CALCULUS OF THE TORSOR OF INERTIA FORCES FOR A RIGID SOLID BODY IN GENERAL MOTION

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Abstract: The calculus of the torsor of the inertia forces for a rigid solid body in general motion is a very important matter for its next dynamic study. In other words if we aim to perform the dynamic survey of a rigid solid body we must determine first the torque of its inertia forces about an arbitrary point. For this reason the present paper deals with the calculus of its elements: the resultant force vector and the resultant moment vector.

Key words: torque, inertia forces, dynamic study.

NOMENCLATURE

C – the domain occupied by the rigid solid body
$T_1(O_1x_1y_1z_1)$ – the body fixed reference frame
O_1 – the origin of the body fixed reference frame
$\mathbf{r}_{O_1}$ – column matrix associated to the position vector $\bar{r}_{O_1}$
$\tilde{\mathbf{r}}_{O_1}$ – anti-symmetric matrix associated to the column matrix $\bar{r}_{O_1}$
$T(Oxyz)$ – the fixed reference frame
O – the origin of the fixed reference frame
A – an arbitrary point of the rigid solid body
x_1, y_1, z_1 – coordinates of the arbitrary point “A” relatively to the body fixed reference frame $T_1(O_1x_1y_1z_1)$
$\bar{r}$ – position vector of the point A relatively to the fixed reference frame $T(Oxyz)$
$\mathbf{r}$ – column matrix associated to the position vector $\bar{r}$
$\tilde{\mathbf{r}}$ – anti-symmetric matrix associated to the column matrix $\mathbf{r}$
$\mathbf{r}_A$ – column matrix associated the position vector $\bar{r}_A$ relatively to the body fixed reference frame $T_1(O_1x_1y_1z_1)$
$\tilde{\mathbf{r}}_A$ – anti-symmetric matrix associated to the position vector $\bar{r}_A$

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C_1 – mass center of the rigid solid body

ξ_1, η_1, ζ_1 – coordinates of the mass center C_1 relatively to the body fixed reference frame $T_1(O_1 x_1 y_1 z_1)$

$\bar{r}_{C_1}$ – position vector of the point "C_1" in projections on the axes of the body fixed reference frame $T_1(O_1 x_1 y_1 z_1)$

$\mathbf{r}_{C_1}$ – column matrix associated to the position vector $\bar{r}_{C_1}$ in its projections on the axis of body fixed reference frame $T_1(O_1 x_1 y_1 z_1)$

$\tilde{\mathbf{r}}_{C_1}$ – anti-symmetric matrix associated to the column vector $\mathbf{r}_{C_1}$

m_1 – the mass of the rigid solid body

I_3 – unit matrix of the third order

$\mathbf{S}_{O_1}$ – column matrix associated to the polar static moment vector $\bar{S}_{O_1}$

$\tilde{\mathbf{S}}_{O_1}$ – anti-symmetric matrix associated to the polar static moment vector $\bar{S}_{O_1}$

$\bar{v}_{O_1}$ – velocity vector of the point "O_1" in its projections on the $T_1(O_1 x_1 y_1 z_1)$

$\mathbf{v}_{O_1}$ – column matrix associated to the velocity vector $\bar{v}_{O_1}$

$\tilde{\mathbf{v}}_{O_1}$ – anti-symmetric matrix associated to the column matrix $\mathbf{v}_{O_1}$

$\bar{a}_{O_1}$ – acceleration vector of the point "O_1" in its projections on the $T_1(O_1 x_1 y_1 z_1)$

$\mathbf{a}_{O_1}$ – column matrix associated to the acceleration vector $\bar{a}_{O_1}$

$\bar{a}_A$ – acceleration vector of the arbitrary point "A" of the rigid solid body in its projections on the body fixed reference frame $T_1(O_1 x_1 y_1 z_1)$

$\mathbf{a}_A$ – column matrix associated to the acceleration vector $\bar{a}_A$

$\bar{\omega}_1$ – angular velocity vector of the rigid solid body in its projections on the axes of the body fixed reference frame $T_1(O_1 x_1 y_1 z_1)$

$\boldsymbol{\omega}_1$ – column matrix associated to the angular velocity vector $\bar{\omega}_1$ of the rigid solid body

$\tilde{\boldsymbol{\omega}}_1$ – anti-symmetric matrix associated to the angular velocity vector $\bar{\omega}_1$

$\bar{\varepsilon}_1$ – angular velocity vector of the rigid solid body in its projections on the axes of the body fixed reference frame $T_1(O_1 x_1 y_1 z_1)$

$\boldsymbol{\varepsilon}_1$ – column matrix associated to the angular acceleration vector $\bar{\varepsilon}_1$ of the rigid solid body

$\tilde{\boldsymbol{\varepsilon}}_1$ – anti-symmetric matrix associated to the angular acceleration vector $\bar{\varepsilon}_1$ of the rigid solid body

$d\bar{F}_i$ – elementary inertia force vector

$d\mathbf{F}_i$ – column matrix associated to the elementary inertia force vector

dm – elementary mass of small infinite value of the arbitrary point "A" of the rigid solid body

$\overline{R}^{i}$ – resultant force vector of the inertia forces in projections on the axes of the body fixed reference frame $T_1(O_1 x_1 y_1 z_1)$

$\mathbf{R}^{i}$ – column matrix associated to the resultant force vector of inertia forces $\overline{R}^{i}$

$\mathbf{M}^{i}_{O_1}$ – column matrix associated to the resultant moment vector of inertia forces

$\boldsymbol{\tau}^{i}_{O_1}$ – the torque of inertia forces relatively to the point O_1

^{T} – indicates the transposition matrix operation

1. INTRODUCTION

We will consider the rigid solid body in general motion as it is shown in Fig. 1.

