ANALYTICAL APPROXIMATION SOLUTION OF NONLINEAR BLASIUS PROBLEM

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Abstract. A new alternative technique of the Optimal Auxiliary Functions Methods (OAFM) is proposed and applied to solve nonlinea rdifferential equations of the Blasius problem. The proposed procedure is very effective and convenient and does not require linearization or small parameters. The main advantage of this approach consists in that it provides a covenient way to control convergence of the approximate solution in a very rigorous way. The solution obtained using the present procedure is in a very good agreement with numerical results and some well-known results, which prove that OAFM is a power tool for ononlinear problems, very efficient and accurate.

Keyword: Blasius equation, Optimal Auxiliary Functions Method, Nonlinear differential equation, Optimal parameters.

1. INTRODUCTION

Approximate analytical solutions of the nonlinear differential equation are useful when exact analytical solutions are too difficult or imposible to obtain or when the work to find a numerical solution cannot be justified. In science and engineering there exist many nonlinear differential equations and even strongly nonlinear problems which are still very difficult to solve.

In general the study of nonlinear differential equations is restricted to a variety of special classes of equations and the method to find the solutions usually involves one or more techniques to active analytical approximation to the solutions. Therefore, many researchers and scientists have recently paid much attention to find and develop approximate solutions. Perturbation methods are well established tools to study diverse aspects of nonlinear problems [1, 2]. Howerer the use of peturbation theory in many important practical problems is invalid, or it simply breaks down for parameters beyond a certain specified range.

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Therefore, new analytical techniques should be develop to overcome these short – commings. Such a new technique should work over a large range of parameters and yield accurate analytical approximate solution beyond the coverage and ability of the classical perturbation methods.

It is noted that serval methods, have been used to obtain approximation solution for strongly nonlinear problems. An interesting approach which combines the harmonic balance method and linearisation method of nonlinear oscillation equations was proposed in [3]. There also exists a wide range of literature dealing with approximate solutions for nonlinear problems with large parameters by using a mixture of methodologies: the variationed iteration method [4], optimal homotopy asymptotic method [5, 6], some modified Lindstedt-Poincare methods [7], the method of weighted residuals [8], and so on.

Many problems arising in technique are defined in unbounded domains, and steady flow of the non-Newtonian fluids has attracted considerable attention in the last years, because of its several aplications in various fields of science and engineering. Blasius equation is one of the basic equations of fluid dynamics. Blasius equation descriebed the velocity profile of the fluid in the boundary layer theory on a half-infinite interval. The Blasius equation is the “mother” of all boundary layer in fluid mechanics and describes the steady two dimensional laminar boundary layer that forms a semi-infinite plate which is held parallel to a constant uni-directional flow. A broad class of analytical solution methods are used to handle this problem. Adomian decomposite method is applied by Wang [9] to obtain an approximate solution for classical Blasius equation. Fazio [10] introduced a numerical parameter and require to on extended scaling group in involving this parameter. The method of gradient is applied by Borşa [11] in the study of the thin film flows. The [4/3] Padé aproximant for the derivative is modified by Ahmed and Al-Barakati [12] so that the resulting expression has the required asymptotic behavior. Momentum laminar boundary layers of an incompressible fluid either about a moving plate in a quiscent ambient fluid (Sakiads flow) and the flow induced over a resting flat-plate by a uniform free stream (Blasius flow) are investigated numerically by Cortell Bataller [13], for each case via Runge-Kutta algorithm along with shooting procedure. Parand et al. [14] proposed on approach based on the first kind of Bessel functions collocations method, reducing the solution of a a nonlinear problem to the solution of a system of nonlinear algebraic equations. Marinca and Herişanu [15] applied Optimal Auxiliary Functions Method to solve the nonlinear differential equation of Blasius problem. Robin [16] presented three uniform rational algebraic approximations to the Blasius velocity profile, along with corresponding uniform approximations to the Blasius function. Akgűl [17] considered the reproducing kernel method to determine numerical approximation for Blasius equation. Najafi [18] converted the nonlinear Blasius equation to a nonlinear Volterra integral equation satisfying the condition of the quasiliniarization scheme. The solutions of the obtained liniar integral equation are approximated by the collocation method.

