SUBCLASS OF DIFFERENTIAL LINEAR EQUATIONS WITH AN IMPOSED AND PERIODIC SOLUTION

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Abstract. The imposed and periodic solution is an even function with a finite number of Fourier coefficients and a mean value of zero. The differential equations having as solution this imposed function are specified. The given function multiplies with time and with a characteristic coefficient, so it is the oscillating term of the second fundamental solution of the differential equation. The singular integration of the characteristic coefficient is determined. A differential system is specified and integrated for calculation of the periodic term of the second fundamental solution. When the characteristic coefficient is zero, the second fundamental solution is also the periodic solution.

Key words: Second order ordinary equation, Dynamic system, Parametric resonance.

1. INTRODUCTION

Mathieu's equation which commonly occurs in non-linear vibrational problems and Sturm Liouville's problems have been investigated in various papers, including references [1], [11], [12] and [13]. The present work is strongly related with the results obtained in [3] on this subject.

The unspecified functions $Q(t)$ and $r(t)$ have the period π and 2π respectively. The fundamental solutions (x,u), (y,v) and (z,w) verify two systems that are not autonomous [1, 2]. As a working hypothesis, the imposed solution (x,u) has a period of 2π.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=u,\quad \frac{\mathrm{d}u}{\mathrm{d}t}=-Qx,\quad x(0)=1,\quad u(0)=0\,, \tag{1}$$

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$$\frac{\mathrm{d}y}{\mathrm{d}t}=v,\quad \frac{\mathrm{d}v}{\mathrm{d}t}=-Q\,y,\quad y(0)=0,\quad v(0)=1\,. \tag{2}$$

The relatively arbitrary function $m(t)$ has the period 2π and $(\cos t)u\,/\,x=r(t)$ has no singularities.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=u,\quad \frac{\mathrm{d}u}{\mathrm{d}t}=-Q\,x-\ m(u\cos t-r\,x),\quad x(0)=1,\quad u(0)=0\,, \tag{3}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=w,\quad \frac{\mathrm{d}w}{\mathrm{d}t}=-\,Q\,z-m(w\cos t-r\,z),\quad z(0)=0,\quad w(0)=1\,. \tag{4}$$

The functions y_p and z_p will have period 2π, if and only if the coefficients σ and ρ will have certain characteristic values [3].

$$y_p=y-\sigma t x,\ \ z_p=z-\rho t x\,. \tag{5}$$

The problem consists in determining the expressions of the constant characteristic coefficients σ and ρ, specifying the non-autonomous and inhomogeneous systems that these periodic components verify and building their analytical solutions.

2. THE FIRST INHOMOGENEOUS SYSTEM AND THE CHARACTERISTIC COEFFICIENT

By derivation it results

$$v_p=v-\sigma x-\sigma t u\,. \tag{6}$$

The first inhomogeneous system is

$$\frac{\mathrm{d}y_p}{\mathrm{d}t}=v_p,\quad \frac{\mathrm{d}v_p}{\mathrm{d}t}=-Q\,y_p-2\,\sigma\,u,\quad y_p(0)=0,\quad v_p(0)=1-\sigma\,. \tag{7}$$

In [3], the Q function was chosen and defined. The functions ξ and Q have the following expressions:

$$\xi(t)=1-2p\cos^2 t+\ (p^2-k)\ \cos^4 t,\qquad \xi_0=(1-p)^2-k=\xi(0)\,, \tag{8}$$

$$Q(t)=1+\frac{4}{\xi(t)}\cdot\left\{3p-\left[4p+5(p^2-k)\right]\cos^2 t+6(p^2-k)\cos^4 t\right\}. \tag{9}$$

Explicit dependence on the real parameters with small values p and k is omitted. The solution (x,u) is:

$$x(t)=\frac{\cos t}{\xi_0}\cdot\xi(t)\,, \tag{10}$$

$$u(t)=-\frac{\sin t}{\xi_0}\cdot\eta(t),\quad \eta(t)=1-6p\cos^2 t+5\,(p^2-k)\cos^4 t\,. \tag{11}$$

The characteristic coefficient σ is an integral with parameters, [3].

$$\sigma=\frac{2\xi_0^2}{\pi}\int_0^{\frac{\pi}{2}}\frac{2p-(p^2-k)\cos^2 t}{\xi(t)}\left[1+\frac{1}{\xi(t)}\right]\mathrm{d}t \tag{12}$$

