DIRECT AND INVERSE PROBLEMS FOR AN INHOMOGENEOUS MEDIUM

LUCIANA MAJERCSIK[1], LIGIA MUNTEANU[2]

Abstract. The plane problem for an inhomogeneous medium is analyzed in this paper by using the cnoidal representation of the material inhomogeneity. This representation approximates the distribution of elastic properties in the material. Direct problem is finding the stress distribution in an inhomogeneous body when the elastic moduli of the material are known, and the inverse problem is searching for the elastic moduli of the inhomogeneous material for a given state of stress. The inverse problem is exercised to an inhomogeneous shallow shell subjected to uniformly distributed external load.

Key words: inhomogeneous medium, direct and inverse problems, shallow shell.

1. INTRODUCTION

When the governing equations of an elastic body are solved for initial and boundary conditions in terms of displacement and/or stress vectors, the problem is called the direct problem. When one or more of the information or conditions for solving the direct problem are partially or entirely unknown then an inverse problem can be stated to find the unknowns from specified or measured responses [1–4]. An inverse problem of interest in the elasticity theory is to find the elastic moduli of an inhomogeneous material, or to find the model parameters, or small defects, cavities or cracks into material [5–10]. The inverse problem is motivated by the desire to overcome a lack of information concerning the properties of the elastic body. The most of the inverse problems are ill posed and more difficult to solve than the direct problems [11–13].

In this paper we solve the inverse problem of finding the elastic moduli of an inhomogeneous material for a given representation of the material inhomogeneity. Torlin [14] used the polynomials in cosine function for describing the material inhomogeneity. In this paper, the inhomogeneity is expressed as polynomials of cnoidal functions. The choice of cnoidal functions is motivated by the fact that the

[1] Transilvania University of Brașov, Romania
[2] Institute of Solid Mechanics of the Romanian Academy

Ro. J. Techn. Sci. – Appl. Mechanics, Vol. 64, N° 1, P. 73–80, Bucharest, 2019

governing equation can admit analytical-numerical solutions. The existence of the such solutions of the direct problem ensures the stability of the inverse problem. The cnoidal functions are much richer than the trigonometric or hyperbolic functions because these functions depend on the modulus $0 \le m \le 1$ which can vary from $m = 0$ to obtain a sine function or a cosine function, to $m = 1$ to obtain a sech function or a tanh function. For $m = 0.5$ the Stokes function is obtained for example [13].

2. FORMULATION OF THE PROBLEMS

The isotropic elastic modulus E and the Poisson's ratio ν of an inhomogeneous medium are defined as [14]

$$E = E_0 f_1(r,\theta), \quad \nu = \nu_0 f_2(r,\theta), \tag{1}$$

where r,θ are polar coordinates, and f_1, f_2 are functions known only in a finite number of points describing the inhomogeneity of the medium. In the direct problem these functions are known. The governing equation with respect to the stress function is written under the form [14]

$$\begin{aligned}
&f_1^2\nabla^2\nabla^2\varphi - 2f_1\left(f_{1,r}(\nabla^2 r)_{,r} + \frac{1}{r^2} f_{1,\theta}(\nabla^2\varphi)_{,\theta} \right) - \frac{1}{r}\left[f_1\left(f_{1,r} + \frac{1}{r} f_{1,\theta\theta} \right) - 2\frac{1}{r} f_{1,\theta}^2 \right] L_1\varphi - \\
&-\left[f_1 f_{1,r} - 2f_{1,r}^2 \right] L_2\varphi - 2\frac{1}{r^2}\left(1+\nu_0 f_2\right) \times \left[f_1\left(f_{1,rr} - \frac{1}{r} f_{1,\theta} \right) - 2f_{1,r} f_{1,\theta} \right] L_3\varphi - \\
&-\nu_0 f\left(\frac{1}{r} f_1 f_{2,rr} - 2f_{1,r} f_{2,r} \right) \times \left(L_1 + \nu_0 f_2 \frac{\partial}{\partial r^2} \right)\varphi + \frac{\nu_0}{r^2}\left(f_1 f_{2,\theta\theta} - 2f_{1,\theta} f_{2,\theta} + r^2 f_1 f_{2,rr} \right)\varphi_{,rr} + \\
&+2\frac{\nu_0}{r^2}\left(f_1 f_{2,r\theta} + f_{1,\theta} f_{2,r} + f_{1,r} f_{2,\theta} + \frac{1}{r} f_1 f_{2,\theta\theta} \right) L_3\varphi + E_0^2 f_1^4 = 0,
\end{aligned} \tag{2}$$

