}$$

$$\lambda_1\left(\beta_1 \cos\lambda_1\sqrt{p}l_1 - \beta_2 \sin\lambda_1\sqrt{p}l_1 + \beta_3 \mathrm{ch}\lambda_1\sqrt{p}l_1 + \beta_4 \mathrm{sh}\lambda_1\sqrt{p}l_1\right) = \lambda_2(\gamma_1 + \gamma_3) \tag{7.13}$$

$$\alpha_1 \sin\lambda_1\sqrt{p}l_1 + \alpha_2 \cos\lambda_1\sqrt{p}l_1 + \alpha_3 \mathrm{sh}\lambda_1\sqrt{p}l_1 + \alpha_4 \mathrm{ch}\lambda_1\sqrt{p}l_1 = \gamma_6 \tag{7.14}$$

$$\beta_1 \sin\lambda_1\sqrt{p}l_1 + \beta_2 \cos\lambda_1\sqrt{p}l_1 + \beta_3 \mathrm{sh}\lambda_1\sqrt{p}l_1 + \beta_4 \mathrm{ch}\lambda_1\sqrt{p}l_1 = \gamma_6. \tag{7.15}$$

The last three conditions are obtained considering the balance of an infinitesimal mass element containing the point P.

We have: the sum of shear forces on the two branches MP and NP occurring in point P is equal to the axial force of the beam PR in P:

$$T_1 + T_2 = S. \tag{16}$$

The shear force that occurs in a section of the beam at a distance x from the left end of the beam and the bending moment are given by (12) and (14). Substituting in (16) yields the final equilibrium condition of shear:

$$\begin{aligned}&(\alpha_1+\beta_1)\cos\lambda_1\sqrt{p}l_1-(\alpha_2+\beta_2)\sin\lambda_1\sqrt{p}l_1-(\alpha_3+\beta_3)\mathrm{ch}\lambda_1\sqrt{p}l_1-\\&-(\alpha_4+\beta_4)\mathrm{sh}\lambda_1\sqrt{p}l_1=\frac{\delta_2E_2A_2}{E_1I_{z1}\lambda_1^3\sqrt{p}}\gamma_1.\end{aligned} \tag{7.16}$$

If we note:

$$a_3=-\frac{\delta_2E_2A_2}{E_1I_{z1}\lambda_1^3\sqrt{p}},$$

we can write that:

$$\begin{aligned}&-(\alpha_1+\beta_1)\cos\lambda_1\sqrt{p}l_1+(\alpha_2+\beta_2)\sin\lambda_1\sqrt{p}l_1+(\alpha_3+\beta_3)\mathrm{ch}\lambda_1\sqrt{p}l_1+\\&+(\alpha_4+\beta_4)\mathrm{sh}\lambda_1\sqrt{p}l_1=a_3\gamma_1\end{aligned} \tag{7.16'}$$

Similarly, an equilibrium condition can be obtained for the bending moments:

$$S_1 + S_2 = T \tag{17.1}$$

$$M^{b1} + M^{b2} = M^{b}. \tag{17.2}$$

The torsional bending moment occurring in a section of the beam at a distance x from the left end of the beam is given by (12), (13) and (15). Substituting (12) in (17.1) and (17.2), the balance conditions for the bending moments are obtained:

$$E_1A_1\frac{\partial u_{MP}(l_1)}{\partial x}+E_1A_1\frac{\partial u_{NP}(l_1)}{\partial x}=-E_2I_{z2}\frac{\partial^3 v_{PR}(0)}{\partial x^3} \tag{16}$$

$$E_1I_{z1}\frac{\partial^2 v_{MP}(l_1)}{\partial x^2}+E_1I_{z1}\frac{\partial^2 u_{NP}(l_1)}{\partial x^2}=E_2I_{z2}\frac{\partial^2 v_{PR}(0)}{\partial x^2}. \tag{17}$$

If we consider (13) and (15), the final conditions for the equilibrium of the sectional forces and bending moments (16) and (17) become:

$$(\alpha_5+\beta_5)\cos\delta_1pl_1-(\alpha_6+\beta_6)\sin\delta_1pl_1=-\frac{E_2I_{z2}\lambda_2^3}{E_1A_1\delta_1}\sqrt{p}(-\gamma_1+\gamma_3) \tag{7.17}$$

$$-(\alpha_1+\beta_1)\sin\lambda_1\sqrt{p}l_1-(\alpha_1+\beta_1)\cos\lambda_1\sqrt{p}l_1+(\alpha_1+\beta_1)\mathrm{sh}\lambda_1\sqrt{p}l_1+$$
$$+(\alpha_1+\beta_1)\mathrm{ch}\lambda_1\sqrt{p}l_1=\frac{E_2I_{z2}\lambda_2^2}{E_1A_1\lambda_1^2}(-\gamma_2+\gamma_4). \tag{7.18}$$

