CONSIDERATIONS CONCERNING YIELD CRITERIA INSENSITIVE TO HYDROSTATIC PRESSURE

ADRIAN SANDOVICI[1], PAUL-DORU BARSANESCU[2]

Abstract. For distinguishing between pressure insensitive and pressure sensitive criteria, the form of yield surface with invariants is preferred. It is known that when this function can be formulated only in terms of the stress deviator invariants, it describes a pressure-insensitive material behaviour. But expression of criteria in terms of invariants may be a difficult goal in some particular cases. In the present paper is mathematically demonstrated that all criteria developed for standard plastic materials, in other words homogeneous, isotropic, ductile (without so-called strength differential or SD effect) materials, loaded in plane state of stress (2D space), are insensitive to hydrostatic pressure.

Key words: plasticity, yield criteria, pressure insensitive, biaxial state of stress, invariants.

1. INTRODUCTION

Strength theory of materials is an important interdisciplinary field of research. It includes different criteria (yield criteria, failure criteria etc.) which are widely used in engineering and material science. Many scientists have studied yielding or fracture behaviour of materials in uniaxial or multiaxial stress conditions. Under the complex state of stress, important changes in responses occur from one material to another.

Any criterion or yield condition can be represented graphically as a yield surface in the stress space. *Haigh* and *Westgaard* have introduced the notion of *limit surface* in three dimensional space of principal stresses. The fundamental postulate concerning the yield surfaces was introduced by *Drucker*, *Bishop* and *Hill* [1]. This is an important contribution to the development of strength theories.

When the state of stress is on the yield surface, the material has reached its yield point. The material is elastic for any point (corresponding to a state of stress) situated inside of the yield surface (the yield function $f < 0$) and in elasto-plastic field for any point situated outside of it.

[1] "Gheorghe Asachi" Technical University of Iasi, Department of Mathematics and Informatics, B-dul Carol I, no. 11, 700506, Iasi, Romania.

[2] "Gheorghe Asachi" Technical University of Iasi, Department of Mechanical Engineering, Mechatronics and Robotics, B-dul Dimitrie Mangeron no. 61-63, 700050, Iasi, Romania.

It is very difficult to determine experimentally the entire yield surface of any material. However, there are many mathematical models of yield surfaces. *Mao-hong Yu* [1] classified the strength theories into three categories, according to the number of shear stresses taken into consideration:

- Single shear stress (including *Mohr-Coulomb*, *Tresca*, *Hoek-Brown* etc.);
- Twin shear stress (*Hill*, *Yu*);
- Octahedral shear stress (*von Mises*, *Burzynski*, *Drucker-Prager* etc.).

For some materials the stress at which the yield occurs is dependent on the level of hydrostatic pressure applied on the specimens, during the traction or compression test. Such materials are called sensitive to hydrostatic pressure.

The yielding of solid metals is generally unaffected by hydrostatic pressure (there are exceptions as iron, some high strength steels and high strength alloys). But the behaviour of most non-metallic materials (polymers, ceramics, concrete, geomaterials etc.) is dependent of hydrostatic pressure [1, 2, 3, 4]. Action of hydrostatic pressure increases the strength of the materials. In fact isotropic materials do not fail under engineering hydrostatic pressure conditions [5].

There are specific yielding criteria for each these two categories of materials. All criteria which predict that yielding is independent on the hydrostatic pressure are based on the assumption that the yield stress in tension and compression are equal (materials without SD effect). This assumption is reasonable for many ductile metallic materials, but inaccurate for some other materials, which have a tension-compression strength assymetry (SD effect). But the simple fact that a yielding criterion admits this hypothesis, does not offer the guarantee that it is insensitive to hydrostatic pressure.

Many triaxial-stress experiments were developed in order to obtain the initial and subsequent yield surfaces determined by hydrostatic pressure and many criteria have been proposed.

The equation of the yield surface can be written in different equivalent forms. For isotropic and homogeneous materials among the most popular are the following two:

$$f\left(\sigma_1,\sigma_2,\sigma_3\right)=0 \tag{1}$$

and

$$f\left(I_1,J_2,J_3\right)=0, \tag{2}$$

where $\sigma_1,\sigma_2,\sigma_3$ are the principal stresses, I_1 is the first (axiatoric) invariant of the stress tensor, J_2 and J_3 are the second and third invariants of the deviatoric stress tensor.