If k is not zero, then by composition, the following real or complex constants depend on the p and k.

$$a=\sqrt{1-p-\sqrt{k}}\,,\ b=\sqrt{1-p+\sqrt{k}}\,,\ A=p+\frac{p^2+k}{2\sqrt{k}},\ B=2p-A\,, \tag{13}$$

$$C=A\left(2+\frac{B}{\sqrt{k}}+\frac{A}{2a^2}\right),\quad D=B\left(2-\frac{A}{\sqrt{k}}+\frac{B}{2b^2}\right),$$
$$\beta_1=\xi_0^2\frac{C}{a},\quad \beta_2=\xi_0^2\frac{D}{b},\quad \sigma=\beta_1+\beta_2. \tag{14}$$

Also, in the paper [3], the graphs of the periodic analytical components (y_p,v_p) were drawn. When σ is zero, the system has all the solutions periodic, the solution (x,u) being the imposed one. Otherwise, by imposing $\sigma(p,k)=0$, the system will have all the solutions periodic, but the imposed solution $(x,u)(p,k)$ changes since the parameters p and k depend on each other.

3. THE SECOND INHOMOGENEOUS SYSTEM. THE CHARACTERISTIC COEFFICIENT

By derivation from formula (5) it results

$$w_p=w-\rho x-\rho t u\,. \tag{15}$$

Then if σ is not zero, we will consider a new inhomogeneous system that accepts the unchanged solution (x,u), the parameters having values independent on each other.

$$\frac{\mathrm{d}z}{\mathrm{d}t}=w,\quad \frac{\mathrm{d}w}{\mathrm{d}t}=-\left[Q-m\,r\right]z-m\,w\cos t,\quad z(0)=0,\quad w(0)=1,\tag{16}$$

$$\begin{gathered}\frac{\mathrm{d}z_p}{\mathrm{d}t}=w_p,\quad \frac{\mathrm{d}w_p}{\mathrm{d}t}=-\left[Q-m\,r\right]z_p-m(w_p+\rho x)\cos t-2\rho u,\\ z_p(0)=0,\quad w_p(0)=1-\rho.\end{gathered}\tag{17}$$

Let r be the function with the following property

$$u\cos t-r(t)x=0,\quad r(t)=-\frac{\sin t}{\xi(t)}\cdot\eta(t)\,.\tag{18}$$

In this case, the first and second systems have the same periodic fundamental solution (x,u). Let M be the "kinetic moment" or the determinant of the fundamental matrix [1].

$$M(t)=x\,w-u\,z,\quad M(0)=1\,.\tag{19}$$

The derivative of this function finally has the expression

$$\frac{\mathrm{d}M}{\mathrm{d}t}=-m(t)M\cos t\,.\tag{20}$$

The moment will have the expression

$$M(t)\;=\;\exp\left[-\int_{[0,t]}m(t)(\cos t)\mathrm{d}t\right].\tag{21}$$

We will continue and accept a simplification of the working hypothesis. The expression of the M moment will result. The small real parameter δ specifies a family of systems:

$$m(t)=\frac{2\delta\sin t}{1+\delta\cos^2 t}\qquad\Rightarrow\quad M(t)=\frac{1+\delta\cos^2 t}{1+\delta}\,.\tag{22}$$

Therefore, solution z verifies the inhomogeneous linear differential equation:

$$x\frac{\mathrm{d}z}{\mathrm{d}t}-u\,z=M\,.\tag{23}$$

We consider the z_p component for which periodicity is required.

$$z = z_p + \rho t x \quad \Rightarrow \quad x\left(\frac{\mathrm{d}z_p}{\mathrm{d}t} + \rho x\right) - u z_p = M, \quad z_p(0) = 0. \tag{24}$$

Let G be the constant variable of integration

$$z_p = G x \quad \Rightarrow \quad \frac{\mathrm{d}G}{\mathrm{d}t} = \frac{M}{x^2} - \rho = \frac{1}{1+\delta}\cdot\frac{1}{x^2} + \frac{\delta}{1+\delta}\cdot\frac{\cos^2 t}{x^2} - \rho. \tag{25}$$

Especially for $\delta = 0$, the moment has the value one.

$$y = y_p + \sigma t x, \quad y_p = C x \quad \Rightarrow \quad \frac{\mathrm{d}C}{\mathrm{d}t} = \frac{1}{x^2} - \sigma. \tag{26}$$