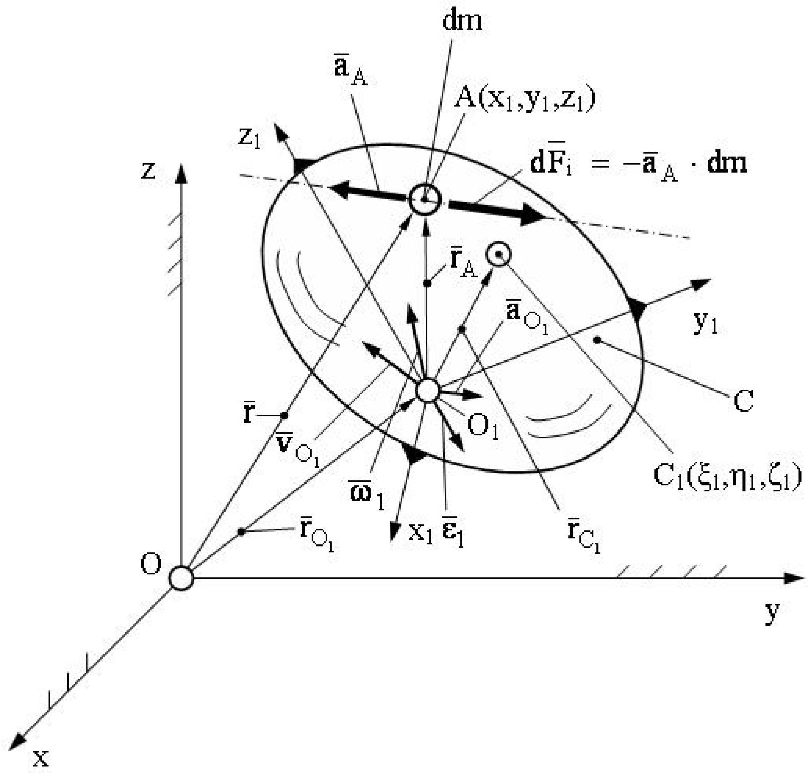

Fig. 1 – Solid rigid body in general motion.

We aim to perform the calculus of the torque of the inertia forces which acts upon it during its motion.

2. CALCULUS OF THE RESULTANT OF INERTIA FORCES

We will consider an arbitrary point belonging to the rigid solid body. This point will be denoted with "A". The elementary mass of the material point "A" is "d*m*". The elementary inertia force which acts on the arbitrary point A, according to d'Alembert principle may be written in matrix form as following [1–3, 9–17]:

$$\mathrm{d}\mathbf{F}_{\mathrm{i}} = -\mathbf{a}_{\mathrm{A}} \cdot \mathrm{d}m. \tag{1}$$

The resultant force vector could be determined by performing the triple integral on the domain "C" occupied by the rigid solid body [8]:

$$\mathbf{R}^{\mathrm{i}} = \int_{\mathrm{C}} \mathrm{d}\mathbf{F}_{\mathrm{i}} = -\int_{\mathrm{C}} \mathbf{a}_{\mathrm{A}} \cdot \mathrm{d}m. \tag{2}$$

In the relation above the acceleration of the point "A" may be written using the Euler's equation for the distribution of accelerations as following [1–3]:

$$\mathbf{a}_{\mathrm{A}} = \mathbf{a}_{\mathrm{O}_1} + \tilde{\boldsymbol{\varepsilon}}_1 \cdot \mathbf{r}_{\mathrm{A}} + \tilde{\boldsymbol{\omega}}_1 \cdot \left(\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{r}_{\mathrm{A}}\right). \tag{3}$$

Taking into account (3), relation (2) becomes:

$$\mathbf{R}^{\mathrm{i}} = -\int_{\mathrm{C}} \mathbf{a}_{\mathrm{O}_1} \cdot \mathrm{d}m - \int_{\mathrm{C}} \tilde{\boldsymbol{\varepsilon}}_1 \cdot \mathbf{r}_{\mathrm{A}} \cdot \mathrm{d}m - \int_{\mathrm{C}} \tilde{\boldsymbol{\omega}}_1 \cdot \left(\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{r}_{\mathrm{A}}\right) \cdot \mathrm{d}m. \tag{4}$$

But the acceleration of the point "O_1" may be written as following [9-11]:

$$\mathbf{a}_{\mathrm{O}_1} = \dot{\mathbf{v}}_{\mathrm{O}_1} + \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_{\mathrm{O}_1}, \qquad \dot{\mathbf{v}}_{\mathrm{O}_1} = \mathrm{d}\left(\mathbf{v}_{\mathrm{O}_1}\right)\big/\mathrm{d}t. \tag{5}$$

Taking into account the relation (5) we will obtain the following result:

$$\int_{\mathrm{C}} \mathbf{a}_{\mathrm{O}_1} \cdot \mathrm{d}m = \dot{\mathbf{v}}_{\mathrm{O}_1} \cdot \underbrace{\left(\int_{\mathrm{C}} \mathrm{d}m\right)}_{m_1} + \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_{\mathrm{O}_1} \cdot \underbrace{\left(\int_{\mathrm{C}} \mathrm{d}m\right)}_{m_1}. \tag{6}$$

The first integral of the relationship (4) may be written in the following form:

$$\int_{\mathrm{C}} \mathbf{a}_{\mathrm{O}_1} \cdot \mathrm{d}m = \mathbf{M}_1 \cdot \dot{\mathbf{v}}_{\mathrm{O}_1} + \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{M}_1 \cdot \mathbf{v}_{\mathrm{O}_1} \tag{7}$$

where:

$$\mathbf{M}_1 = m_1 \cdot \mathbf{I}_3, \qquad \mathbf{I}_3 = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]. \tag{8}$$

The second integral of the relationship (4) may be written in the following equivalent form:

$$\int_C \tilde{\boldsymbol{\varepsilon}}_1 \cdot \mathbf{r}_A \cdot \mathrm{d}m = \tilde{\boldsymbol{\varepsilon}}_1 \cdot \underbrace{\int_C \mathbf{r}_A \cdot \mathrm{d}m}_{\mathbf{S}_{O_1}} = \tilde{\boldsymbol{\varepsilon}}_1 \cdot \mathbf{S}_{O_1} = \tilde{\mathbf{S}}_{O_1}^{T} \cdot \dot{\boldsymbol{\omega}}_1 \tag{9}$$

where:

$$\tilde{\boldsymbol{\varepsilon}}_1 = \dot{\tilde{\boldsymbol{\omega}}}_1 = \mathrm{d}\left(\tilde{\boldsymbol{\omega}}_1\right)/\mathrm{dt}. \tag{10}$$

