A PERTURBATION SOLUTION TO VINTI'S DYNAMICAL PROBLEM

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Abstract. For the short times in which Vinti's analytical solution is suitable for real Earth orbital problems, we propose to replace the exact orbital propagation algorithm by an analytical perturbation approximation. In practice, the approximation provides analogous accuracy to that of the exact solution, is also free from essential small divisors, and is of straightforward evaluation. As a side effect, our approach presents another example of how the truncation of an integrable problem can lead to the illusion of chaotic behavior.

Key words: artificial satellite theory, intermediaries, oblate spheroid, integrability, perturbations, Vinti's dynamical problem.

1. INTRODUCTION

Vinti's spheroidal potential, which is integrable in oblate spheroidal coordinates, provides an accurate intermediary solution to the artificial satellite problem [1, 2]. In particular, it accounts for the full effect of the second zonal harmonic of the Geopotential and for an important part of the remaining zonal potential. The original formulation constrained in practice the applicability of the solution to bodies with equatorial symmetry. However, soon after publication, Vinti's dynamical model was identified with the classical problem of two fixed centers under the simple transformation that connects prolate and oblate spheroidal coordinates (see chap. 17, §13 of [3], and also the discussions on p. 592 of the same book). This equivalence inspired adaptations of the two-fixed-center problem for the exact inclusion in this integrable model of the effect of the third zonal harmonic coefficient, in addition to the second one [4]. In turn, the modifications of the two-fixed-center problem prompted Vinti to further refine his spheroidal model in order to successfully include this later effect [5, 6].

In practice, the involved analysis required by Vint's approach, make that his solutions is used for operational purposes by only a minority of researchers [7, 8, 9]. Still, the fact that some comparisons [10, 11] showed Vinti's method more accurate than widely used analytical solutions, like Brouwer's perturbation theory [12, 13], and that different algorithms are freely available for implementation [14], makes Vinti's method to enjoy some revival these days [15, 16, 17, 18].

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Among the acclaimed merits of Vinti's solution are that, being exact, the solution is valid for any time, contrary to the limited intervals in which analytical perturbation solutions apply, and is free from singularities at the critical inclination. However, Vinti's solution is just an approximation of the real dynamics undergone by Earth orbits. In particular, since it misses secular contributions of the Geopotential its validity is constrained de facto to short-time propagation intervals. By the same reason it is unable to reproduce the actual libration of the perigee of orbits in the close vicinity of the critical inclination. If Vinti's solution is supplemented with the needed gravitational effects in order to reflect this last behavior, then the divisors related to the critical inclination necessarily arise in the perturbation equations [19]. It could not be otherwise due to the essential character of the critical inclination resonance [20, 21, 22, 23].

On the other hand, like it is clearly pointed out in [3, p. 573], the particularization of Brouwer's solution for the value of the fourth zonal harmonic requested by Vinti's integrable model makes the critical divisors of Brouwer's long-period corrections to cancel (see also [14, p. 251]). Hence, we propose to replace the involved implementation of an orbital propagation program based on Vinti's solution by a particularization of Brouwer's solution to the case of Vinti's expansion of the zonal harmonic potential. Cursory tests on different types of Earth orbits show that initial conditions propagated in the Vinti model yield orbits that can depart on the km level from corresponding orbits of a full zonal model in time intervals that range from one to several days. Therefore, for the short times in which Vinti's solution is valid in practice, a low-order approach of the perturbation solution achieves analogous merits, in this way replacing the intricate exact solution with a radically simpler algorithm. For common Earth orbits, we checked that higher orders of the perturbation approach are able to mimic Vinti's exact solution within the μm level in time intervals extending up to one month, yet the computational burden is notably higher in this last case.

On a different aspect, the particularization of the Geopotential coefficients to the values of the harmonic expansion coefficients of Vinti's spheroidal model supplies another instance in which the perturbation solution only adopts the behavior typical of an integrable system, showing non-isolated equilibria, when the degree of the truncation tends to infinity [24].[1] Indeed, by increasing the perturbation order of each particular truncation of Vinti's zonal harmonics expansion, the degeneracy of the equilibria at the critical inclination [26] is broken to show stable and unstable isolated equilibria, the latter providing indications of chaotic behavior in the original model [27]. Since we chose a *reverse* normalization to ease construction of the perturbation solution [28], this is only shown here indirectly, through the appearance of small divisors related to the critical inclination.