Reference paper [3] shows that the previous equation leads to the value σ of the formula (12) or (14). Explicit expressions of function C and component y_p are also specified in [3]. G's equation is:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \left(\frac{\mathrm{d}C}{\mathrm{d}t} + \sigma\right)\cdot\frac{1}{1+\delta} + \frac{\delta}{1+\delta}\cdot\frac{\cos^2 t}{x^2} - \rho. \tag{27}$$

Let θ be the constant with the property

$$\rho = \frac{\sigma + \theta\delta}{1+\delta}. \tag{28}$$

Thus

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{1}{1+\delta}\cdot\left[\frac{\mathrm{d}C}{\mathrm{d}t} + \left(\frac{\cos^2 t}{x^2} - \theta\right)\delta\right]. \tag{29}$$

Function G has the following representation with a new unknown function F.

$$G = \frac{C + F\delta}{1+\delta}, \quad \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\cos^2 t}{x^2} - \theta. \tag{30}$$

The periodic component z_p has the expression

$$z_p = G x = \frac{y_p + F x \delta}{1+\delta}. \tag{31}$$

The function $y_p = Cx$ is periodic [3]. The function Gx should be periodic so that $F(t)x(t)$ must be a periodic function, too. The mean value of the function $F(t)$ according to formula (30) must be zero (see [4–6]). The integrated function depends only on $\cos^2 t$. The identity below is derived according to the formula (10).

$$\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2 t}{x(t)^2}\,\mathrm{d}t = \frac{2\xi_0^2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\xi(t)^2}\,\mathrm{d}t\,. \tag{32}$$

The value of the characteristic constant ρ can be calculated with formulas (14) and (28).

4. CALCULATION OF PERIODIC COMPONENT z_p

The function y_p is specified in the reference [3] for the calculation of the function $F(t)$. We transform the integration variable.

$$s = \tan t, \quad \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\cos^2 t}, \quad \cos^2 t = \frac{1}{s^2+1}\,. \tag{33}$$

The function F is expressed by the function H .

$$F(t) = H(s), \quad \frac{\mathrm{d}H}{\mathrm{d}s} = \frac{\mathrm{d}F}{\mathrm{d}t}\cdot\frac{1}{s^2+1} = \frac{\xi_0^2}{\xi(\mathrm{t})^2}\cdot\frac{1}{s^2+1} - \frac{\theta}{s^2+1}\,. \tag{34}$$

Consider formula (13). Therefore the algebraic identities result:

$$\begin{aligned}\frac{1}{\xi(\mathrm{t})} &= \frac{1}{1-2p\cos^2 t + (p^2-k)\cos^4 t} = \frac{(s^2+1)^2}{(s^2+a^2)(s^2+b^2)} = \\ &= 1 + \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}.\end{aligned} \tag{35}$$

By deriving function H it results the expression of a rational function.

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \xi_0^2\left(1 + \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}\right)^2 \cdot \frac{1}{s^2+1} - \frac{\theta}{s^2+1} \tag{36}$$

Let the following constants be:

$$R_1=\frac{A^2}{2a^2(1-a^2)},\quad R_3=\frac{1}{1-a^2}\cdot\left[R_1(1-3a^2)+2A+\frac{2AB}{b^2-a^2}\right],$$
$$\beta_3=\xi_0^2R_3\cdot\frac{1}{a},\tag{37}$$

$$R_2=\frac{B^2}{2b^2(1-b^2)},\quad R_4=\frac{1}{1-b^2}\cdot\left[R_2(1-3b^2)+2B-\frac{2AB}{b^2-a^2}\right],$$
$$\beta_4=\xi_0^2R_4\cdot\frac{1}{b}.\tag{38}$$

Let be the functions defined by

$$g(s)=\frac{R_1s}{s^2+a^2}+\frac{R_2s}{s^2+b^2},\quad f(s)=\frac{R_3}{s^2+a^2}+\frac{R_4}{s^2+b^2}.\tag{39}$$

The following identity is satisfied.

$$\left(1+\frac{A}{s^2+a^2}+\frac{B}{s^2+b^2}\right)^2\cdot\frac{1}{s^2+1}=\frac{R_0}{s^2+1}+f(s)+\frac{\mathrm{d}}{\mathrm{d}s}g(s)\tag{40}$$