The third integral from relation (4) may be written as followings:

$$\int_C \tilde{\boldsymbol{\omega}}_1 \cdot \left(\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{r}_A\right) \cdot \mathrm{d}m = \tilde{\boldsymbol{\omega}}_1 \cdot \left[\tilde{\boldsymbol{\omega}}_1 \cdot \underbrace{\left(\int_C \mathbf{r}_A \cdot \mathrm{d}m \right)}_{\mathbf{S}_{O_1}} \right] = \tilde{\boldsymbol{\omega}}_1 \cdot \left(\tilde{\mathbf{S}}_{O_1}^{T} \cdot \boldsymbol{\omega}_1 \right). \tag{11}$$

Taking into account the relations (7), (9) and (11), the resultant of inertia forces may be written in matrix form as follows:

$$\mathbf{R}^{\mathrm{i}} = -\mathbf{N}_1 \cdot \dot{\mathbf{v}}_1 - \mathbf{L}_1 \cdot \mathbf{v}_1. \tag{12}$$

In the relation (12) the expressions of the matrices $\mathbf{N}_1$ and $\mathbf{L}_1$ are the followings:

$$\mathbf{N}_1 = \left[\mathbf{M}_1 \;\vdots\; \tilde{\mathbf{S}}_{O_1}^{T} \right] \tag{13}$$

$$\mathbf{L}_1 = \left[\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{M}_1 \;\vdots\; \tilde{\boldsymbol{\omega}}_1 \cdot \tilde{\mathbf{S}}_{O_1}^{T} \right] \tag{14}$$

where:

$$\tilde{\boldsymbol{\omega}}_1 = \left[\begin{array}{c:c:c} 0 & -\omega_{z_1} & \omega_{y_1} \\ \hdashline \omega_{z_1} & 0 & -\omega_{x_1} \\ \hdashline -\omega_{y_1} & \omega_{x_1} & 0 \end{array} \right] \tag{15}$$

and

$$\tilde{\mathbf{S}}_{O_1} = \left[\begin{array}{c:c:c} 0 & -m_1 \cdot \zeta_1 & m_1 \cdot \eta_1 \\ \hdashline m_1 \cdot \zeta_1 & 0 & -m_1 \cdot \xi_1 \\ \hdashline -m_1 \cdot \eta_1 & m_1 \cdot \xi_1 & 0 \end{array} \right]. \tag{16}$$

3. CALCULUS OF THE RESULTANT MOMENT OF INERTIA FORCES

The resultant moment vector will be determined using the following matrix formula [9–17]:

$$\mathbf{M}_{O_1}^{i} = \int_C \tilde{\mathbf{r}}_A \cdot \mathrm{d}\mathbf{F}_i = -\int_C \tilde{\mathbf{r}}_A \cdot \mathbf{a}_A \cdot \mathrm{d}m \tag{17}$$

where:

$$\tilde{\mathbf{r}}_A = \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix}. \tag{18}$$

Using the relation (3) the relation (17) becomes:

$$\mathbf{M}_{O_1}^{i} = -\int_C \tilde{\mathbf{r}}_A \cdot \mathbf{a}_{O_1} \cdot \mathrm{d}m - \int_C \tilde{\mathbf{r}}_A \cdot \left(\tilde{\boldsymbol{\varepsilon}}_1 \cdot \mathbf{r}_A\right) \cdot \mathrm{d}m - \int_C \tilde{\mathbf{r}}_A \cdot \left[\tilde{\boldsymbol{\omega}}_1 \cdot \left(\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{r}_A\right)\right] \cdot \mathrm{d}m. \tag{19}$$

The first integral of the relationship (19) may be written as followings:

$$\int_C \tilde{\mathbf{r}}_A \cdot \mathbf{a}_{O_1} \cdot \mathrm{d}m = \int_C \tilde{\mathbf{r}}_A \cdot \left(\dot{\mathbf{v}}_{O_1} + \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_{O_1}\right) \cdot \mathrm{d}m \tag{20}$$

$$\int_C \tilde{\mathbf{r}}_A \cdot \left(\dot{\mathbf{v}}_{O_1} + \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_{O_1}\right) \cdot \mathrm{d}m = \underbrace{\left(\int_C \tilde{\mathbf{r}}_A \mathrm{d}m\right)}_{\tilde{\mathbf{S}}_{O_1}} \cdot \dot{\mathbf{v}}_{O_1} + \underbrace{\left(\int_C \tilde{\mathbf{r}}_A \mathrm{d}m\right)}_{\tilde{\mathbf{S}}_{O_1}} \cdot \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_{O_1}. \tag{21}$$

The relationship (21) may be written as followings:

$$\int_C \tilde{\mathbf{r}}_A \cdot \left(\dot{\mathbf{v}}_{O_1} + \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_{O_1}\right) \cdot \mathrm{d}m = \tilde{\mathbf{S}}_{O_1} \cdot \dot{\mathbf{v}}_{O_1} + \tilde{\mathbf{S}}_{O_1} \cdot \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_{O_1}. \tag{22}$$

The second integral of the relationship (19) may be written as followings:

$$\int_C \tilde{\mathbf{r}}_A \cdot \left(\tilde{\boldsymbol{\varepsilon}}_1 \cdot \mathbf{r}_A\right) \cdot \mathrm{d}m = \left(\int_C \left(\tilde{\mathbf{r}}_A \cdot \tilde{\mathbf{r}}_A^{\mathrm{T}}\right) \cdot \mathrm{d}m\right) \cdot \boldsymbol{\varepsilon}_1 = \mathbf{J}_{O_1} \cdot \boldsymbol{\varepsilon}_1 \tag{23}$$

where:

$$\mathbf{J}_{O_1} = \int_C \left(\tilde{\mathbf{r}}_A \cdot \tilde{\mathbf{r}}_A^{\mathrm{T}}\right) \cdot \mathrm{d}m. \tag{24}$$