where the coma means the differentiation with respect to the specified variable, and the operators are defined as

$$\begin{gathered}
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \quad L_1 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \nu_0\frac{\partial^2}{\partial\theta^2} - \nu_0 f_2\frac{\partial^2}{\partial r^2}, \\
L_2 = \frac{\partial^2}{\partial r^2} - \nu_0 f_2\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \right), \quad L_3 = \frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}.
\end{gathered} \tag{3}$$

The solutions $\varphi_i, i = 1,2,3$, of (2) are the stress functions $\sigma_r, \sigma_\theta, \sigma_{r\theta}$, which depend on the inhomogeneity of the medium

$$\varphi_1(r,\theta,f_1,f_2) = \sigma_r, \quad \varphi_2(r,\theta,f_1,f_2) = \sigma_\theta, \quad \varphi_3(r,\theta,f_1,f_2) = \sigma_{r\theta}. \tag{4}$$

The stress functions are determined by using the cnoidal method [13] under the form

$$\varphi_i = L_i\left(\log\Theta_i(r\ ,\theta, f_i)\right), \quad i = 1,2,3\,, \tag{5}$$

where Θ is the *theta function* defined as

$$\Theta_i = 1 + r^i f_i \mathrm{cn}(m_i, \theta)\,, \quad i = 1,2,3\,. \tag{6}$$

In the following, we suppose that the functions f_1, f_2 are known in a finite number of points

$$f_1 = \sum_{n=1}^{N} a_n r^n \mathrm{cn}(m,\theta)\,, \qquad f_2 = \sum_{n=1}^{N} b_n r^n \mathrm{cn}(m,\theta)\,. \tag{7}$$

Introducing (7) in (5) and (6), the solutions of (2) result as

$$\varphi_i = \sum_{l=1}^{n} \alpha_{li} r^l \mathrm{cn}^2[\theta; m_{il}]\,. \tag{8}$$

The constants α_{li}, $i = 1,2,3$, $l = 1,2,...,n$, depend in a known way on a_n and b_n, $n = 1,2,...,N$, and the boundary conditions of the specific problem.

The inverse problem consists in determination of the values of parameters a_n and b_n, $n = 1,2,...,N$, corresponding to a given state of stress.

The solution of the direct problem (8) allows solving the inverse problem.

3. THE SHALLOW SHELL

The inverse problem is exercised to an inhomogeneous shallow shell subjected to uniformly distributed external load. A shallow shell is a curved slab whose thickness h is small compared with its other dimensions and compared with its principal radii of curvature [15, 16]. In the bending theory of cylindrical shell roofs, the shallow shell model is commonly used [17].

The governing stress equation of the shallow shell is (2). The structural behavior of the shallow shell is analogous to those of a beam with a hollow cross section. The dimensions of the shell are the length *L*, the width is *d* and the thickness is *h*.

We suppose that the shallow shell is loaded with uniformly distributed external load at the ends $x = 0$ and $x = L$ (Fig. 1).

We solve the inverse problem by using the solutions (8) for $n = 3$. The stress functions $\varphi_1 = \sigma_r$, $\varphi_2 = \sigma_\theta$ and $\varphi_3 = \sigma_{r\theta}$ are

$$\varphi_1 = 3.2r\mathrm{cn}^2(\theta,0.4) + 4.09r^2\mathrm{cn}^2(\theta,0.23) + 6.02r^3\mathrm{cn}^2(\theta,0.34)\,,$$
$$\varphi_2 = 7.03r\mathrm{cn}^2(\theta,0.74) - 0.39r^2\mathrm{cn}^2(\theta,0.44) + 7.93r^3\mathrm{cn}^2(\theta,0.58)\,, \tag{9}$$
$$\varphi_3 = 0{,}34r\mathrm{cn}^2(\theta,0.9) - 1.93r^2\mathrm{cn}^2(\theta,0.75) + 4.32r^3\mathrm{cn}^2(\theta,0.49)\,.$$

The simulations have indicated that for $n > 3$ there are no workable improvements to solutions.