According to formulas (13) the constant R_0 is zero. By deriving the function H , according to formula (36), the equivalent expression results

$$R_0=\left(1-\frac{A}{1-a^2}-\frac{B}{1-b^2}\right)^2\equiv 0,$$
$$\frac{\mathrm{d}H}{\mathrm{d}s}=\xi_0^2\left(f(s)+\frac{\mathrm{d}}{\mathrm{d}s}g(s)\right)-\frac{\theta}{s^2+1}.\tag{41}$$

The unknown function $H(s)$ has two $h_3(s)$ and $h_4(s)$ terms.

$$H(s)=h_3(s)+h_4(s),$$
$$h_3(s)=\xi_0^2\,g(s)=\xi_0^2\,s\left(\frac{R_1}{s^2+a^2}+\frac{R_2}{s^2+b^2}\right),\tag{42}$$

$$\frac{\mathrm{d}h_4}{\mathrm{d}s}=\xi_0^2\,f(s)-\frac{\theta}{s^2+1}=\beta_3\frac{a}{s^2+a^2}+\beta_4\frac{b}{s^2+b^2}-\frac{\theta}{s^2+1}.\tag{43}$$

The expression (32) results from the formula (34).

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}F}{\mathrm{d}t}dt=\frac{2}{\pi}\int_0^{\infty}\frac{\mathrm{d}H}{\mathrm{d}s}\mathrm{d}s=\beta_3+\beta_4-\theta=0 \tag{44}$$

The product $x\times h_3$ has the final expression, according to formulas (10), (36) and (42):

$$\begin{aligned}x(t)h_3(s)&=\frac{(s^2+a^2)(s^2+b^2)\cos t}{\xi_0(s^2+1)}\xi_0^2 s\left(\frac{R_1}{s^2+a^2}+\frac{R_2}{s^2+b^2}\right)=\\&=\xi_0\frac{R_1(s^2+b^2)+R_2(s^2+a^2)}{(s^2+1)}\sin t=\\&=\xi_0\left[(R_1+R_2)\sin^2 t+(R_1b^2+R_2a^2)\cos^2 t\right]\sin t,\end{aligned} \tag{45}$$

$$\begin{aligned}&x(t)h_3(\tan t)=\xi_0\left(R_1+R_2+R_6\cos^2 t\right)\sin t,\\&R_6=R_1(b^2-1)+R_2(a^2-1).\end{aligned} \tag{46}$$

The function Fx of formula (31) has the expression

$$F(t)x(t)=H(s)x(t)=x(t)h_3(s)+x(t)h_4(s). \tag{47}$$

Let us consider the periodic functions

$$\begin{aligned}\gamma_4(t)&=\frac{1}{1+\delta}\cdot\left[\frac{(\beta_1+\beta_3\delta)a}{1+(a^2-1)\cos^2 t}+\frac{(\beta_2+\beta_4\delta)b}{1+(b^2-1)\cos^2 t}\right],\\C_4(t)&=\int_0^t\left[\gamma_4(t)-\frac{\sigma+\theta\delta}{1+\delta}\right]\mathrm{d}t.\end{aligned} \tag{48}$$

In the case of δ equal to zero, $\gamma_4(t)$ is reduced to $\gamma(t)$, and $C_4(t)$ would be equal to $C_2(t)$ from reference [3]. If δ would tend to infinity $\gamma_4(t)$ would correspond to the solution of equation (43). Because y_p has two components, so z_p will also have two components.

$$\begin{aligned}&z_p(t)=z_{p_1}(t)+z_{p_2}(t),\qquad z_{p_2}(t)=x(t)C_4(t),\\&z_{p_1}(t)=\frac{y_{p1}(t)}{1+\delta}+\frac{\delta}{1+\delta}\cdot\xi_0\left(R_1+R_2+R_6\cos^2 t\right)\sin t.\end{aligned} \tag{49}$$

The y_{p_1} component has the expression, [3].