[1] Note that while higher-order degeneracy is a usual hint of integrability, the discovery of all possible integrable cases through normalization is in no way guaranteed [25].

2. VINTI'S DYNAMICAL PROBLEM

For bodies with axial symmetry, the gravitational potential $\mathscr{V}$ is given by the usual expansion in zonal harmonics

$$\mathscr{V} = \frac{\mu}{r} \sum_{n\geq 0} \frac{R_\oplus^n}{r^n} J_n P_{n,0}(\sin\varphi). \tag{1}$$

where the physical coefficients μ, for the gravitational parameter, $R_\oplus$, for the body's equatorial radius, and J_n, for the zonal harmonics coefficient of degree n, characterize the gravitational field; r is distance from the body's center of mass, φ is latitude, and $P_{n,0}$ denotes the Legendre polynomial of degree n. The motion of a particle in this gravitational field is non-integrable in general, but in the particular case in which

$$J_{2n} = (-1)^{n+1} J_2^n, \qquad J_{2n+1} = (-1)^n J_1 J_2^n, \tag{2}$$

Vinti found that the Hamiltonian

$$\mathscr{H} = \mathscr{T} + \mathscr{V}, \tag{3}$$

where $\mathscr{T}$ is the kinetic energy, is separable in oblate spheroidal coordinates [19].[2] Hence, Vinti's integrable potential provides an intermediary solution to the artificial satellite problem that keeps *all* the J_2 effects. On the contrary, most intermediaries of the main problem cope with first order effects of J_2 only, which, besides, are limited to the contribution of the secular terms and commonly miss a part of the periodic effects (see, for instance [29, 30, 31, 32, 33, 34]).

Vinti's exact solution to the non-expanded version of the potential $\mathscr{V}$ involves special functions, and the implementation of an orbit propagator based on it is not an easy task. Alternatively, given that Vinti's solution is just an approximation of the artificial satellite problem, the solution of Eq. (3) can itself be approached by perturbations. Indeed, the condition in Eq. (2), shows the role of J_2 as a small parameter. In consequence, Eq. (3) is naturally arranged in the form of a perturbation Hamiltonian

$$\mathscr{H} = \sum_{m\geq 0} \frac{J_2^m}{m!} \mathscr{H}_m. \tag{4}$$

For the Earth, J_1 is negligible and, therefore, the original Vinti's solution cannot deal with the effects of the odd zonal harmonics of the Geopotential. While this lack was later amended by Vinti [5] we constrain here to the case of equatorial symmetry for simplicity, a case that is known as "Vinti's dynamical problem" [7].

The perturbation solution is obtained by extending the integrals of the unperturbed Hamiltonian to the whole perturbation problem. That is, by complete Hamiltonian normalization of Eq. (4) up to some truncation order of the small parameter. As customary, we choose $\mathscr{H}_{0,0} = -\mu/(2a)$, the Keplerian Hamiltonian, in which a is the orbit semi-major axis, like the zeroth-order integrable part.

[2] In fact, what Vinti found was a general closed form potential in oblate spheroidal coordinates satisfying Laplace's equation, and yielding separability of the Hamilton-Jacobi equation. After expansion in Legendre polynomials, he particularized the values of the parameters to yield Eq. 2.

The projection of the angular momentum vector along the Earth's rotation axis is an integral of the original problem due to the axial symmetry of the zonal potential (1), which is, therefore, of two degrees of freedom. Then, up to some truncation order of the small parameter, the total normalization can then be approached in different ways. The most common approach is to proceed stepwise by first finding a transformation of variables that removes the short-period effects, thus converting the semi-major axes into a formal integral, and then finding a second transformation that removes the remaining long-period effects, in this way turning the total angular momentum into the third (formal) integral. This is the classical way of approaching the artificial satellite problem by perturbations [12, 35, 36]. Carrying out a preprocessing of the Hamiltonian [33, 37] may be advisable when approaching the perturbation solution [38, 39]. Conversely, normalizing the total angular momentum first provides real advantages in the computation of higher orders [40, 41, 42]. We choose this latter approach, in which, in addition, we do it in the style of [28] without the customary Hamiltonian preprocessing.[3]