The elastic properties of the thin shell are approximated by

$$E = E_0\sum_{j=1}^{N} a_j r^j\,, \quad \nu = \nu_0\sum_{j=1}^{N} b_j r^j\,, \tag{10}$$

where a_j, b_j, $j = 1,2,...,N$, are unknown parameters determined from (9).

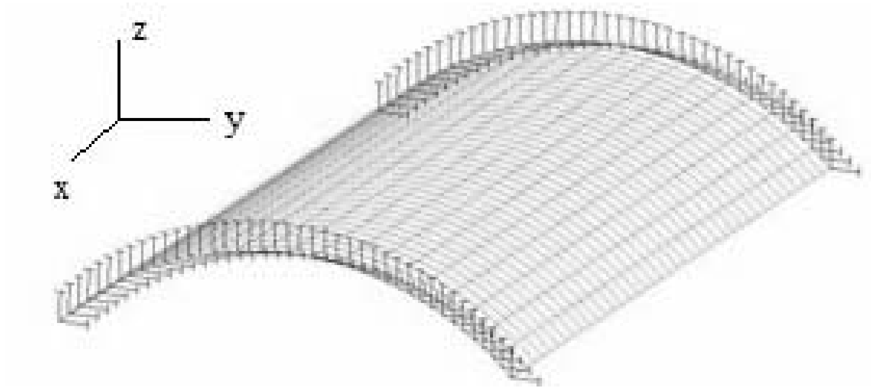

Fig. 1 – Illustration of the thin shell subjected to external load.

The simulation was carried out for 30 points in which the functions f_1 and f_2 given by (7) are known. The stability of the inverse problem is good.

For 30 points and $N = 5$ the solution of the inverse problem is

$$E = E_0(1 - 1.873r + 4.32r^2 - 1.56r^3 + 3.77r^4 - 0.87r^5)\,,$$
$$\nu = \nu_0(1 + 0.33r + 2.02r^2 - 2.06r^3 - 1.17r^4 + 1.65r^5)\,. \tag{11}$$

For 50 points and $N = 5$ the solutions is

$$E = E_0(1 - 1.54r + 4.33r^2 - 1.50r^3 + 3.75r^4 - 0.81r^5)\,,$$
$$\nu = \nu_0(1 + 0.56r + 2.03r^2 - 2.03r^3 - 1.12r^4 + 1.64r^5)\,. \tag{12}$$

The length of the shell is $L = 25$ cm, the width $d = 8$ cm, and the thickness $h = 0.5$ mm. Also, we have taken $\nu_0 = 0.334$ and $E_0 = 69\times 10^9\,\mathrm{GPa}$.

Simulations have indicated that for $N > 5$ and 50 points there are no significant improvements to values of E and ν (Fig. 2).