2. THE GOVERNING EQUATION OF MOTION

The governing equation of motion for the steady incompresible two-dimensional boundary layer equation for continuity and momentum can be summarized as [13, 15]:

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0\,, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\vartheta\frac{\partial^2 u}{\partial y^2}\,, \tag{2}$$

in which ϑ is kinematic viscosity. The boundary condition for the velocity field are

$$u=v=a \text{ at } y=0, u=U \text{ at } u=0\,, \tag{3}$$

$$u\to U_\infty \text{ at } y\to\infty\,. \tag{4}$$

It δ is the boundary-layer thickness and L is natural length scale, then balancing between viscosity and convective inertia it results the scaling argument

$$\frac{U^2}{L}\cong\vartheta\frac{U}{\delta^2}\,. \tag{5}$$

From the scaling argument it is apparent that the boundary layer grows the downstream coordinate x, e.q.:

$$\delta(x)\approx\left(\frac{\vartheta x}{u}\right)^{\frac{1}{2}}. \tag{6}$$

Introducing a similarity variable η and a dimensionless stream function $f(\eta)$ as

$$\eta=y\left(\frac{\vartheta x}{U}\right)^{\frac{1}{2}};\quad \frac{u}{U}=f';\quad v=\frac{1}{2}\sqrt{\frac{U\vartheta}{x}}\left(\eta f'-f\right), \tag{7}$$

where prime denotes differentation with respect to η. From Eqs. (7), we obtain:

$$\frac{\partial u}{\partial x}=-\frac{U}{2}\frac{\eta}{x}f'';\quad \frac{\partial v}{\partial x}=\frac{U}{2}\frac{\eta}{x}f''\,. \tag{8}$$

The equation of continuity (1) is satisfied identically and follows that

$$\frac{\partial u}{\partial y}=Uf''\sqrt{\frac{U}{\vartheta x}};\ \frac{\partial^2 u}{\partial y^2}=\frac{U^2}{\vartheta x}f'''. \tag{9}$$

After simple manipulations, we obtain Blasius equation:

$$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)=0, \tag{10}$$

subject to the boundary conditions:

$$f(0)=0,\ f'(0)=0,\ f'(\infty)=1. \tag{11}$$

3. AN ALTERNATIVE TO THE OPTIMAL AUXILIARY FUNCTIONS METHOD

The nonlinear differential Eq. (10) with the boundary conditions (11) can be written in a more general form [20, 21].

$$L\left[f(\eta)\right]+N\left[f(\eta)\right]=0, \tag{12}$$

where L is a linear operator and N is a nonlinear, subject to the boundary/initial condition:

$$B\left(f(\eta),\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\right)=0. \tag{13}$$

In order to determine an analytic approximate solution of Eq. (12) and (13), we suppose that the approximate solution $\left(\bar{f}(\eta)\right)$ can be expressed in the following form only with we components:

$$\bar{f}(\eta)=f_0(\eta)+f_1(\eta,C_i),\ i=1,2,\ldots,p, \tag{14}$$

where the initial approximation $\left(f_0(\eta)\right)$ and first approximation $\left(f_1(\eta,C_i)\right)$ will be determined as follows. Inverting Eq. (14) into Eq. (12), one get:

$$L\left[f_0(\eta)\right]+L\left[f_1(\eta,C_i)\right]+N\left[f_0(\eta)+f_1(\eta,C_i)\right]=0. \tag{15}$$

The initial approximation can be obtained from the following linear equation

$$L\left[f_0(\eta)\right]=0,\; B\left(f_0(\eta),\frac{\mathrm{d}f_0(\eta)}{\mathrm{d}\eta}\right)=0 \tag{16}$$

and the first approximation is obtained from the equation:

$$L\left[f_1(\eta,C_i)\right]+N\left[f_0(\eta)+f_1(\eta,C_i)\right]=0,\; B\left(f_1(\eta),\frac{\mathrm{d}f_1(\eta)}{\mathrm{d}\eta}\right)=0 \tag{17}$$

Now, the nonlinear term from the least equation is expanded in the form:

$$N\left[f_0(\eta)+f_1(r,C_i)\right]=N\left[f_0(\eta)\right]+\sum_{k\geq 1}\frac{f_1^k(r,C_i)}{k!}N^{(k)}\left[f_0(\eta)\right] \tag{18}$$

where $k!=1\cdot 2\cdot 3\cdot .....\cdot k$ and $N^{(k)}$ means the differentiation of order k of nonlinear function N.