In view of the scaling of the zonal harmonic coefficients in Eq. (2), and for the actual value of the Earth's J_2 coefficient, which is $\mathscr{O}(10^{-3})$, we find that the standard numerical precision is exceeded beyond $J_{10} = J_2^5$ and, in consequence, truncate the original perturbation Hamiltonian (4) to the order $m = 5$. Accordingly, a fifth-order perturbation solution should achieve numerical precision in long time intervals.

3. NORMALIZATION OF THE TOTAL ANGULAR MOMENTUM

The Hamiltonian reduction is approached in Delaunay variables, which are the action-angle variables of the Kepler problem. Thus, $\mathscr{H} = \mathscr{H}(\vec{x}, \vec{X}; J_2)$, where the coordinates $\vec{x} = (\ell, g, h)$ denote the mean anomaly, the argument of the perigee, and the right ascension of the ascending node, respectively, and their conjugate momentum $\vec{X} = (L, G, H)$ denote the Delaunay action, the total angular momentum, and the projection of the angular momentum vector along the Earth's rotation axis, respectively. Due to the axial symmetry of the zonal potential (1), h is an ignorable coordinate in the original Hamiltonian (3)–(2). That is,

$$\mathscr{H} = \mathscr{H}(\ell, g, -, L, G, H; J_2),$$

thus immediately showing the preservation of H. Next, a first transformation $\mathscr{T}_1 : (\vec{x}', \vec{X}'; J_2) \mapsto (\vec{x}, \vec{X})$ is computed such that, up to some truncation order k, it makes the argument of the perigee ignorable in the transformed Hamiltonian $\mathscr{H} \circ \mathscr{T}_1 = \mathscr{K}$. Namely,

$$\mathscr{K} = \mathscr{H}(\vec{x}(\vec{x}', \vec{X}'; J_2), \vec{X}(\vec{x}', \vec{X}'; J_2); J_2) \equiv \sum_{m=0}^{k} \frac{J_2^m}{m!} \mathscr{K}_m(\ell', L', G', H) + \mathscr{O}(J_2^{k+1}). \quad (5)$$

[3] It deserves to be mentioned that the complete Hamiltonian reduction can be achieved at once, with a single canonical transformation [43]. This latter procedure may alleviate the computational burden in the evaluation of the analytical solution and still remains to be fully explored.

The transformation $\mathscr{T}_1$ takes the form

$$\vec{x} = \vec{x}' + \sum_{m=1}^{k} \frac{J_2^m}{m!} \delta_m(\vec{x}', \vec{X}'), \qquad \vec{X} = \vec{X}' + \sum_{m=1}^{k} \frac{J_2^m}{m!} \Delta_m(\vec{x}', \vec{X}'). \tag{6}$$

However, to get rid of the singularities shown by circular and equatorial orbits when using Delaunay variables, we compute the transformation in polar coordinates $\vec{x} = (r, \theta, \nu)$, standing for the radial distance, the argument of the latitude θ, and the longitude of the node $\nu \equiv h$, as well as their conjugate momenta $\vec{X} = (R, \Theta, N)$ denoting the radial velocity, $\Theta = G$, and $N = H$. In addition to avoiding singularities, the evaluation of the corrections in polar variables is commonly found much more efficient than when using other sets of variables [44, 45, 46, 47, 48].