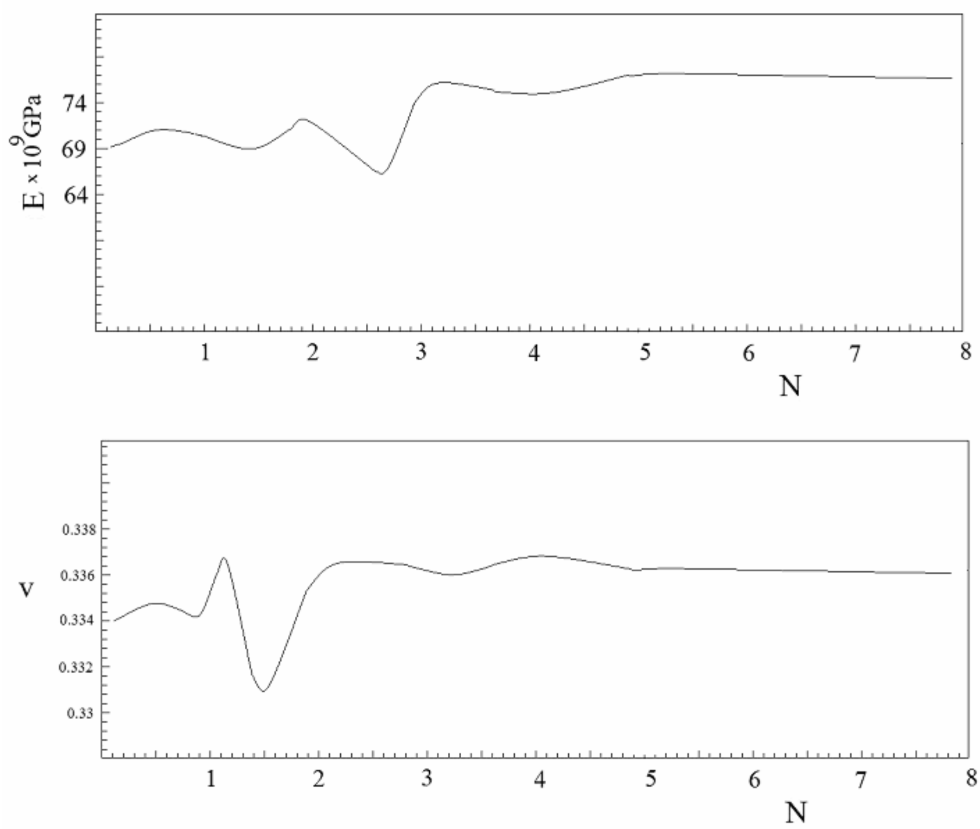

Fig. 2 – Variation of E and ν with respect to N.

The stress functions $\sigma_r, \sigma_\theta, \sigma_{r\theta}$ depend on the inhomogeneity of the medium. Variations with the distance x, of the stress field σ_r, σ_θ and $\sigma_{r\theta}$ of the thin shell subjected to external load are presented in Figs. 3–5, for two inhomogeneous cases $\nu_0 = 0.334$, $E_0 = 69 \times 10^9\,\text{GPa}$ and $\nu_0 = 0.3$, $E_0 = 81 \times 10^9\,\text{GPa}$, and the homogeneous case $\nu = 0.334$, $E = 69 \times 10^9\,\text{GPa}$.

The variation of $\sigma_{r\theta}$ in the nonhomogeneous cases is not so oscillating than σ_r and σ_θ which have a pronounced oscillating character. The graphs show the importance of neomogenities in the body and the influence of the way of formulating of these neomogeneities on the stress field. In the center of the homogeneous structure we observe large values of the stresses compared to the cases of inhomogeneity. This was also observed in the works [14] and [15].

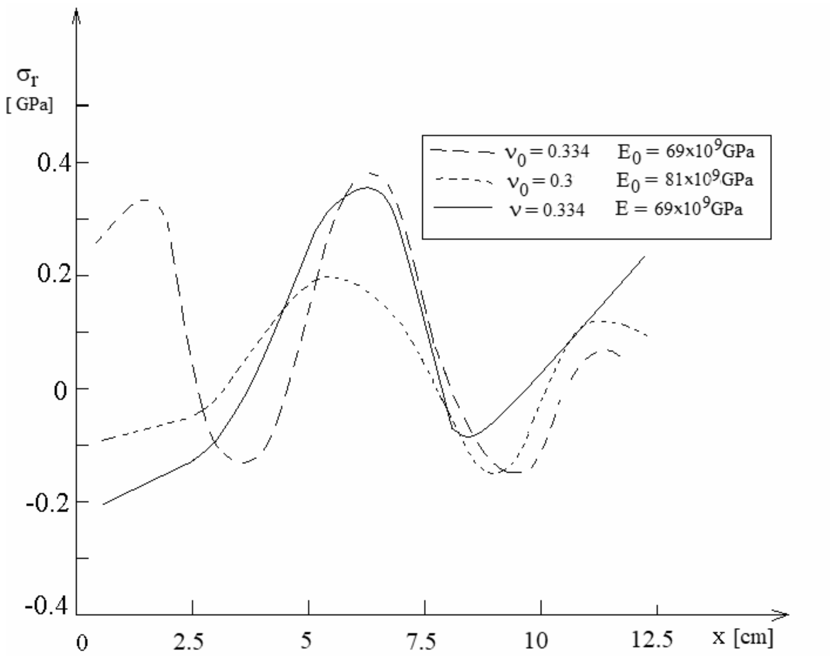

Fig. 3 – Variation of the stress σ_r with distance for two inhomogeneous cases compared to the homogeneous case.