If we note: $a_2=-\frac{E_2I_{z2}\lambda_2^3}{E_1A_1\delta_1}\sqrt{p}$ and $a_1=\frac{E_2I_{z2}\lambda_2^2}{E_1A_1\lambda_1^2}$, (7.17) and (7.18) become:

$$(\alpha_5+\beta_5)\cos\delta_1pl_1-(\alpha_6+\beta_6)\sin\delta_1pl_1=a_2(-\gamma_1+\gamma_3) \tag{7.17}$$

$$\begin{gathered}-(\alpha_1+\beta_1)\sin\lambda_1\sqrt{p}l_1-(\alpha_1+\beta_1)\cos\lambda_1\sqrt{p}l_1+(\alpha_1+\beta_1)\mathrm{sh}\lambda_1\sqrt{p}l_1+\\+(\alpha_1+\beta_1)\mathrm{ch}\lambda_1\sqrt{p}l_1=a_1(-\gamma_2+\gamma_4).\end{gathered} \tag{7.18}$$

To determine the constants which provide the imposed end conditions we must solve the homogeneous linear system (7.1–7.18), in order to determine constants $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6$, $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6$, $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6$.

In the following, we will note:

$$\{BC\}=\begin{Bmatrix}\alpha\\ \beta\\ \gamma\end{Bmatrix}= \tag{19}$$

$$=[\ \alpha_1\ \ \alpha_1\ \ \alpha_1\ \ \alpha_1\ \ \alpha_1\ \ \alpha_1\ \ \beta_1\ \ \beta_1\ \ \beta_1\ \ \beta_1\ \ \beta_1\ \ \beta_1\ \ \gamma_2\ \ \gamma_2\ \ \gamma_2\ \ \gamma_2\ \ \gamma_2\ \ \gamma_2\ ]^{\mathrm{T}}$$

being the vector of the integration constants.

If we set the condition that the system should have a zero determinant, we will now get the natural frequencies (the eigenvalues) of the mechanical system:

If we note:

$$\mathbf{A}_{11}=\begin{bmatrix}0 & -1 & 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ \sin\lambda_1\sqrt{p}l_1 & \cos\lambda_1\sqrt{p}l_1 & \mathrm{sh}\lambda_1\sqrt{p}l_1 & \mathrm{ch}\lambda_1\sqrt{p}l_1 & 0 & 0\\ \cos\lambda_1\sqrt{p}l_1 & -\sin\lambda_1\sqrt{p}l_1 & \mathrm{ch}\lambda_1\sqrt{p}l_1 & \mathrm{sh}\lambda_1\sqrt{p}l_1 & 0 & 0\\ 0 & 0 & 0 & 0 & \sin\delta_1p & \cos\delta_1p\end{bmatrix} \tag{19}$$

$$\mathbf{A}_{13}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \frac{\lambda_2}{\lambda_1} & 0 & \frac{\lambda_2}{\lambda_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{20}$$

$$\mathbf{A}_{31}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\delta_1 p & -\sin\delta_1 p \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\lambda_1\sqrt{p}l_1 & -\sin\lambda_1\sqrt{p}l_1 & -\mathrm{ch}\lambda_1\sqrt{p}l_1 & -\mathrm{sh}\lambda_1\sqrt{p}l_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin\lambda_1\sqrt{p}l_1 & -\cos\lambda_1\sqrt{p}l_1 & \mathrm{sh}\lambda_1\sqrt{p}l_1 & \mathrm{ch}\lambda_1\sqrt{p}l_1 & 0 & 0 \end{bmatrix} \tag{21}$$

$$\mathbf{A}_{33}=\begin{bmatrix} \sin\lambda_2\sqrt{p}l_2 & \cos\lambda_2\sqrt{p}l_2 & \mathrm{sh}\lambda_2\sqrt{p}l_2 & \mathrm{ch}\lambda_2\sqrt{p}l_2 & 0 & 0 \\ \cos\lambda_2\sqrt{p}l_2 & -\sin\lambda_2\sqrt{p}l_2 & \mathrm{ch}\lambda_2\sqrt{p}l_2 & \mathrm{sh}\lambda_2\sqrt{p}l_2 & 0 & 0 \\ 0 & -a_1 & 0 & 0 & \sin\delta_2 p & \cos\delta_2 p \\ 0 & 0 & 0 & 0 & -a_3 & 0 \\ a_2 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 \end{bmatrix} \tag{22}$$