3.1. Partially normalized Hamiltonian

Up to the fourth order of the normalization, the Hamiltonian terms of the (partially) normalized Hamiltonian (5) take the general form

$$\mathscr{K}_m = \frac{\mu}{r} \frac{p^2}{r^2} \chi^m \sum_{n=0}^{2(m-1)} \frac{p^n}{r^n} I_{m,n}(s) \eta^{2(m-1)-2\lfloor(1+n)/2\rfloor}, \tag{7}$$

where the symbol $\lfloor\#\rfloor$ denotes integer division, $\eta = G'/L'$ is the eccentricity function, $c = H/G'$ is the cosine of the inclination, $s = (1-c^2)^{1/2}$ is the sine of the inclination, and $I_{m,n}$ are inclination polynomials of degree $2n$; $I_{1,0} = 3s^2 - 2$, $I_{m,1} = 0$, and the remaining coefficients are listed in Table 1. We abbreviate $\chi = \chi(p) \equiv \frac{1}{4}(R_\oplus/p)^2$ where $p = G^2/\mu$ denotes the conic parameter, which is constant after normalization of the total angular momentum G'. Beyond the fourth order, Eq. (7) must be supplemented with additional terms.

Table 1
Inclination polynomials $I_{m,k}$ in Eq. (7)

$_{2,0}: \frac{3}{2}s^2(5s^2-4)$	$_{4,0}: \frac{9}{8}s^2(1469s^6-3016s^4+1872s^2-320)$
$_{2,2}: -36s^4+48s^2-12$	$_{4,2}: 12s^2(805s^6-1318s^4+570s^2-36)$
$_{3,0}: 9s^2(s^2-1)(9s^2-4)$	$_{4,3}: -\frac{9}{4}s^2(17343s^6-31836s^4+17216s^2-2432)$
$_{3,2}: \frac{1}{2}s^2(275s^4-246s^2-24)$	$_{4,4}: -\frac{15}{4}s^2(773s^6-2348s^4+1632s^2-192)$
$_{3,3}: -\frac{9}{2}s^2(253s^4-342s^2+96)$	$_{4,5}: \frac{3}{4}s^2(119657s^6-248128s^4+155592s^2-27456)$
$_{3,4}: \frac{5}{2}(503s^6-858s^4+408s^2-48)$	$_{4,6}: -\frac{105}{4}(2257s^8-5088s^6+3720s^4-960s^2+64)$

Remark that the critical inclination divisor $5s^2 - 4$ remains absent from these expressions, which besides are notably simpler than corresponding ones in [28], where the apparently simpler yet non-integrable main problem was normalized analogously. On the contrary, it can be checked that this offending divisor will always appear in the normalized Hamiltonian (5), starting at order k, if a truncation $m = k-1$ of the original Hamiltonian (4) is approached by perturbations up to the order k or higher. Appearance of these divisors disclose a resonance of the dynamical model represented by the early truncated Hamiltonian, and prevent application of the normalized solutions in resonant regions.

In a reduced phase space with the short-period terms removed, the resonant areas are bounded by homoclinic/heteroclinic trajectories, which link unstable equilibria, and correspond to regions of librational motion of the perigee [21, 26, 49]. The existence of isolated unstable equilibria in the reduced problem is related with the existence of unstable periodic orbits in the original model [50], which in turn imply the existence of chaos [51, 52]. However, in the current case this would be only an illusion due to the early truncation of the original integrable problem [24].

3.2. Transformation equations

The first order corrections are

$$\delta_1 r = \chi p s^2 \left(\frac{p}{r} \cos 2\theta + \frac{pR}{\Theta} \sin 2\theta \right) \tag{8}$$

$$\delta_1 \theta = \chi \left[\eta^2 \left(\frac{3}{2} s^2 - 1 \right) \sin 2\theta - 2\frac{p}{r}(s^2-1) \left(\frac{pR}{\Theta} \cos 2\theta - \frac{p}{r} \sin 2\theta \right) \right] \tag{9}$$

$$\delta_1 \nu = -\chi \frac{p}{r} c \left[\left(\frac{rpR^2}{\Theta^2} - \frac{p}{r} - 2 \right) \sin 2\theta + 2\frac{pR}{\Theta} \cos 2\theta \right] \tag{10}$$

$$\Delta_1 R = -\chi \frac{\Theta}{r} \frac{p}{r} s^2 \left[\left(\frac{p}{r} + 1 \right) \sin 2\theta - \frac{pR}{\Theta} \cos 2\theta \right] \tag{11}$$

$$\Delta_1 \Theta = \chi \Theta \frac{p}{r} s^2 \left[\left(\frac{p}{r} - \frac{rpR^2}{\Theta^2} + 2 \right) \cos 2\theta + 2\frac{pR}{\Theta} \sin 2\theta \right] \tag{12}$$

where primes in the right hand have been dropped for brevity, $c = N/\Theta$ in the polar variables, and $\eta = \left[(pR/\Theta)^2 + (-1+p/r)^2 \right]^{1/2}$.