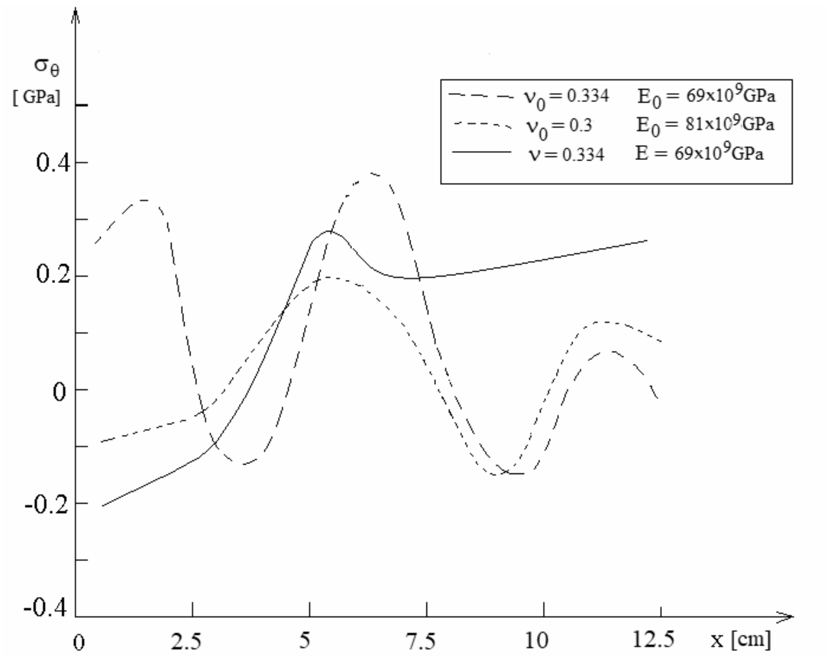

Fig. 4 – Variation of the stress σ_θ with distance for two inhomogeneous cases compared to the homogeneous case.

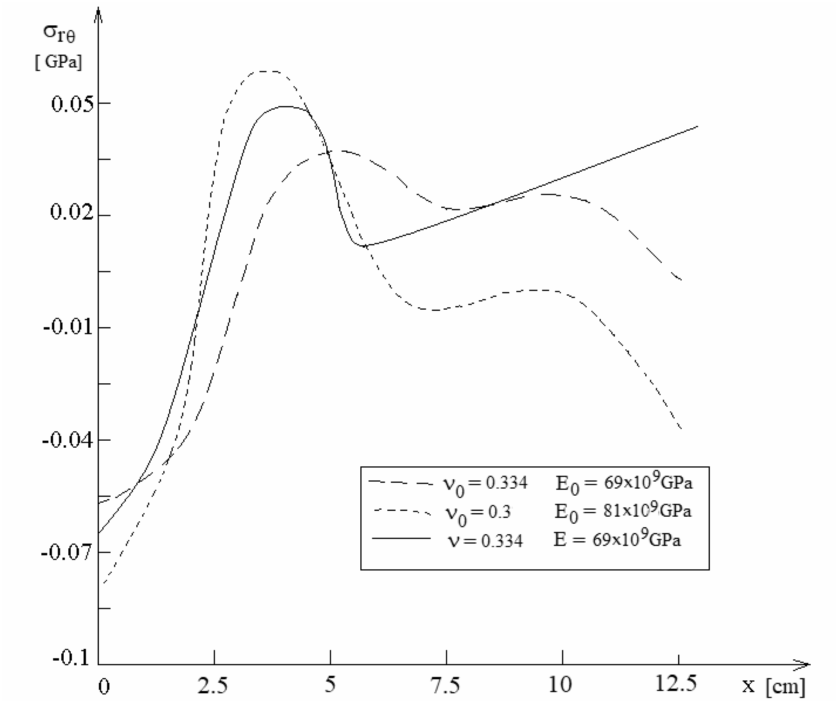

Fig. 5 – Variation of the stress $\sigma_{r\theta}$ with distance for two inhomogeneous cases compared to the homogeneous case.