In general, the solution of the linear Eq. (16) can be expressed by:

$$f_0(\eta)=\sum_{i=1}^{m_1}a_i h_i(\eta), \tag{19}$$

where the coefficients a_i, the function h_i and the positive integer m_1 are known.

The nonlinear operator N calculated for $f_0(\eta)$ may be written as:

$$N\left[f_0(\eta)\right]=\sum_{i=1}^{m_2}b_i g_i(\eta), \tag{20}$$

where the coefficients b_i, the functions $g_i(\eta)$ and positive integer m_2 are known, and all depend on the initial approximation $f_0(\eta)$ and also on the nonlinear operator N.

In what follows, we do not solve Eq. (17), but it should be emphasised that from the theory of differential equations, taking into consideration the method of variation of parameters, Cauchy method, the method of influence functions, the operator method and so on [19] it is more convenient to consider the unknown function $f_1(\eta,C_i)$ as dependent of $f_0(\eta)$ and $N\left[f_0(\eta)\right]$. More precisely $f_1(\eta,C_i)$ can be written in the form:

$$f_1(\eta,C_k)=\sum_{k=1}^{p}C_k F_k(h_i,g_i), \tag{21}$$

where C_k are several (p) unknown parameters at this moment and F_k are auxiliary functions depending on the function h_i and g_i defined by Eqs. (19) and (20) respectivelly.

We have a great freedom to choose the values of the integer positive p and the auxiliary functions F_k. Note that the boundary/initial conditions could be fulfilled by the Eq. (21), so that finally, Eq. (14) responds to all boundary/initial conditions given by Eq. (13). We can not demand that $f_1(\eta, C_i)$ given by Eq. (21) is a solution of Eq. (17), but $\overline{f}(\eta)$ given by the Eq. (14) is the solution of Eqs. (13) and (14).

For instance if $f_0(\eta)$ and $N\left[f_0(\eta)\right]$ are trigonometric functions, then $f_1(\eta, C_i)$ is a combination of the trigonometric functions. More precisely if

$$f_0 = 4\sin\alpha\eta + 7\cos\alpha\eta \;\left(a_1 = 4,\, a_2 = 7,\, h_1 = \sin\alpha\eta,\, h_2 = \cos\alpha\eta,\, m_1 = 2\right)$$

and

$$N\left[f_0(\eta)\right] = 5\sin 2\alpha\eta + 13\cos 3\alpha\eta$$

$$\left(b_1 = 5,\, b_2 = 13,\, g_1 = \sin 2\alpha\eta,\, g_2 = \cos 3\alpha\eta,\, m_2 = 2\right),$$

then

$$f_1(\eta) = C_1\sin\alpha\eta + C_2\sin 2\alpha\eta + C_3\cos 3\alpha\eta + C_4\cos 5\alpha\eta$$

$$\left(F_1 = \sin\alpha\eta,\, F_2 = \sin 2\alpha\eta,\, F_3 = \cos 3\alpha\eta,\, F_4 = \cos 5\alpha\eta,\, p = 4\right).$$

Similarly if $f_0(\eta)$ and $N\left(f_0(\eta)\right)$ are polynomial functions

$$\left(f_0(\eta) = 3\eta + 5\eta^2,\, N\left(f_0(\eta)\right) = 7\eta^3 + 9\eta^4 + 11\eta^5\right),$$

then

$$f_1(\eta, C_i) = C_1\eta + C_2\eta^2 + C_3\eta^3 + C_4\eta^4 + C_5\eta^5 + \ldots.$$