$$K_1 = \frac{1}{2}\left(\frac{A^2}{a^2} + \frac{B^2}{b^2}\right) - 2p,$$

$$K_2 = p^2 - k - \frac{1}{2}\left[\left(p - \sqrt{k}\right)\frac{A^2}{a^2} + \left(p + \sqrt{k}\right)\frac{B^2}{b^2}\right], \tag{50}$$

$$y_{p_1}(t) = \xi_0\left(1 + K_1\cos^2 t + K_2\cos^4 t\right)\sin t\,.$$

The w_p function becomes

$$\begin{aligned} w_p(t) &= w_{p_1}(t) + w_{p_2}(t), \\ w_{p_2}(t) &= u(t)C_4(t) + x(t)\,[\gamma_4(t) - \rho], \\ w_{p_1}(t) &= v_{p_1}(t)/(1+\delta), \\ v_{p_1}(t) &= \xi_0[1 - 2K_1 + (3K_1 - 4K_2)\cos^2 t + 5K_2\cos^4 t]\cos t. \end{aligned} \tag{51}$$

The functions z_p and w_p represent the periodic solution of the inhomogeneous system (17), (18), (22).

$$\begin{aligned} &R(t) = Q(t) - m(t)\,r(t) = Q(t) + \frac{2\delta\,\eta(t)\sin^2 t}{\xi(\mathrm{t})(1+\delta\cos^2 t)}, \\ &P(t) = m(t)\cos t = \frac{\delta\sin 2t}{1+\delta\cos^2 t}, \\ &\frac{\mathrm{d}z_p}{\mathrm{d}t} = w_p, \quad \frac{\mathrm{d}w_p}{\mathrm{d}t} = -R(t)\,z_p - P(t)\left[w_p + \rho\,x(t)\right] - 2\rho\,u(t), \\ &z_p(0) = 0, \quad w_p(0) = 1 - \rho. \end{aligned} \tag{52}$$

Therefore, the *Floquet*'s expression of the fundamental matrix of the homogeneous system (3) and (4) becomes

$$\begin{aligned} \Phi(t) &= \begin{bmatrix} x & z \\ u & w \end{bmatrix} = \begin{bmatrix} x & z_\mathrm{p} \\ u & w_\mathrm{p} + \rho x \end{bmatrix} + \rho t \begin{bmatrix} 0 & x \\ 0 & u \end{bmatrix} = \\ &= \begin{bmatrix} x & z_\mathrm{p} \\ u & w_\mathrm{p} + \rho x \end{bmatrix} \exp\left(\begin{bmatrix} 0 & \rho \\ 0 & 0 \end{bmatrix} t\right), \\ \rho &= [\beta_1 + \beta_2 + (\beta_3 + \beta_4)\delta]/(1+\delta). \end{aligned} \tag{53}$$

In order to obtain the analytical solutions of the system (4) we used the working hypothesis (22) for which the determinant $M(t)$ of the fundamental matrix is a certain rational function in relation to the useful variable $s = \tan(t)$ (see [7–9]). The analytical solution of the system (4) is analogously obtained if the determinant given by the formulas (22) has the following expression:

$$M_1(t) := (1 + \delta_1 \cos^2 t + \delta_2 \cos^4 t + ...) / (1 + \delta_1 + \delta_2 + ...).$$

The parameters are small real numbers. Function $m_1(t)$ will be calculated according to (20). The decomposition into simple fractions and obtaining analytical solutions requires longer time calculations.

5. THE SECOND HOMOGENEOUS SYSTEM WITH PERIODIC SOLUTIONS

The following program allows the specification of the expressions of the coefficients $P(t)$ and $R(t)$ of a differential, linear and homogeneous system. The system has an imposed and periodic solution (x,u). For a certain value of the parameter δ the system will have the second periodic solution (z,w).

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = -R(t)x - P(t)u, \quad x(0) = 1, \quad u(0) = 0, \tag{54}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = w, \quad \frac{\mathrm{d}w}{\mathrm{d}t} = -R(t)z - P(t)w, \quad z(0) = 0, \quad w(0) = 1. \tag{55}$$

The values of the parameters k and p are initialized in the MATHCAD program [10].

$$k := 0.2, \qquad p := -0.03. \tag{56}$$

We write the expressions of the following constants: ξo according to formula (8); $a, b, A, B, C, D, \beta_1, \beta_2$, and σ according to formulas (13) and (14); $R_1, R_2, R_3, R_4, \beta_3, \beta_4$ according to formulas (37) and (38) inclusive θ according to formula (44). The specific value $\delta = \delta(k, p)$ is updated so that the characteristic constant ρ given by formula (28) is zero.

$$\delta := -\frac{\sigma}{\theta}, \qquad \rho := \frac{\sigma + \theta\delta}{1 + \delta}. \tag{57}$$