Although the form presented in Eq. (1) is more used then Eq. (2), for distinguishing pressure insensitive or pressure sensitive criteria, the form with invariants J_2, J_3 and I_1 is preferred. In the case of pressure-insensitive material behaviour, the first invariant I_1 has no influence in Eq. (2) [6].

Many mathematical models for incompressible material behavior or pressure insensitive criteria (e. g. *von Mises*, *Tresca*, *Drucker*, *Yu*, *Schmidt-Ishlinsky* etc.), as well as pressure-sensitive criteria (*Mohr-Coulomb*, *Beltrami*, *Drucker-Prager*, *Mirolyubov*, *Burzynski-Yagn*, *Burzynski-Torre*, *Pisarenko-Lebedev, Christensen* etc.) are used. According to *Holm Altenbach* et al. [7], in the main part, pressure-insensitive, pressure-sensitive and combined models are separated. But many pressure dependent yield criteria are based on the *von Mises* or *Huber* criterion [8].

The *von Mises* criterion can be written in the form:

$$J_2 = k^2; \;\; k = \frac{\sigma_y}{\sqrt{3}} \tag{3}$$

where σ_y is tensile yield stress, which is the same (in absolute value) for tension and compression. This criterion is applied to ductile metals and alloys.

In order to include the influence of pressure dependency on *von Mises* criterion, *Hu* and *Pae* added in Eq. (3) a second term, depending on I_1 invariant [8, 9]:

$$J_2 = k^2 + \sum_{i=1}^{n} \alpha_i I_1^i. \tag{4}$$

In terms of invariants, the *Drucker-Prager* criterion has the following form:

$$\sqrt{J_2} + \frac{1}{\sqrt{3}} \cdot \frac{\sigma_C - \sigma_T}{\sigma_C + \sigma_T} I_1 = \frac{2}{\sqrt{3}} \cdot \frac{\sigma_C \sigma_T}{\sigma_C + \sigma_T} \tag{5}$$

where σ_T and σ_C are the uniaxial tensile and compressive strengths, respectively, which are determined by experiments [10]. It is noticeable that this is a particular form of Eq. (4)

$$\sqrt{J_2} = A + BI_1 \tag{6}$$

where the constants A and B are determined by experiments.

Drucker-Prager criterion can be used for polymers, foams, concrete, rocks, soils and other pressure-dependent materials.

The *Christensen*'s criterion is a linear combination of J_2 and I_1 invariants [10]:

$$\frac{3}{\sigma_T \sigma_C} J_2 + \left(\frac{1}{\sigma_T} - \frac{1}{\sigma_C} \right) I_1 = 1 \quad \text{for} \quad 0 \le \frac{\sigma_T}{\sigma_C} \le 1. \tag{7}$$

It can be used for homogeneous and isotropic materials, in order to evaluate the transition of fracture from brittle to ductile and viceversa.

Both *Drucker-Prager* and *Christensen* depend on I_1 and are pressure-dependent criteria. They are used for materials with SD effect. But both *Drucker-Prager* and *Christensen* criteria are reduced to *von Mises* criterion when $\sigma_C = \sigma_T$ (i.e. materials without SD effect).

Mohr-Coulomb criterion is widely used in rock and soil mechanics and generally to describe the constitutive behaviours of granular materials. It is usually applied to materials for which the compressive strength far exceeds the tensile strength. As a function of principal stresses, this criterion can be written

$$\frac{\sigma_1}{\sigma_{LT}} - \frac{\sigma_3}{\sigma_{LC}} = 1 \tag{8}$$

where σ_L is the limit normal stress (yielding or ultimate stress) and indices T and C refer to tension and compression, respectively. This criterion is pressure-sensitive and it ignores the intermediate principal stress. For ductile materials

$$\sigma_{LT} = \sigma_{LC} = \sigma_y \tag{9}$$

and consequently *Mohr-Coulomb* criterion is reduced to *Tresca* criterion

$$\sigma_y = |\sigma_1 - \sigma_3| \tag{10}$$

Tresca criterion can be also expressed by the deviatoric invariants [11]:

$$4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 = 0; \quad k = \frac{\sigma_y}{2} \tag{11}$$

Equation (11) is much more complicated than Eq. (10) and for this reason it is not used for engineering calculations. But it has the important merit to demonstrate that *Tresca* criterion is insensitive to hydrostatic pressure. Thereby *Mohr-Coulomb* criterion can be considered to be a pressure-modified *Tresca* criterion [12]. This criterion is applied to ductile metals and alloys, as well as *von Mises*.