In the case when $f_0(\eta) = 3\ln\eta + 3\eta^2 + 7\eta^3$ and $N\left(f_0(\eta)\right) = \frac{2}{\eta} + \frac{7}{\eta^2} + 5\ln\eta$, then $f_0(\eta, C_i)$ is a combination of the function, which appear within $f_0(\eta)$ and $N\left(f_0(\eta)\right)$,

$$f_1(\eta, C_i) = C_1\ln\eta + C_2\eta + \frac{C_3}{\eta} + \frac{C_4}{\eta^3} + C_5\eta\ln\eta + C_6\eta^3\ln\eta + \cdots$$

Now, the parameters $C_1, C_2, \ldots C_p$ which appear into Eq. (21), can be optimally identified via various procedures such as the least square method, the

Galerkin method, the collocation method, the Ritz method, the Kantorovich method or by minimizing the square residual error.

$$J\left(C_1,C_2,...,C_p\right)=\int_{(D)} R^2\left(\eta,C_1,C_2,....,C_p\right)\mathrm{d}\eta\ , \tag{22}$$

where D is domain of interest and the residual R is given by:

$$R\left(\eta,C_i\right)=L\left[\,\overline{f}\left(\eta,C_i\right)\right]+N\left[\,\overline{f}\left(\eta,C_i\right)\right],\ i=1,2,...,p\ . \tag{23}$$

If $R\left(\eta,C_i\right)=0$, then $\overline{f}\left(\eta,C_i\right)$ happens to be an exact solution of Eqs. (12) and (13). The unknown parameters $C_1,C_2,...,C_p$ can be identified from the conditions:

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=...=\frac{\partial J}{\partial C_p}=0\ . \tag{24}$$

With these p parameters known (namely henceforward optimal convergence-control parameters or convergence-control parameters), the approximate solution $\overline{f}\left(\eta\right)$ given by Eq. (14) is well determined.

Note that our technique contains the optimal auxiliary functions F_k, which provides us with a simple way to adjust and control the convergence of the approximate solution. It is very important to properly choose these auxiliary functions F_k. It will shown that our procedure is a powerfull tool to solving nonlinear problems without small or large parameters.

4. APPROXIMATE SOLUTIONS OF THE BLASIUS PROBLEM BY MEANS OF THE ALTERNATIVE TO THE OAFM

In what follows, we apply our technique to obtain on approximate solution of Eqs. (10) and (11). The initial approximation $f_0\left(\eta\right)$ which verifies the boundary conditions (11) can be chosen in the form:

$$f_0\left(\eta\right)=\eta-k\ln\left(1-\frac{\eta}{k}\right) \tag{25}$$

where k is an arbitrary positive unknown parameter. Taking into consideration Eq. (25), we define the linear operator in the forms (the liniar operator would not be unique):

$$L\left[f(\eta)\right]=f'''+\frac{2f''}{k\left(1+\frac{\eta}{k}\right)} \tag{26}$$

or

$$L\left[f(\eta)\right]=f'''-\frac{2f'}{k^2\left(1+\frac{\eta}{k}\right)^2}+\frac{2}{k^2\left(1+\frac{\eta}{k}\right)^2}. \tag{27}$$

If we consider only the linear operator given through Eq. (26), we obtain the nonlinear operator as:

$$N\left[f(\eta)\right]=\frac{1}{2}f(\eta)f''(\eta)-\frac{2f''}{k\left(1+\frac{\eta}{k}\right)}. \tag{28}$$

Substituting Eq. (25) into Eq. (28), it holds that:

$$N\left[f_0(\eta)\right]=\frac{\eta}{k^2\left(1+\frac{\eta}{k}\right)^3}-\frac{\eta}{2k\left(1+\frac{\eta}{k}\right)^2}+\frac{1}{2\left(1+\frac{\eta}{k}\right)}\ln\left(1+\frac{\eta}{k}\right). \tag{29}$$

For the first approximation f_1, the boundary conditions are

$$f_0(0)=f_1'(0)=f'(\infty)=0. \tag{30}$$

From the section 3 and from Eqs. (29) and (30), we have the freedom to choose the first approximation in the following form:

$$f_1(\eta,C_i)=\ln\left(1+C_1\eta^2+C_2\eta^3+...+C_p\eta^{p+1}\right) \tag{31}$$

where $C_1,C_2,...,C_p$ are unknown parameters.