the system's matrix, **S** becomes:

$$\mathbf{S}_{18\times18}=\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{11} & \mathbf{A}_{13} \\ \mathbf{A}_{31} & \mathbf{A}_{31} & \mathbf{A}_{33} \end{bmatrix} \tag{23}$$

and the system (7) can be written as:

$$[\mathbf{S}][\mathbf{C}]=\{\mathbf{0}\} \tag{24}$$

The condition that the system should have a zero determinant is:

$$\det(\mathbf{S})=0 \tag{24$'$}$$

and solving this equation will give us the eigenvalues of the system of differential equations (7.1–7.18).

4. EIGENVALUES AND EIGENMODES

The following will highlight a property of eigenvalues of such a system. Let us consider only one of the identical beams (MP or NP), to be free in M (or N) and clamped in P.

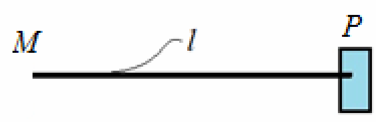

Fig. 2 – One single beam.

The transverse vibrations are described by the differential equation (1), written in the form:

$$\frac{\partial^4 v}{\partial x^4} + \frac{\rho_1 A_1}{E_1 I_{z1}} \frac{\partial^2 v}{\partial t^2} = 0, \tag{25}$$

which has the solution previously presented:

$$v(x,t) = \Phi(x)\sin(pt + \theta),$$

with:

$$\Phi_{MP}(x) = \alpha_1 \sin\lambda_1\sqrt{p}x + \alpha_2 \cos\lambda_1\sqrt{p}x + \alpha_3 \mathrm{sh}\lambda_1\sqrt{p}x + \alpha_4 \mathrm{ch}\lambda_1\sqrt{p}x. \tag{26}$$

For the longitudinal vibrations, the equation which describes the section x of the beam MP (or NP) is (7)

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho_1}{E_1} \frac{\partial^2 u}{\partial t^2} = 0. \tag{27}$$

The solution of the differential equation (27), previously determined in the paper (relation (11)) will be:

$$u(x,t) = \Psi(x)\sin(pt + \theta)$$

$$\Psi_{MP}(x) = \alpha_5 \sin\delta_1 px + \alpha_6 \cos\delta_1 px. \tag{28}$$

The M end is free, while the P end is fixed: $M_{MP}^{b}(0,t)=0$; $T_{MP}(0,t)=0$; $S_{MP}(0,t)=0$; $v_{PR}(l,t)=0$; $v'_{PR}(l,t)=0$; $u_{PR}(l,t)=0$.

Considering these conditions for the solutions (26) and (28), the constants α_i, $i=\overline{1,6}$ can be determined from the linear homogenous system:

$$\left[\mathbf{A}_{11}\right]\left\{\mathbf{C}\right\}=\mathbf{0}. \tag{29}$$

where $\left[\mathbf{A}_{11}\right]$ is the matrix defined by (19). The condition $\det(\mathbf{A}_{11})=0$ allows us to find the eigenvalues for the beam MP (or NP). After calculus, we obtain the well-known equations:

$$\cos\lambda_1\sqrt{p}l_1\,\mathrm{ch}\lambda_1\sqrt{p}l_1=1 \quad \text{and} \quad \mathrm{tg}\lambda_1\sqrt{p}l_1=1 \tag{30}$$

which, for the set of real data, gives the eigenvalues for one of the beams (MP or NP) fixed in R.

We have the following theorem:

THEOREM 1. *The eigenvalues of free at one end and fixed at the other end symmetrical beams are also eigenvalues of the whole mechanical system.*

Proof: It should be emphasized that $\det(\mathbf{A}_{11})=0$ implies $\det(\mathbf{S})=0$. In work [15] this is proven in a more general case. It results that the property exists for the particular case that we have in this work. $\mathbf{A}_{11}$ and $\mathbf{S}$ are defined by the relations (19) and (23) respectively.

Hence, the eigenvalues of the single beams, clamped at one endpoint and free at the other, are also eigenvalues of the entire system, clamped in P and having the free endpoints M and N.