But the right member of the previous relation may be written as follows:

$$\int_C \left(\tilde{\mathbf{r}}_A \cdot \tilde{\mathbf{r}}_A^T\right) \cdot \mathrm{d}m = \left[\begin{array}{c|c|c} J_{x_1} & -J_{x_1y_1} & -J_{x_1z_1} \\ \hline -J_{x_1y_1} & J_{y_1} & -J_{y_1z_1} \\ \hline -J_{x_1z_1} & -J_{y_1z_1} & J_{z_1} \end{array}\right] \tag{25}$$

$$J_{x_1} = \int_C \left(y_1^2 + z_1^2\right) \cdot \mathrm{d}m \tag{26}$$

$$J_{y_1} = \int_C \left(x_1^2 + z_1^2\right) \cdot \mathrm{d}m \tag{27}$$

$$J_{z_1} = \int_C \left(x_1^2 + y_1^2\right) \cdot \mathrm{d}m \tag{28}$$

$$J_{x_1y_1} = \int_C x_1 \cdot y_1 \cdot \mathrm{d}m \tag{29}$$

$$J_{y_1z_1} = \int_C y_1 \cdot z_1 \cdot \mathrm{d}m \tag{30}$$

$$J_{x_1z_1} = \int_C x_1 \cdot z_1 \cdot \mathrm{d}m. \tag{31}$$

The quantities J_{x_1}, J_{y_1} and J_{z_1} defined by relations (26)–(28) represent the axial moments of inertia of the rigid solid body relatively to the axis of the body fixed reference frame $T_1(O_1x_1y_1z_1)$.

The quantities $J_{x_1y_1}$, $J_{y_1z_1}$ and $J_{x_1z_1}$ defined by relations (29)–(31) represent the products of inertia of the rigid solid body relatively to the pairs of planes belonging to the body fixed reference frame $T_1(O_1x_1y_1z_1)$.

The third integral of the relationship (19) may be written as following:

$$\int_C \tilde{\mathbf{r}}_A \cdot \left[\tilde{\boldsymbol{\omega}}_1 \cdot \left(\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{r}_A\right)\right] \cdot \mathrm{d}m = \tilde{\boldsymbol{\omega}}_1 \cdot \left(\int_C \left(\tilde{\mathbf{r}}_A \cdot \tilde{\mathbf{r}}_A^T\right) \cdot \mathrm{d}m\right) \cdot \boldsymbol{\omega}_1. \tag{32}$$

Taking into account the relations (24) and (25) the relation (32) becomes:

$$\int_C \tilde{\mathbf{r}}_A \cdot \left[\tilde{\boldsymbol{\omega}}_1 \cdot \left(\tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{r}_A\right)\right] \cdot \mathrm{d}m = \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{J}_{O_1} \cdot \boldsymbol{\omega}_1 \tag{33}$$

$$\boldsymbol{\omega}_1 = \left[\begin{array}{c|c|c} \omega_{x_1} & \omega_{y_1} & \omega_{z_1} \end{array}\right]^T. \tag{34}$$

The resultant moment of the inertia forces may be written in the following final form:

$$\mathbf{M}_{\mathrm{O}_1}^{\mathrm{i}} = -\mathbf{N}_2 \cdot \dot{\mathbf{v}}_1 - \mathbf{L}_2 \cdot \mathbf{v}_1 \tag{35}$$

$$\mathbf{N}_2 = \left[\, \tilde{\mathbf{S}}_{\mathrm{O}_1} \;\vdots\; \mathbf{J}_{\mathrm{O}_1} \,\right] \tag{36}$$

$$\mathbf{L}_2 = \left[\tilde{\mathbf{S}}_{\mathrm{O}_1} \cdot \tilde{\boldsymbol{\omega}}_1 \;\vdots\; \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{J}_{\mathrm{O}_1} \right] \tag{37}$$

$$\dot{\mathbf{v}}_1 = \left[\, \dot{\mathbf{v}}_{\mathrm{O}_1}^{\mathrm{T}} \;\vdots\; \dot{\boldsymbol{\omega}}_1^{\mathrm{T}} \,\right]^{\mathrm{T}} \tag{38}$$

$$\dot{\mathbf{v}}_{\mathrm{O}_1} = \left[\, \dot{v}_{\mathrm{O}_1 x_1} \;\vdots\; \dot{v}_{\mathrm{O}_1 y_1} \;\vdots\; \dot{v}_{\mathrm{O}_1 z_1} \,\right]^{\mathrm{T}} \tag{39}$$

$$\dot{\boldsymbol{\omega}}_1 = \left[\, \dot{\omega}_{x_1} \;\vdots\; \dot{\omega}_{y_1} \;\vdots\; \dot{\omega}_{z_1} \,\right]^{\mathrm{T}} = \boldsymbol{\varepsilon}_1 \tag{40}$$

$$\mathbf{v}_1 = \left[\, \mathbf{v}_{\mathrm{O}_1}^{\mathrm{T}} \;\vdots\; \boldsymbol{\omega}_1^{\mathrm{T}} \,\right]^{\mathrm{T}} \tag{41}$$

$$\mathbf{v}_{\mathrm{O}_1} = \left[\, v_{\mathrm{O}_1 x_1} \;\vdots\; v_{\mathrm{O}_1 y_1} \;\vdots\; v_{\mathrm{O}_1 z_1} \,\right]^{\mathrm{T}} \tag{42}$$

$$\boldsymbol{\omega}_1 = \left[\, \omega_{x_1} \;\vdots\; \omega_{y_1} \;\vdots\; \omega_{z_1} \,\right]^{\mathrm{T}} \tag{43}$$

$$\boldsymbol{\varepsilon}_1 = \left[\, \varepsilon_{x_1} \;\vdots\; \varepsilon_{y_1} \;\vdots\; \varepsilon_{z_1} \,\right]^{\mathrm{T}}. \tag{44}$$

Taking into account the relationships (12) and (35) the torque of the inertia forces may be written in the following matrix form:

$$\boldsymbol{\tau}_{\mathrm{O}_1}^{\mathrm{i}} = -\mathbf{M}_{\mathrm{O}_1} \cdot \dot{\mathbf{v}}_1 - \boldsymbol{\Omega}_1 \cdot \mathbf{v}_1. \tag{45}$$