In conclusion, the direct problem is a problem of analysis and the inverse problem is a problem of synthesis. The results depend on how the material inhomogeneity is modeled.

5. CONCLUSIONS

The paper analyses the direct and inverse problems for the plane elasticity of an inhomogeneous medium. The cnoidal representation of the material inhomogeneity is used to solve both problems. Direct problem aims to find the stress distribution in an inhomogeneous body when the elastic moduli of the material are known, and the inverse problem is searching for the elastic moduli of the inhomogeneous material for a given state of stress. Both problems are exercised to an inhomogeneous shallow shell subjected to uniformly distributed external load at the ends. The cnoidal representation of the material inhomogeneity leads to analytical-numerical solutions for the governing stress equation, which ensure the stability of the inverse problem.

Received on November 7, 2018

REFERENCES

1. ELLABIB, A., NACHAOUI, A., *An iterative approach to the solution of an inverse problem in linear elasticity*, Mathematics and Computers in Simulation, **77**, *2–3*, pp. 189–201, 2008.
2. AMMARI, H., KANG, H. *Reconstruction of small inhomogeneities from boundary measurements*, Springer-Verlag, 2004.
3. TARDIEU, N., CONSTANTINESCU, A., *On the determination of elastic coefficients from indentation experiments*, Inverse Problems, **16**, *3*, pp. 577–588, 2000.
4. TIKHONOV, A.N., ARSENIN, V.Y., *Solutions to ill-posed problems*, Winston-Wiley, New York, 1977.
5. SOKOLOWSKI, J., ZOCHOWSKI, A., *On the topological derivative in shape optimization*, SIAM J. Control Optim., **37**, pp. 1251–1272, 1999.
6. SOKOLOWSKI, J., ZOLESIO, J.P., *Introduction to shape optimization*. *Shape sensitivity analysis*, Springer series in Computational Mathematics, vol. 16, Springer-Verlag, 1992.
7. NINTCHEU FATA, S., GUZINA, B.B., *A linear sampling method for near-field inverse problems in elastodynamics*, Inverse Problems, **20**, *3*, pp. 713–736, 2004.
8. NINTCHEU FATA, S., GUZINA, B.B., BONNET, M., *A computational basis for elastodynamic cavity identification in a semi-infinite solid*, Comp. Mech., **32**, pp. 370–380, 2003.
9. JADAMBA, B., KHAN, A.A., RACITI, F., *On the inverse problem of identifying Lamé coefficients in linear elasticity*, Computers & Mathematics with Applications, **56**, *2*, pp. 431–443, 2008.
10. GOCKENBACH, M.S., KHAN, A., *A convex objective functional for elliptic inverse problems*, in: *Mathematical Models and Methods for Real World Systems* (eds. K.M. Furuti, M.Z. Nashed, A.H. Siddiqi), pp. 389–419, Chapman & Hall/CRC, 2005.
11. HU, G., LI, P., LIU, X., ZHAO, Y., *Inverse source problems in electrodynamics*, Inverse Problems and Imaging, **12**, pp. 1411–1428, 2018.
12. CHIROIU, V., CHIROIU, C., *Inverse problems in mechanics* (in Romanian), 2003.
13. MUNTEANU, L., DONESCU, St., *Introduction to soliton theory: Applications to mechanics*, Book Series Fundamental Theories of Physics, Vol. 143, Kluwer Academic Publishers, Dordrecht, Boston (Springer Netherlands), 2004.
14. TORLIN, V.N., *Direct and inverse problem of the plane theory of elasticity of a inhomogeneous body*, Prikladnaya Mekhanika, **12**, *8*, pp. 49–52, August, 1976.
15. BILLINGTON, DAVID P., *Thin shell concrete structures*, McGraw-Hill, New York, 1965.
16. MIHAILESCU, M., CHIROIU, V., *Advanced mechanics on shells and intelligent structures*, Edit. Academiei Române, 2004.
17. GHEBRESELASIE, M.H., *Structural Analysis of Thin Concrete shells*, Master Thesis, Trondheim Norwegian University of Science and Technology, 2015.