To extend the direct transformation (6) to the second order of J_2, we need to refine the corrections provided by Eqs. (8)–(12) with additional terms. We obtain,

$$\begin{aligned}
\delta_2 r = {} & \chi^2 p s^2 \frac{1}{2} \left\{ \frac{p}{r}(5s^2-4)\eta^2 - \frac{p^2}{r^2}(21s^2-16) \right. \\
& -8\frac{p}{r} c^2 \left(2\frac{p^2}{r^2} - 4\frac{p}{r} + 3\eta^2 \right) \cos 2\theta + \frac{p}{r} \left(2\frac{p^2}{r^2} + \frac{p}{r} + 2\eta^2 \right) s^2 \cos 4\theta \\
& \left. -16 \left(\eta^2 + \frac{p^2}{r^2} \right) c^2 \sigma \sin 2\theta + \left(\eta^2 + 2\frac{p^2}{r^2} \right) s^2 \sigma \sin 4\theta \right\}
\end{aligned}$$

$$
\begin{aligned}
\delta_2\theta &= \chi^2\left\{\left[\frac{\eta^2}{2}\frac{p^2}{r^2}\left(16c^2+s^4\right)-2\frac{p^3}{r^3}c^2(5s^2+4)+\left(\frac{\eta^4}{8}+\frac{p^4}{r^4}\right)(8c^2+s^4)\right]\right.\\
&\quad\times\sin 4\theta-2\left[c^2(7s^2-2)\eta^2\left(\eta^2+2\frac{p^2}{r^2}\right)+\frac{p^3}{r^3}(35s^4-46s^2+12)\right.\\
&\quad\left.+4\frac{p^4}{r^4}s^2c^2\right]\sin 2\theta+8\frac{p}{r}c^2\left[\eta^2(3s^2-1)+\frac{p^2}{r^2}s^2\right]\sigma\cos 2\theta\\
&\quad\left.-\frac{p}{r}\left[4\eta^2c^2+\frac{p^2}{r^2}(s^4+8c^2)\right]\sigma\cos 4\theta-\frac{p}{r}\eta^2s^2(3s^2-2)\sigma\right\}\\
\delta_2\nu &= \chi^2c\left\{-8\frac{p}{r}\eta^2(2s^2-1)\sigma\cos 2\theta-2\frac{p}{r}(s^2-2)\left(\eta^2+2\frac{p^2}{r^2}\right)\sigma\cos 4\theta\right.\\
&\quad+\left[\eta^2(8s^2-4)\left(\eta^2+2\frac{p^2}{r^2}\right)-8\frac{p^3}{r^3}(7s^2-3)\right]\sin 2\theta\\
&\quad\left.+\left[(4s^2-8)\left(\frac{1}{8}\eta^4+\eta^2\frac{p^2}{r^2}+\frac{p^4}{r^4}\right)+\frac{p^3}{r^3}(6s^2+8)\right]\sin 4\theta\right\}\\
\Delta_2R &= \chi^2\frac{\Theta}{r}\frac{p}{r}s^2\left\{\left[\frac{p}{r}(6s^2-4)-\frac{\eta^2}{2}(5s^2-4)\right]\sigma+8\left(\frac{p^2}{r^2}-\frac{p}{r}-\frac{\eta^2}{2}\right)\right.\\
&\quad\times c^2\sigma\cos 2\theta+\left[8c^2\eta^2-8c^2\frac{p^3}{r^3}+\frac{p^2}{r^2}(34s^2-28)\right]\sin 2\theta\\
&\quad\left.+s^2\left(\frac{p^2}{r^2}-\frac{p}{r}+\eta^2\right)\sigma\cos 4\theta-\left(\frac{p^3}{r^3}+7\frac{p^2}{r^2}+3\frac{p}{r}\frac{\eta^2}{2}+\frac{\eta^2}{2}\right)s^2\sin 4\theta\right\}\\
\Delta_2\Theta &= \chi\Theta s^2\left\{-2\frac{p^3}{r^3}(11s^2-8)-\eta^4(3s^2-2)+4\frac{p}{r}\left(\frac{p^3}{r^3}+\frac{\eta^2}{2}\right)s^2\sigma\sin 4\theta\right.\\
&\quad+\left(4\frac{p^4}{r^4}+6\frac{p^3}{r^3}+4\frac{p^2}{r^2}\eta^2+\frac{\eta^4}{2}\right)s^2\cos 4\theta-8\frac{p}{r}\eta^2c^2\sigma\sin 2\theta\\
&\quad\left.-4\left[\eta^4c^2+2\frac{p^2}{r^2}\eta^2c^2+\frac{p^3}{r^3}(7s^2-6)\right]\cos 2\theta\right\}
\end{aligned}
$$