The quantities involved in relation (45) have the following expressions:

$$\mathbf{M}_{\mathrm{O}_1} = \left[\begin{array}{c|c} \mathbf{M}_1 & \tilde{\mathbf{S}}_{\mathrm{O}_1}^{\mathrm{T}} \\ \hline \tilde{\mathbf{S}}_{\mathrm{O}_1} & \mathbf{J}_{\mathrm{O}_1} \end{array} \right] \tag{46}$$

$$\boldsymbol{\Omega}_1 = \left[\begin{array}{c|c} \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{M}_1 & \tilde{\boldsymbol{\omega}}_1 \cdot \tilde{\mathbf{S}}_{\mathrm{O}_1}^{\mathrm{T}} \\ \hline \tilde{\mathbf{S}}_{\mathrm{O}_1} \cdot \tilde{\boldsymbol{\omega}}_1 & \tilde{\boldsymbol{\omega}}_1 \cdot \mathbf{J}_{\mathrm{O}_1} \end{array} \right]. \tag{47}$$

4. GENERAL EQUATIONS OF MOTION OF A SOLID RIGID BODY

Using the mathematical expression of the torque of inertia forces given by the relation (45), the general equations of motion for a rigid solid body in general motion may be deduced. Thus, by performing the algebraic sum between the torque of inertia forces and the torque of the active forces we will obtain the following matrix equation [5–8]:

$$\boldsymbol{\tau}^{\mathrm{i}}_{\mathrm{O}_1} + \boldsymbol{\tau}^{\mathrm{a}}_{\mathrm{O}_1} = \underbrace{\mathbf{0}}_{6\times 1} = \left[0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0\right]^{\mathrm{T}}. \tag{48}$$

By replacing the relation (45) in relation (48) we will obtain the following matrix relation:

$$\mathbf{M}_{\mathrm{O}_1} \cdot \dot{\mathbf{v}}_1 = -\mathbf{\Omega}_1 \cdot \mathbf{v}_1 + \boldsymbol{\tau}^{\mathrm{a}}_{\mathrm{O}_1}. \tag{49}$$

The general equations of motion for a rigid solid body subjected to constraints may be written as following:

$$\boldsymbol{\tau}^{\mathrm{i}}_{\mathrm{O}_1} + \boldsymbol{\tau}^{\mathrm{a}}_{\mathrm{O}_1} + \boldsymbol{\tau}^{\mathrm{c}}_{\mathrm{O}_1} = \underbrace{\mathbf{0}}_{6\times 1} = \left[0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0\right]^{\mathrm{T}}. \tag{50}$$

Replacing the relation (45) in relation (50) we will obtain:

$$\mathbf{M}_{\mathrm{O}_1} \cdot \dot{\mathbf{v}}_1 = -\mathbf{\Omega}_1 \cdot \mathbf{v}_1 + \boldsymbol{\tau}^{\mathrm{a}}_{\mathrm{O}_1} + \boldsymbol{\tau}^{\mathrm{c}}_{\mathrm{O}_1}. \tag{51}$$

In relations (48–51), $\boldsymbol{\tau}^{\mathrm{a}}_{\mathrm{O}_1}$ represents the torque of the active forces relatively to the point O_1 and $\boldsymbol{\tau}^{\mathrm{c}}_{\mathrm{O}_1}$ represents the torque of the constraint forces relatively to the same point O_1. When we study the dynamics of a solid rigid body we have to determine the position of the solid rigid body at certain moment "*t*". Therefore, we will add to the differential equations of motion (49) and (51) the following differential equation written in matrix form:

$$\dot{\mathbf{x}}_1 = \mathbf{A} \cdot \mathbf{v}_1. \tag{52}$$

In the relation (52) the quantities involved have the followings matrix expressions:

$$\dot{\mathbf{x}}_1 = \left[\dot{x}_{\mathrm{O}_1} \mid \dot{y}_{\mathrm{O}_1} \mid \dot{z}_{\mathrm{O}_1} \mid \dot{\psi}_1 \mid \dot{\theta}_1 \mid \dot{\varphi}_1\right]^{\mathrm{T}} \tag{53}$$

$$\mathbf{v}_1 = \left[v_{\mathrm{O}_1 x_1} \mid v_{\mathrm{O}_1 y_1} \mid v_{\mathrm{O}_1 z_1} \mid \omega_{x_1} \mid \omega_{y_1} \mid \omega_{z_1}\right]^{\mathrm{T}} \tag{54}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_1 & \underset{3\times3}{\mathbf{0}} \\ \underset{3\times3}{\mathbf{0}} & \mathbf{T} \end{bmatrix} \tag{55}$$

$$\mathbf{R}_1 = \mathbf{\Psi}_1 \cdot \mathbf{\Theta}_1 \cdot \mathbf{\Phi}_1 \tag{56}$$

$$\mathbf{\Psi}_1 = \begin{bmatrix} \cos\psi_1 & -\sin\psi_1 & 0 \\ \sin\psi_1 & \cos\psi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{57}$$

$$\mathbf{\Theta}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{58}$$

$$\mathbf{\Phi}_1 = \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 \\ \sin\varphi_1 & \cos\varphi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{59}$$

$$\mathbf{T} = (\mathbf{T}_1 \cdot \mathbf{T}_2)^{-1} = \mathbf{T}_2^{-1} \cdot \mathbf{T}_1^{-1} \tag{60}$$

$$\mathbf{T}_1 = \begin{bmatrix} \sin\varphi_1 & \cos\varphi_1 & 0 \\ \cos\varphi_1 & -\sin\varphi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{61}$$

$$\mathbf{T}_2 = \begin{bmatrix} \sin\theta_1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos\theta_1 & 0 & 1 \end{bmatrix}. \tag{62}$$

In relations (57–62), ψ_1, θ_1 and φ_1 represent Euler's angles.