The analytical approximation of Eqs. (10) and (11) is obtained from Eqs. (14), (25) and (31):

$$\begin{aligned}\bar{f}(\eta,C_i)&=f_0(\eta)+f_1(\eta,C_i)=\\&=\eta-k\ln\left(1+\frac{\eta}{k}\right)+\ln\left(1+C_1\eta^2+...C_p\eta^{p+1}\right).\end{aligned} \tag{32}$$

The optimal convergence control parameters are determined by using the condition (24) where:

$$R(\eta,C_i)=\overline{f}'''(\eta,C_i)+\frac{1}{2}\overline{f}(\eta,C_i)\overline{f}''(\eta,C_i),\quad i=1,2,\cdots,p\quad, \tag{33}$$

and $\overline{f}(\eta,C_i)$ is given in Eq. (32).

In what follows we will shown that the accuracy of the results obtained by an alternative to the OAFM is growing along with increasing the number p of parameters C_i.

For $p=6$, we obtain:

$$\begin{gathered}k=4.136440558322; C_1=0.0451056233184; C_2=0.01996802807\\ C_3=-0.00335917554; C_4=0.001665797515;\\ C_5=-0.0002536093067; C_6=0.00001213043235256.\end{gathered} \tag{34}$$

Table 1

Comparison between the approximate solution given by Eq. (35) and numerical result [22]

η	$\overline{F}_{numeric}(\eta)$, [22]	$\overline{F}(\eta)$, Eq.(35)	error $=\lvert F_{num}(\eta)-\overline{F}(\eta)\rvert$
0	0	0	0
0.2	0.0066412	0.0066418582	6.38E-05
0.6	0.0597215	0.059749865	2.83E-05
1	0.1655717	0.165584803	1.31E-05
1.4	0.3229815	0.322975577	5.92E-06
2	0.6500243	0.650065389	4.11E-05
2.4	0.9222901	0.922399411	1.01E-04
3	1.3968082	1.396900384	9.21E-05
3.4	1.7469501	1.746963516	1.34E-05
4	2.3057464	2.305857328	1.11E-04
5	3.28329	3.284382105	1.01E-03
6	4.27964	4.279822606	1.82E-04
7	5.27926	5.27547322	3.78E-03
8	6.27923	6.279480302	2.51E-04

In this case the approximate solution given by Eqs. (32) and (34) of the Eqs. (10) and (11) becomes:

$$\begin{aligned}\overline{f}(\eta)&=\eta-4.13644055832\ln(1+0.241753745999\eta)+\\&+\ln(1+0.0451056233184\eta^2+0.0199680280744\eta^3-\\&-0.00335917554\eta^4+0.00166579751\eta^5-\\&-0.0002536093067\eta^6+0.00001283043235256\eta^7).\end{aligned} \tag{35}$$

In *Table 1* we present a comparison between the approximate solution given by Eq. (35) with numerical results [22] for some values of variable η and the corresponding errors.

For $p = 8$, we obtain:

$$\begin{gathered} k = 2.500137843922;\ C_1 = -0.03410933979297; \\ C_2 = 0.054471864056; \\ C_3 = -0.0176773550246;\ C_4 = 0.004895234662; \\ C_5 = -0.00088409414;\ C_6 = 8.98134158273 \cdot 10^{-5}; \\ C_7 = -4.5347403075 \cdot 10^{-6}; C_8 = 8.365909375 \cdot 10^{-8}. \end{gathered} \tag{36}$$

The approximative solution given by Eqs. (32) and (36) of the Blasius eqiation (10) and (11) is:

$$\begin{aligned} &\bar{f}(\eta) = \eta - 2.50137843922 \ln(1 + 0.3999779463356\eta) + \\ &+ \ln(1 - 0.03410933979297\eta^2 + 0.05447186056181\eta^3 - \\ &-0.017677355024\eta^4 + 0.004895237662\eta^5 - \\ &-0.00088409414\eta^6 + 8.98134158273 \cdot 10^{-5}\eta^7 - \\ &-4.53347403075 \cdot 10^{-6}\eta^8 + 8.365909375 \cdot 10^{-8}\eta^9). \end{aligned} \tag{37}$$

In Table 2 we present a comparison between our approximate solution (37) and numerical results.