2. THE YIELD SURFACE IN 2D SPACE

Many criteria are used to describe the behaviour of different materials. To demonstrate that a criterion is pressure insensitive or pressure sensitive, it must be written in form with invariants. This work may be difficult in some particular cases, as it was for *Tresca* criterion, for example.

Mao-hong Yu et al. [4] have determined the three principal stresses as functions of invariants J_2, J_3 and I_1. The question is if the principal stresses can be written only in function of deviatoric invariants, at least for biaxial state of stress (in 2D space).

The following demonstration is based on the assumptions that the material is homogeneous and isotropic, without SD effect (namely the yield stress in tension and compression are equal and they are equally treated). Another demonstration was featured in [13].

The Cauchy stress invariants I_1, I_2, I_3 and respectively the invariants of the deviatoric part of the Cauchy stress J_1, J_2, J_3 are [14]:

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{12}$$

$$\begin{aligned} J_1 &= 0 \\ J_2 &= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] \\ J_3 &= \frac{1}{81}\left[(2\sigma_1-\sigma_2-\sigma_3)^3 + (2\sigma_2-\sigma_1-\sigma_3)^3 + (2\sigma_3-\sigma_1-\sigma_2)^3\right] \end{aligned} \tag{13}$$

Between the two groups of invariants there are the following relations [15]:

$$\begin{aligned} J_1 &= I_1 - I_1 = 0 \\ J_2 &= \frac{1}{3}I_1^2 - I_2 \\ J_3 &= \frac{2}{27}I_1^3 - \frac{1}{3}I_1I_2 + I_3. \end{aligned} \tag{14}$$

For biaxial state of stress ($\sigma_2 = 0$) one can easily obtain the following relationships between the two groups of invariants:

$$J_2 = \frac{1}{3}I_1^2 - I_2 \tag{15}$$

$$J_3 = \frac{2}{27}I_1^3 - \frac{1}{3}I_1I_2 = \frac{1}{3}I_1\left(\frac{2}{9}I_1^2 - I_2\right). \tag{16}$$

Let $x = (\sigma_1 - \sigma_3)^2$ and define $y = x - 2J_2$. It is easily seen that

$$x = (\sigma_1-\sigma_3)^2 = (\sigma_1+\sigma_3)^2 - 4\sigma_1\sigma_3 = I_1^2 - 4I_2$$

so that

$$y = x - 2J_2 = \ = I_1^2 - 4I_2 - \frac{2}{3}\left(I_1^2 - 3I_2\right) = \frac{1}{3}I_1^2 - 2I_2.$$

One successively has

$$y^3-3yJ_2^2+27J_3^2-2J_2^3=\left(\frac{1}{3}I_1^2-2I_2\right)^3-3\left(\frac{1}{3}I_1^2-2I_2\right)\cdot J_2^2+27J_3^2-2J_2^3=$$

$$=\left(\frac{1}{3}I_1^2-2I_2\right)^3-\frac{1}{9}\left(I_1^2-6I_2\right)\cdot\left(I_2^2-3I_2\right)^2+3\cdot I_1^2\left(\frac{2}{9}I_1^2-I_2\right)^2-\frac{2}{27}\cdot\left(I_1^2-3I_2\right)^3=0$$

Consequently, y verifies the following equation:

$$y^3-3yJ_2^2+27J_3^2-2J_2^3=0. \tag{17}$$

With the substitutions $z=\dfrac{y}{J_2}$ and $a=\dfrac{27J_3^2-2J_2^3}{J_2^3}$ the equation (17) can be written as:

$$z^3-3z+a=0 \tag{18}$$

One successively has $4J_2^3\geq 27J_3^2\Leftrightarrow 4\left(\frac{1}{3}I_1^2-I_2\right)^3\geq 27\left(\frac{2}{27}I_1^3-\frac{1}{3}I_1I_2\right)^2\Leftrightarrow$

$\Leftrightarrow 9\left(\dfrac{I_1}{3}\right)^2\geq 4I_2\Leftrightarrow I_1^2\geq 4I_2\Leftrightarrow\left(\sigma_1+\sigma_3\right)^2\geq 4\sigma_1\cdot\sigma_3\Leftrightarrow\left(\sigma_1-\sigma_3\right)^2\geq 0$, so that $a\leq 2$. On the other hand, $\dfrac{4J_2^3-27J_3^2}{J_2^3}\leq 4\Leftrightarrow 4J_2^3-27J_3^2\leq 4J_2^3\Leftrightarrow 27J_3^2\geq 0$, which implies that $a\geq -2$.