5. EQUATIONS OF MOTION OF A SOLID RIGID BODY IN THE PARTICULAR CASE OF PLANE MOTION

In this paragraph we will present the form of the general equations of motion in the case of plane motion. As it was shown, the relations (49) and (52) represent the most general equations of motion for a free solid rigid body written in matrix form. Further on we will write these equations in the case of the plane motion of a solid rigid body. When a solid rigid body describes a plane motion the following relationships between kinematical parameters may be written:

$$v_{\mathrm{O}_1 z_1} = 0 \tag{63}$$

$$\omega_{x_1} = \omega_{y_1} = 0 \tag{64}$$

$$\dot{v}_{\mathrm{O}_1 z_1} = 0 \tag{65}$$

$$\dot{\omega}_{x_1} = \dot{\omega}_{y_1} = 0 \tag{66}$$

$$z_{\mathrm{O}_1} = 0 \tag{67}$$

$$\psi_1 = \theta_1 = 0 \tag{68}$$

$$\dot{z}_{\mathrm{O}_1} = 0 \tag{69}$$

$$\dot{\psi}_1 = \dot{\theta}_1 = 0. \tag{70}$$

Taking into account the relationships (63–70) the relations (49) and (52) become:

$$\mathbf{M}_{\mathrm{O}_1} \cdot \mathbf{L}_\tau \cdot \dot{\mathbf{q}}_1 = -\mathbf{\Omega}_1 \cdot \mathbf{L}_\tau \cdot \mathbf{q}_1 + \boldsymbol{\tau}^{\mathrm{a}}_{\mathrm{O}_1} \tag{71}$$

$$\mathbf{L}_\tau \cdot \dot{\boldsymbol{\alpha}} = \mathbf{A} \cdot \mathbf{L}_\tau \cdot \mathbf{q}_1 \tag{72}$$

$$\mathbf{L}_\tau = \left[\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]^{\mathrm{T}} \tag{73}$$

$$\dot{\mathbf{q}}_1 = \left[\dot{v}_{\mathrm{O}_1 x} \;\middle|\; \dot{v}_{\mathrm{O}_1 y} \;\middle|\; \dot{\omega}_{x_1}\right]^{\mathrm{T}} \tag{74}$$

$$\mathbf{q}_1 = \left[v_{\mathrm{O}_1 x} \;\middle|\; v_{\mathrm{O}_1 y} \;\middle|\; \omega_{z_1}\right]^{\mathrm{T}} \tag{75}$$

$$\dot{\boldsymbol{\alpha}}_1 = \left[\dot{x}_{\mathrm{O}_1} \;\middle|\; \dot{y}_{\mathrm{O}_1} \;\middle|\; \dot{\varphi}_1\right]^{\mathrm{T}}. \tag{76}$$

We will multiply the equations (71) and (72) to the left by the matrix $\mathbf{L}_\tau^{\mathrm{T}}$ and we will obtain:

$$\mathbf{L}_\tau^{\mathrm{T}} \cdot \mathbf{M}_{\mathrm{O}_1} \cdot \mathbf{L}_\tau \cdot \dot{\mathbf{q}}_1 = -\mathbf{L}_\tau^{\mathrm{T}} \cdot \mathbf{\Omega}_1 \cdot \mathbf{L}_\tau \cdot \mathbf{q}_1 + \mathbf{L}_\tau^{\mathrm{T}} \cdot \boldsymbol{\tau}^{\mathrm{a}}_{\mathrm{O}_1} \tag{77}$$

$$\mathbf{L}_\tau^{\mathrm{T}} \cdot \mathbf{L}_\tau \cdot \dot{\boldsymbol{\alpha}}_1 = \mathbf{L}_\tau^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{L}_\tau \cdot \mathbf{q}_1. \tag{78}$$

In the relations (77) and (78) the following notations will be introduced:

$$\tilde{\mathbf{M}}_{\mathrm{O}_1} = \mathbf{L}_\tau^{\mathrm{T}} \cdot \mathbf{M}_{\mathrm{O}_1} \cdot \mathbf{L}_\tau \tag{79}$$

$$\tilde{\mathbf{\Omega}}_1 = \mathbf{L}_\tau^{\mathrm{T}} \cdot \mathbf{\Omega}_1 \cdot \mathbf{L}_\tau \tag{80}$$

$$\tilde{\mathbf{A}} = \mathbf{L}_\tau^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{L}_\tau \tag{81}$$

$$\tilde{\boldsymbol{\tau}}_{\mathrm{O}_1}^{\mathrm{a}} = \mathbf{L}_\tau^{\mathrm{T}} \cdot \boldsymbol{\tau}_{\mathrm{O}_1}^{\mathrm{a}}. \tag{82}$$

Using the notations (79–82) the relations (77) and (78) may be written as follows:

$$\tilde{\mathbf{M}}_{\mathrm{O}_1} \cdot \dot{\mathbf{q}}_1 = -\tilde{\mathbf{\Omega}}_1 \cdot \mathbf{q}_1 + \tilde{\boldsymbol{\tau}}_{\mathrm{O}_1}^{\mathrm{a}} \tag{83}$$

$$\dot{\boldsymbol{\alpha}}_1 = \tilde{\mathbf{A}} \cdot \mathbf{q}_1. \tag{84}$$

By performing the calculus we will obtain:

$$\tilde{\mathbf{M}}_{\mathrm{O}_1} = \left[\begin{array}{c:c:c} m_1 & 0 & -m_1 \cdot \eta_1 \\ \hdashline 0 & m_1 & m_1 \cdot \xi_1 \\ \hdashline -m_1 \cdot \eta_1 & m_1 \cdot \xi_1 & J_{z_1} \end{array}\right] \tag{85}$$

$$\tilde{\mathbf{\Omega}}_1 = \left[\begin{array}{c:c:c} 0 & -\omega_{z_1} \cdot m_1 & -\omega_{z_1} \cdot m_1 \cdot \xi_1 \\ \hdashline \omega_{z_1} \cdot m_1 & 0 & -\omega_{z_1} \cdot m_1 \cdot \eta_1 \\ \hdashline \omega_{z_1} \cdot m_1 \cdot \xi_1 & \omega_{z_1} \cdot m_1 \cdot \eta_1 & 0 \end{array}\right] \tag{86}$$

$$\tilde{\mathbf{A}} = \left[\begin{array}{c:c:c} \cos\varphi_1 & -\sin\varphi_1 & 0 \\ \hdashline \sin\varphi_1 & \cos\varphi_1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right] \tag{87}$$

$$\tilde{\boldsymbol{\tau}}_{\mathrm{O}_1}^{\mathrm{a}} = \left[\begin{array}{c:c:c} R_{x_1}^{\mathrm{a}} & R_{y_1}^{\mathrm{a}} & M_{\mathrm{O}_1 z_1}^{\mathrm{a}} \end{array}\right]^{\mathrm{T}}. \tag{88}$$