The following functions are defined: $\xi(t)$, $Q(t)$ and $\eta(t)$ according to formulas (8), (9) and (11). According to formulas (52) the coefficients of the systems (3) and (4) are

$$P(t) := \frac{\delta \cdot \sin(2\cdot t)}{1+\delta\cdot\cos(t)^2}, \quad R(t) := Q(t) + \frac{2\cdot\delta\cdot\eta(t)\cdot\sin(t)^2}{\xi(t)\cdot\left[1+\delta\cdot\cos(t)^2\right]}. \tag{58}$$

Numerical x,u,z,w solutions are uppercase. These solutions check the following system, where the integration interval is divided into N parts.

$$\mathrm{Co} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1-\rho \end{bmatrix} \quad \mathrm{D}(t,u) := \begin{bmatrix} u_1 \\ -R(t)\cdot u_0 - P(t)\cdot u_1 \\ u_3 \\ -R(t)\cdot u_2 - P(t)\cdot(u_3 + \rho\cdot u_0) - 2\rho\cdot u_1 \end{bmatrix} \tag{59}$$

$$N := 1024$$

$$S := \mathrm{rkfixed}(\mathrm{Co}, 0, 4, \pi, N, \mathrm{D}).$$

The columns of the solution matrix ***S*** represent the numerical values of the corresponding values of the functions x,u,z and w.

$$X := S^{<1>} \quad U := S^{<2>} \quad Z := S^{<3>} \quad W := S^{<4>} \tag{60}$$

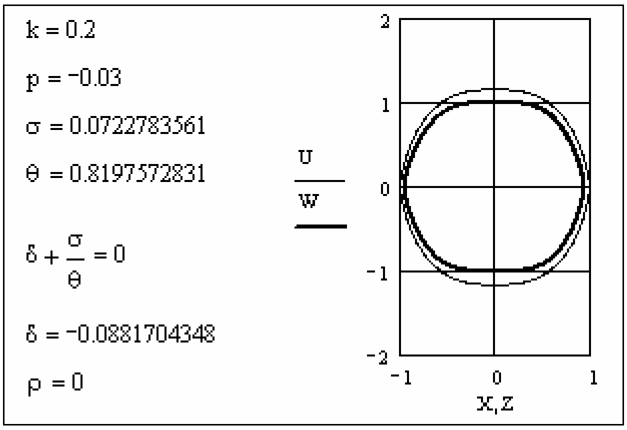

Fig. 1 – Graphs (X,U) and (Z,W) show periodicity.

The period is 2π although the length of the integration interval was 4π. The function $P(t)$ is an odd continuous function. *If* ρ *is nonzero then we have periodic components* $Z = z_p$ and $W = w_p$.

6. THE SECOND EXEMPLE OF THE HOMOGENEOUS SYSTEM

In this example the following determinant is considered which does not decompose into simple fractions.

$$M_{\mathrm{o}}(t) = 1+\delta_{\mathrm{o}}\left|\sin t\right|. \tag{61}$$

Returning to equations (23), (24) and (25) we have the following formulas:

$$x\frac{\mathrm{d}z_{\mathrm{o}}}{\mathrm{d}t} - u\,z_{\mathrm{o}} = M_{\mathrm{o}}\,,\quad z_{\mathrm{o}} = z_{\mathrm{o\,p}} + \rho_{\mathrm{o}}\,t\,x,\quad z_{\mathrm{o\,p}} = G_{\mathrm{o}}\,x,$$
$$\frac{\mathrm{d}G_{\mathrm{o}}}{\mathrm{d}t} = \frac{1+\delta_{\mathrm{o}}\left|\sin t\right|}{x^2} - \rho_{\mathrm{o}}\,. \tag{62}$$

The characteristic coefficient has the expression:

$$\rho_{\mathrm{o}} = \sigma + \theta_{\mathrm{o}}\delta_{\mathrm{o}},\quad \theta_{\mathrm{o}} = \frac{1}{\pi}\cdot\xi_{\mathrm{o}}^2\cdot\int_0^{\pi}\left[\frac{\sin t}{\xi(t)^2} - 1\right]\cdot\frac{1}{\cos^2 t}\mathrm{d}t\,. \tag{63}$$

The elimination of the singularity from the moment $\pi/2$ results in the expression of the corresponding definite integral.