Consequently,

$$-2\leq a\leq 2. \tag{19}$$

The real function $f(z)=z^3-3z+a$ has a maximum value equal to $a+2$ in $z=-1$ and a minimum local value equal to $a-2$ in $z=1$. Using (19) one concludes that the equation (18) has three real solutions: $z_1\in(-\infty,-1)$, $z_2\in[-1,1]$ and $z_3\in(1,+\infty)$. These are given by

$$z_i=z_{\theta_i}=2\cos\left(\frac{1}{3}\arccos\left(\frac{a}{2}\right)+\theta_i\right),\quad \theta_i\in\left\{0,\frac{2\pi}{3},\frac{4\pi}{3}\right\},\quad i=1,2,3. \tag{20}$$

Consequently, $\left(\sigma_1-\sigma_3\right)_i=2\sqrt{J_2}\cdot\cos\left(\dfrac{1}{6}\arccos\left(\dfrac{27J_3^2}{2J_2^2}-1\right)+\dfrac{\theta_i}{2}\right)$, $i=1,2,3$ so that, $\sigma_1-\sigma_3$ can be expressed only in terms of J_2 and J_3.

This dependence is denoted by

$$\sigma_1 - \sigma_3 = I\left(J_2, J_3\right) \tag{21}$$

Also $\sigma_1\sigma_3 = \left(\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2\right) - \left(\sigma_1^2 - 2\sigma_1\sigma_3 + \sigma_3^2\right)$ so that

$$\sigma_1\sigma_3 = 3J_2 - I^2\left(J_2, J_3\right). \tag{22}$$

Therefore, σ_1 and $-\sigma_3$ are the solutions of the quadratic equation:

$$\sigma^2 - I\left(J_2, J_3\right)\cdot\sigma + I^2\left(J_2, J_3\right) - 3J_2 = 0. \tag{23}$$

Solving (23) one obtains

$$\begin{cases} \sigma_1 = \dfrac{\sqrt{12J_2 - 3I^2\left(J_2, J_3\right)} + I\left(J_2, J_3\right)}{2}, \\ \sigma_3 = \dfrac{\sqrt{12J_2 - 3I^2\left(J_2, J_3\right)} - I\left(J_2, J_3\right)}{2}. \end{cases} \tag{24}$$

Values of σ_1 and σ_3 given by Eq. (24) can be substituted in Eq. (1), written for 2D space $(\sigma_2 = 0)$, and thus the equivalent form of the yield surface, only in terms of J_2 and J_3, can be obtained. This is true for any function in terms principal stresses σ_1 and σ_3 or, in other word, for every criterion in 2D space, which is based on above assumptions.

According to *D.W.A. Rees* [14], "... when a yielding function is formulated from the stress deviator invariants, it assumes that initial yielding is unaffected by the magnitude of hydrostatic stress." Jacob Lubliner [11] showed that "in a standard material plastic volume change occurs if and only if the yield criterion depends on I_1". The change of volume is associated with hydrostatic pressure. Based on the above assertions, it can be concluded that in the space of principal stresses σ_1 and σ_3, all yield criteria for standard plastic materials are insensitive to hydrostatic pressure.

3. CONCLUSIONS

In this paper has been mathematically demonstrated that in 2d space, all yield criteria dedicated to standard plastic materials are insensitive to hydrostatic pressure and the volume change does not occurs. In such situations, it is no longer necessary to express the yield function in terms of deviatoric invariants in order to show that one criterion is pressure insensitive. If, however, is desired to write criterion in terms of deviatoric invariants, it can be done easily, using principal stresses presented in Eqs. (24), which was deducted in this article.

Testing and modelling of material behaviour under hydrostatic pressure conditions is very useful for a thorough understanding of the yielding and strain hardening of metals and other materials under combined stress.

Received on June 5, 2018

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