The eigenvectors. If we have the matrix (23) and the eigenvalues, for these eigenvalues we can obtain the eigenvectors from the equation (24), written as:

$$\left[\mathbf{S}\right]\left\{\mathbf{\Phi}\right\}=\left\{\mathbf{0}\right\} \tag{24''}$$

We noted with $\left\{\mathbf{\Phi}\right\}$ the vector of constants obtained from the eigenvalues calculated for the condition (24′). The next step will demonstrate the following two theorems:

THEOREM 2. *For the eigenvalues common to the system in Fig. 2 and to the entire system in Fig. 1, the eigenvectors are of the form*:

$$\{\mathbf{\Phi}\} = \begin{Bmatrix} \mathbf{\Phi}_1 \\ -\mathbf{\Phi}_1 \\ \mathbf{0} \end{Bmatrix} \tag{31}$$

(*the components of the eigenmodes corresponding to the two identical beams are skew symmetric*).

Proof: The existence of common eigenvalues is proven by Theorem 1. For the eigenvalues obtained from (30), the following system must be solved:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{11} & \mathbf{A}_{13} \\ \mathbf{A}_{31} & \mathbf{A}_{31} & \mathbf{A}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{\Phi}_s \\ \mathbf{\Phi}_d \\ \mathbf{\Phi}_m \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{32}$$

with

$$\det(\mathbf{A}_{11}) = 0. \tag{33}$$

The condition (33) implies that we can find a vector Φ_s so that:

$$\mathbf{A}_{11}\mathbf{\Phi}_s = \mathbf{0} \tag{34}$$

and then (32) becomes:

$$\mathbf{A}_{13}\mathbf{\Phi}_m = \mathbf{0} \tag{35.a}$$

$$\mathbf{A}_{11}\mathbf{\Phi}_d + \mathbf{A}_{13}\mathbf{\Phi}_m = \mathbf{0} \tag{35.b}$$

$$\mathbf{A}_{31}\left(\mathbf{\Phi}_s + \mathbf{\Phi}_d\right) + \mathbf{A}_{33}\mathbf{\Phi}_m = \mathbf{0}. \tag{35.c}$$

From (35.a), because $\det\left(\mathbf{A}_{13}\right) \neq 0$ we immediately have:

$$\mathbf{\Phi}_m = \mathbf{0} \tag{36}$$

and replacing in (35.c) we obtain $\mathbf{\Phi}_s = -\mathbf{\Phi}_d$, which also verifies (35.b) if we consider (34). If we note $\mathbf{\Phi}_s = \mathbf{\Phi}_1$, we easily obtain (31).

THEOREM 3. *For the other eigenvalues, which are not obtained from Theorem 1, the eigenvectors are of the form*:

$$\{\mathbf{\Phi}\} = \begin{Bmatrix} \mathbf{\Phi}_1 \\ \mathbf{\Phi}_1 \\ \mathbf{\Phi}_3 \end{Bmatrix} \tag{37}$$

(*the components of the eigenmodes corresponding to the two identical beams are symmetric*)

Proof: For the calculated eigenvalues we have to solve the system (32), considering that $\det(\mathbf{A}) \neq 0$, or:

$$\mathbf{A}_{11}\mathbf{\Phi}_s + \mathbf{A}_{13}\mathbf{\Phi}_m = \mathbf{0} \tag{37.a}$$

$$\mathbf{A}_{11}\mathbf{\Phi}_d + \mathbf{A}_{13}\mathbf{\Phi}_m = \mathbf{0} \tag{37.b}$$

$$\mathbf{A}_{31}\left(\mathbf{\Phi}_s + \mathbf{\Phi}_d\right) + \mathbf{A}_{33}\mathbf{\Phi}_m = \mathbf{0} \tag{37.c}$$

By substracting (37.a) from (37.b) we obtain:

$$\mathbf{A}_{11}\left(\mathbf{\Phi}_s - \mathbf{\Phi}_d\right) = \mathbf{0} \tag{38}$$

If $\det\left(\mathbf{A}_{11}\right) \neq 0$, then we have $\mathbf{\Phi}_s - \mathbf{\Phi}_d = \mathbf{0}$ so $\mathbf{\Phi}_s = \mathbf{\Phi}_d = \mathbf{\Phi}_1$.

For the eigenvalues of the system which coincide with those of a single beam, fixed at one end and clamped to the other, the vibration modes are skewsymmetric, the two identical beams vibrate in opposite phase, and the third beam is at rest. For the other eigenvalues, the identical beams have identical vibration modes.

5. CONCLUSIONS

For reasons which relate to design time, material costs and execution time of some mechanical systems, identical parts (in terms of construction) are used. In this way, a device or a machine used in industry can be designed and executed faster and cheaper. The problem which we intend to study in this paper is to determine if these repetitive (symmetrical) structures induce characteristic properties of these systems and if these properties can bring advantages in calculus or design. The paper examined the particular case of a mechanical system consisting of three beams in the plane, two of them being identical, and identified vibration properties that enable ease of calculation.

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