$$\theta\mathrm{o} := \frac{1}{\pi}\cdot\xi_{\mathrm{o}}^2\cdot\int_0^{\pi}\Big[1+\xi(t)\cdot\left[2\cdot p - (p^2-k)\cdot\cos(t)^2\right] - $$
$$-\frac{1}{\sin(t)+1}\Big]\cdot\frac{1}{\xi(t)^2}\,\mathrm{d}t\,. \tag{64}$$

The specific value $\delta\mathrm{o}$ is updated so that the characteristic constant $\rho\mathrm{o}$ given by formula (63) is zero.

$$\delta\mathrm{o} := -\sigma/\theta\mathrm{o},\quad \rho\mathrm{o} = 0\,. \tag{65}$$

The expressions of the functions $m\mathrm{o}(t)$, $P\mathrm{o}(t)$ and $R\mathrm{o}(t)$ result in accordance with formulas (20) and (52).

$$m\mathrm{o}(t) := -\frac{\delta\mathrm{o}\cdot\mathrm{if}(\sin(t)>0,1,-1)}{M\mathrm{o}(t)},\quad P\mathrm{o}(t) := m\mathrm{o}\cdot\cos(t),$$
$$R\mathrm{o}(t) := Q(t) + \left[\frac{1}{M\mathrm{o}(t)} - 1\right]\cdot\frac{\eta(t)}{\xi(t)}. \tag{66}$$

The numerical solutions x,u,z,w verify the following system:

$$\mathrm{Co} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1-\rho o \end{bmatrix}$$

$$\mathrm{Do}(t,v) := \begin{bmatrix} v_1 \\ -R\mathrm{o}(t)\cdot v_0 - P\mathrm{o}(t)\cdot v_1 \\ v_3 \\ -R\mathrm{o}(t)\cdot v_2 - P\mathrm{o}(t)\cdot(v_3+\rho o\cdot v_0) - 2\rho o\cdot v_1 \end{bmatrix} \tag{67}$$

$$S := \mathrm{rkfixed}(\mathrm{Co},0,4,\pi,N,\mathrm{Do}).$$

Therefore, we have

$$x:\ = S_o{}^{<1>},\quad u:\ = S_o{}^{<2>},\quad z:\ = S_o{}^{<3>},\quad w:\ = S_o{}^{<4>}. \tag{68}$$

$k = 0.2$

$p = -0.03$

$\sigma = 0.0722783561$

$\theta o = -0.4624617237$

$\delta o + \dfrac{\sigma}{\theta o} = 0$

$\delta o = 0.1562904612$

$\rho o = 0$

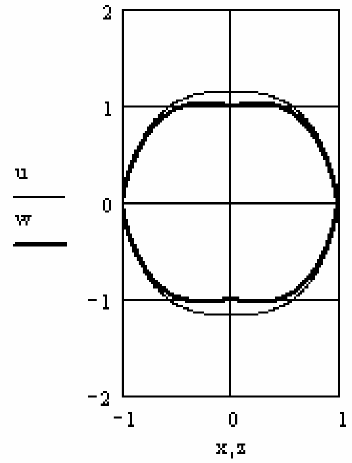

Fig. 2 – Graphs (x,u) and (z,w).

The function $P\mathrm{o}(t)$ has points of discontinuity and has no constant sign.

7. CONCLUSIONS

The contributions of the paper are demonstrations of statements and the derivation of analytical formulas. If Q is a certain set of functions (9), then the linear and homogeneous differential system (1) has an imposed and periodic fundamental solution (x,u) given by formulas (10) and (11). The second fundamental solution has the expression of the structure $(y = y_p + \sigma t x, \quad v = v_p + \sigma x + \sigma t u)$ where σ is a characteristic coefficient, and (y_p, v_p) is a periodic solution of the inhomogeneous system (7). In this system σ has two calculable equivalent expressions. The first is a integral with parameters (12) and the second is the algebraic composition of some functions. In addition, the value $y(2\pi) = 2\pi\sigma$ can be calculated by integrating the system (2) on the interval $[0, 2\pi]$.

If σ is different from zero, the linear and homogeneous differential system (3) in which $m(t)$ is a relatively arbitrary function, also has an imposed and periodic fundamental solution (x,u). The second fundamental solution has the structure expression $(z = z_p + \rho t x, \; w = w_p + \rho x + \rho t u)$ in which ρ is a characteristic coefficient.