Table 2

Comparison between the approximate solution given by Eq. (37) and numerical result [22]

η	$\bar{F}_{numeric}(\eta)$, [22]	$\bar{F}(\eta)$, Eq.(37)	error
0	0	0	0
0.2	0.0066412	0.0066418	6.58E-07
0.6	0.0597215	0.05975296	3.14E-05
1	0.1655717	0.163571192	5.07E-07
1.4	0.3229815	0.32292591	5.56E-05
2	0.6500243	0.649967725	5.65E-05
2.4	0.9222901	0.922443657	1.53E-04
3	1.3968082	1.397106905	2.98E-04
3.4	1.7469501	1.747172681	2.22E-04
4	2.3057464	2.305742988	3.41E-06
5	3.28329	3.283285682	4.32E-06
6	4.27964	4.279634827	5.17E-06
7	5.27926	5.279254032	5.96E-06
8	6.27923	6.279213301	6.05E-06

In Table 3 we present a comparison between our approximate solution (37) with published results [23, 24].

Table 3

Comparison between the present results with other published results

η	$\overline{f}_{numeric}(\eta)$,[22]	$\overline{f}(\eta)$, Eq.(37)	$\overline{f}(\eta)$, [23]	$\overline{f}(\eta)$, [24]
0	0	0	0	0
0.2	0.0066412	0.0066418	0.066409	0.0069699
0.6	0.0597215	0.05975296	0.0597345	0.0626959
1	0.1655717	0.163571192	0.1655715	0.1738016
1.4	0.3229815	0.32292591	0.3229812	0.3391217
2	0.6500243	0.649967725	0.6500224	0.6828833
2.4	0.9222901	0.922443657	0.9222734	0.9691873
3	1.3968082	1.397106905	1.3964712	1.4674133
3.4	1.7469501	1.747172681	1.7451217	1.8335195
4	2.3057464	2.305742988	2.2897787	2.4153361
5	3.28329	3.283285682	-	-
6	4.27964	4.279634827	-	-
7	5.27926	5.279254032	-	-
8	6.27923	6.279213301	-	-

From Tables 1, 2 and *3* it can be seen that the analytical method of Blasius equation and other published results are very accurate and that accuracy of the obtained results by OAFM is growing along with increasing the number of parameters in the auxiliary functions. However, some other method published in [23] and [24] give a good accuracy, but OAFM in by far the best method delivering faster convergence and better accuracy. In our procedure the approximate relations are perfomed in a simple manner by identifying some coefficients and therefore very good approximation are obtained in few terms.

CONCLUSIONS

In this paper, an alternative Optimal Auxiliary Functions Method is employed to propose new analytic approximate solutions for the velocity profile of the fluid in the boundary layer theory on a half-infinite interval - Blasius problem.

In comparison with any other known methods, our technique is based uper an original construction of the solution using a moderate number of the optimal-convergence-control parametric C_i which appear in the so-called auxiliary functions F_k. For the sake of brevity, it is very important to remark that these parameters lead to a high precission, comparing our approximate solutions with numerical results and other mentioned methods. Let us note that the nonlinear diffrential Eq.(10) is reduced to two linear differential equations, which not depend on all terms of the nonlinear equation.

Also the construction of the equation which determine the first approximate $f_0(\eta)$ is not unique. We have a great freedon to choose the number of optimal convergence control parameters, of the auxiliary function $\bar{F}_k$ and some terms from nonlinear operator $N\left[f_0(\eta)\right]$.

The accuracy of the results obtained by OAFM is growing along with increasing the number of the parameters C_i. Our procedure is very effective, explicit and accurate for nonlinear approximation, rapidy converging to the exact solution and provides a simple but rigorous way of controlling and adjusting the convergence of the solution.

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