If the solid rigid body is subjected to constraints the equation of motion (83) becomes:

$$\tilde{\mathbf{M}}_{\mathrm{O}_1} \cdot \dot{\mathbf{q}}_1 = -\tilde{\mathbf{\Omega}} \cdot \mathbf{q}_1 + \tilde{\boldsymbol{\tau}}_{\mathrm{O}_1}^{\mathrm{a}} + \tilde{\boldsymbol{\tau}}_{\mathrm{O}_1}^{\mathrm{c}}. \tag{89}$$

In the equation (89) $\tilde{\boldsymbol{\tau}}_{\mathrm{O}_1}^{\mathrm{c}}$ represents the torque of the constraint forces. In general it has the following expression:

$$\tilde{\boldsymbol{\tau}}_{\mathrm{O}_1}^{\mathrm{c}} = \left[\begin{array}{c:c:c} R_{x_1}^{\mathrm{c}} & R_{y_1}^{\mathrm{c}} & 0 \end{array}\right]^{\mathrm{T}}. \tag{90}$$

6. NUMERICAL APPLICATION

In this paragraph we will present a numerical example namely the dynamic analysis of the physical pendulum in order to illustrate how the general equations of motion can be applied to a very simple case. Whereby make it our task to study the dynamics of physical pendulum. It can be regarded as a solid rigid in plane motion subjected to constraints. We will consider the physical pendulum presented in the figure below (Fig. 2). Its motion is described by differential equations as presented in the previous paragraph. The torque of the active forces may be written as follows:

$$\tilde{\boldsymbol{\tau}}^{\mathrm{a}}_{\mathrm{O}_1} = \left[m_1 \cdot g \cdot \cos\varphi_1 \;\vdots\; -m_1 \cdot g \cdot \sin\varphi_1 \;\vdots\; -m_1 \cdot g \cdot \xi_1 \cdot \sin\varphi_1 \right]^{\mathrm{T}}. \tag{91}$$

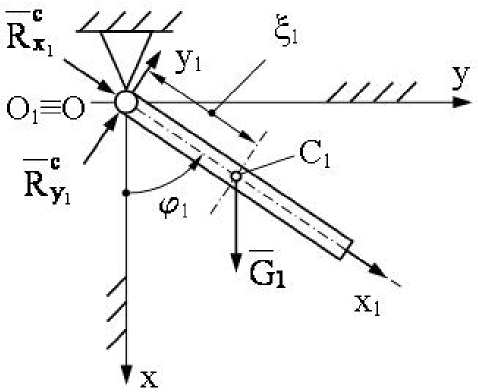

Fig. 2 – Physical pendulum.

The torque of the constraint forces may be written in the following matrix form:

$$\tilde{\boldsymbol{\tau}}^{\mathrm{c}}_{\mathrm{O}_1} = \mathbf{L}_\lambda \cdot \boldsymbol{\lambda} \tag{92}$$

$$\mathbf{L}_\lambda = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array} \right]^{\mathrm{T}} \tag{93}$$

$$\boldsymbol{\lambda} = \left[R^{\mathrm{c}}_{x_1} \;\vdots\; R^{\mathrm{c}}_{y_1} \right]^{\mathrm{T}}. \tag{94}$$

Taking into account the relation (92) the relation (89) becomes:

$$\tilde{\mathbf{M}}_{\mathrm{O}_1} \cdot \dot{\mathbf{q}}_1 = -\tilde{\boldsymbol{\Omega}} \cdot \mathbf{q}_1 + \tilde{\boldsymbol{\tau}}^{\mathrm{a}}_{\mathrm{O}_1} + \mathbf{L}_\lambda \cdot \boldsymbol{\lambda}. \tag{95}$$

The constraint forces are unknown and for this reason they must be removed from the equations of motion. By multiplying the relation (95) to the left with $\tilde{\mathbf{M}}^{-1}_{\mathrm{O}_1}$ matrix we will obtain the followings:

$$\dot{\mathbf{q}}_1 = \tilde{\mathbf{M}}_{O_1}^{-1} \cdot \left(-\tilde{\boldsymbol{\Omega}} \cdot \mathbf{q}_1 + \tilde{\boldsymbol{\tau}}_{O_1}^{a}\right) + \tilde{\mathbf{M}}_{O_1}^{-1} \cdot \mathbf{L}_{\lambda} \cdot \boldsymbol{\lambda}. \tag{96}$$

In the relation (96) we will introduce the following notation:

$$\mathbf{B} = \tilde{\mathbf{M}}_{O_1}^{-1} \cdot \left(-\tilde{\boldsymbol{\Omega}} \cdot \mathbf{q}_1 + \tilde{\boldsymbol{\tau}}_{O_1}^{a}\right). \tag{97}$$

The relation (96) becomes:

$$\dot{\mathbf{q}}_1 = \mathbf{B} + \tilde{\mathbf{M}}_{O_1}^{-1} \cdot \mathbf{L}_{\lambda} \cdot \boldsymbol{\lambda}. \tag{98}$$

By multiplying the relation (98) to the left with $\mathbf{L}_{\lambda}^{T}$ matrix we will obtain the followings:

$$\mathbf{L}_{\lambda}^{T} \cdot \dot{\mathbf{q}}_1 = \mathbf{L}_{\lambda}^{T} \cdot \mathbf{B} + \mathbf{L}_{\lambda}^{T} \cdot \tilde{\mathbf{M}}_{O_1}^{-1} \cdot \mathbf{L}_{\lambda} \cdot \boldsymbol{\lambda}. \tag{99}$$

In the relation (99) we will introduce the following notation:

$$\mathbf{A} = \mathbf{L}_{\lambda}^{T} \cdot \tilde{\mathbf{M}}_{O_1}^{-1} \cdot \mathbf{L}_{\lambda}. \tag{100}$$

Using the relation (100), the relation (99) becomes:

$$\mathbf{L}_{\lambda}^{T} \cdot \dot{\mathbf{q}}_1 = \mathbf{L}_{\lambda}^{T} \cdot \mathbf{B} + \mathbf{A} \cdot \boldsymbol{\lambda}. \tag{101}$$

It is obvious that:

$$\mathbf{L}_{\lambda}^{T} \cdot \dot{\mathbf{q}}_1 = \underbrace{\mathbf{0}}_{2\times 1} = \left[0 \,\vdots\, 0\right]^{T}. \tag{102}$$

Taking into account the relation (102) from the relation (101) it may be determined the unknown matrix $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda} = -\mathbf{A}^{-1} \cdot \left(\mathbf{L}_{\lambda}^{T} \cdot \mathbf{B}\right). \tag{103}$$

By replacing the relation (103) into the relation (95) and taking into account the relationship (84) we will obtain a system of six differential equations with six unknowns which can very easily solved using numerical integration methods. Using MatLab software a computing program has been elaborated and the results presented in the figures below (Fig. 3 and Fig. 4) were obtained.