The determinant M of the fundamental matrix was chosen as a periodic function in according with formula (22) in which a new parameter δ appears. The component z_p is a solution of the inhomogeneous equation (24) in which the coefficient ρ appears. According to formula (25), the characteristic coefficient ρ is equal to the mean value of the periodic function $N(t)$.

$$N(t) \; = \; \left(M / x^2\right)(t) - M(\pi/2)\xi_0{}^2 / \cos^2 t \,.$$

The new expression of the characteristic coefficient is specified by the formula (28) in which the coefficient θ is given by the integral with parameters (32). Using decompositions into rational fractions we obtained the equivalent algebraic formula of the coefficient θ (44). The analytical expressions of the periodic solution (z_p, w_p) are given in formulas (49) and (51). The fundamental matrix is specified in formula (53).

If $\delta = -\sigma / \theta$, the characteristic coefficient $\rho = 0$ such that the linear and homogeneous differential system (59), but except of the initial condition, has all solutions as periodic functions. The graphs of the fundamental solutions are presented in Fig. 1.

But not for any choice of the determinant of the fundamental matrix can be used the decomposition into rational fractions. For example, a M_0 determinant was chosen according to formula (61) where another parameter δo appears. In this case we will have the linear and homogeneous differential system (67). The characteristic coefficient ρ_0 is given by the formula (63) in which the coefficient θ_0 has the expression of the integral with parameters (64).

If $\delta_o = -\sigma_0\theta_0$ the characteristic coefficient $\rho_0 = 0$ such that the linear and homogeneous differential system (67) has all solutions as periodic functions. The graphs of the fundamental solutions are presented in Fig. 2. Homogeneous differential systems (59) and (67) have the same imposed and periodic fundamental solution (x,u). Their fundamental matrices are periodic functions, but they are different from each other.

The solution x in formula (10) can generally depend on the parameters k, p, p_1, p_2 etc.

COMMENT: Let us note that Mathieu's equation and Sturm Liouville's problems are presented in various papers, including references [1], [11], [12] and [13]. We return to equations (1) and (2), we give up the knowledge of the solution (x,u) but we impose a new function $Q(t)$ according to the reference [14] for Mathieu's equation.

$$Q(t) = 4\alpha_1(q) - 16q\cos 2t$$

The eigenvalue $4\alpha_1$ has the following polynomial approximation.

$$4\alpha_1 = 1 + 8q\left(1 - q - q^2 - q^3/3\right)$$

We choose a value q and integrate on the interval $[0, 2\pi]$ the new system (2). The characteristic constant σ will be:

$$q = 0.02, \qquad \sigma = y(2\pi)/(2\pi) = 0.141892\,.$$

With the known value σ, the following system is integrated for six components. Let $N = 1024$.

$$\mathrm{xo} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ \sigma \\ 0 \\ 1-\sigma \end{bmatrix} \qquad F(t,z) := \begin{bmatrix} z_1 \\ -Q(t)\cdot z_0 \\ z_3 \\ -Q(t)\cdot z_2 + 2\cdot\sigma\cdot z_1 \\ z_5 \\ -Q(t)\cdot z_4 - 2\cdot\sigma\cdot z_1 \end{bmatrix}$$

$$Rx := \mathrm{rkfixed}(\mathrm{xo}, 0, 2\pi, N, F).$$

The columns of the matrix F represent the numerical values of the periodic solution $x(t)$, of the periodic component $y_p(t)$ and of the term $t\sigma x(t)$, so that $y = y_p + t\sigma x$.

$$t := Rx^{<0>},\ x := Rx^{<1>},\ t\sigma x := Rx^{<3>},\ yp := Rx^{<5>}.$$

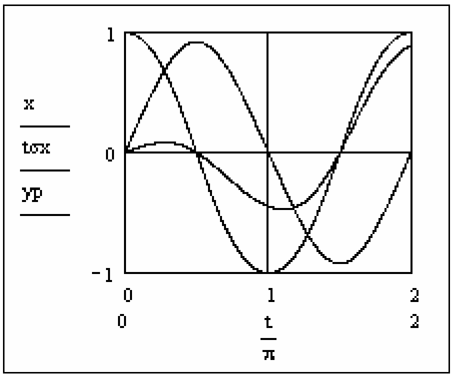

Fig. 3 – Graphs of fundametal solution x and a periodic component y_p.

Knowing the characteristic constant is important for specifying the existence of the second periodic solution.

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