To integrate the differential equations system, the following initial conditions concerning the velocities are required:

$$v_{O_1 x_1}^{0} = v_{O_1 y_1}^{0} = 0 \text{ meter/second} \tag{104}$$

$$\omega_{z_1}^{0} = 0 \text{ radian/second} \tag{105}$$

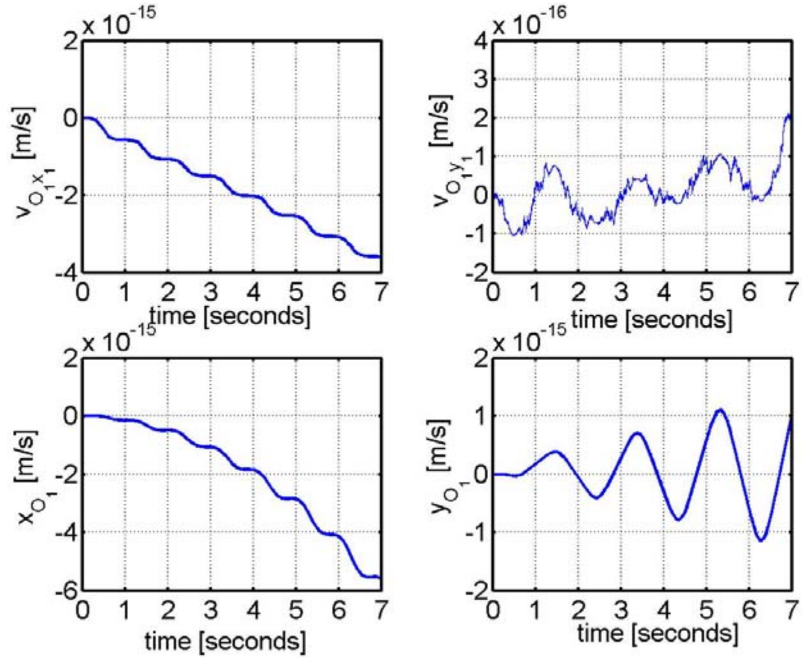

Fig. 3 – Variation of the origin coordinates and velocities projections with respect to time.

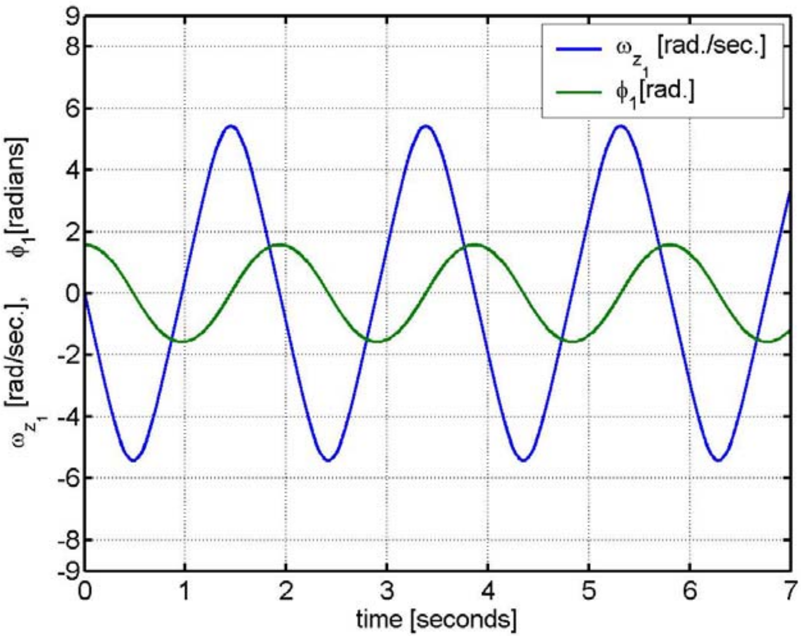

Fig. 4 – Variation of the angular speed and self-rotation angle with respect to time.

The initial conditions concerning the position parameters are the following:

$$x_{O_1}^0 = y_{O_1}^0 = 0\,\text{meter} \tag{106}$$

$$\varphi_1^0 = \pi/2\,\text{radian}. \tag{107}$$

The computing program contains the following input data:

$$m_1 = 10\,\text{kilogram} \tag{108}$$

$$l_1 = 1\,\text{meter} \tag{109}$$

$$J_{z_1} = (1/3)\cdot m_1 \cdot l_1^2 \cong 3.334\,\text{kg}\cdot\text{m}^2 \tag{110}$$

$$\xi_1 = l_1/2 = 0.5\,\text{meter} \tag{111}$$

$$\eta_1 = 0\,\text{meter} \tag{112}$$

$$g = 9.81\,\text{meter/second}^2 \tag{113}$$

$$t_0 = 0\,\text{second} \tag{114}$$

$$t_f = 7\,\text{second}. \tag{115}$$

7. CONCLUSIONS

The present paper presents a detailed deduction of the calculus formula of the inertia forces torque for a solid rigid body in general motion. Matrix writing of inertia force torque expression has the advantage of easily solving the system of differential equations describing the motion of the rigid solid body.

The paper also presents the deduction of general equations of a motion of a free solid rigid body and a solid rigid body subjected to constraints.

The general equations of motion for a solid rigid body may be used for the dynamic study of any mechanical system.

The simple pendulum whose dynamic study has been shown in the research stands only as an example to illustrate the general research method used in the study.

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