where we abbreviated $\sigma = pR/\Theta$, and, like previously, omitted the prime notation for brevity.

The inverse (original to prime variables) transformation is given by

$$
\vec{x}' = \vec{x} + \sum_{m=1}^{k}\frac{J_2^m}{m!}\delta'_m(\vec{x},\vec{X}), \qquad \vec{X}' = \vec{X} + \sum_{m=1}^{k}\frac{J_2^m}{m!}\Delta'_m(\vec{x},\vec{X}). \tag{13}
$$

Because the first order truncation imposes a linearization, the first order corrections (8)–(12) also apply to the inverse transformation, yet in this case they are evaluated in the original variables and subtracted to them. Direct and inverse periodic corrections are no longer the same at the second and higher orders.

We obtain

$$
\begin{aligned}
\delta_2' r &= \chi^2 p s^2 \left\{ 8c^2 \left(\eta^2 + \frac{p^2}{r^2} \right) \sigma \sin 2\theta + 8c^2 \frac{p}{r} \left(\frac{p^2}{r^2} - 2\frac{p}{r} + \frac{3}{2}\eta^2 \right) \cos 2\theta \right. \\
&\quad - s^2 \left(\frac{\eta^2}{2} + 3\frac{p^2}{r^2} \right) \sigma \sin 4\theta - \frac{p}{r} s^2 \left(3\frac{p^2}{r^2} - \frac{9}{2}\frac{p}{r} + 2\eta^2 \right) \cos 4\theta \\
&\quad \left. + \frac{1}{2}\frac{p}{r} \left[\eta^2 (5s^2 - 4) - \frac{p}{r}(21s^2 - 16) \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
\delta_2' \theta &= \chi^2 \left\{ \frac{p}{r} \left[\frac{p^2}{r^2}(9s^4 - 8) + 2\eta^2 (3s^4 - 2) \right] \sigma \cos 4\theta - \frac{p}{r} \eta^2 (3s^2 - 2) s^2 \sigma \right. \\
&\quad - 8c^2 \frac{p}{r} \left[\frac{p^2}{r^2} s^2 + \eta^2 (3s^2 - 1) \right] \sigma \cos 2\theta + 2 \left[\left(2\frac{p^2}{r^2}\eta^2 + \eta^4 \right) c^2 (7s^2 - 2) \right. \\
&\quad \left. + 4\frac{p^4}{r^4} s^2 c^2 + \frac{p^3}{r^3}(35s^4 - 46s^2 + 12) \right] \sin 2\theta - \left[\frac{p^4}{r^4}(9s^4 - 8) \right. \\
&\quad \left. \left. + \frac{p^3}{r^3}(5c^4 + 3) + \frac{p^2}{r^2}\frac{\eta^2}{2}(21s^4 - 16) + \frac{\eta^4}{8}(13s^4 - 8) \right] \sin 4\theta \right\}
\end{aligned}
$$

$$
\begin{aligned}
\delta_2' \nu &= \chi^2 c \left\{ 2\frac{p}{r} \left(\eta^2 + \frac{2p^2}{r^2} \right) (s^2 + 2) \sigma \cos 4\theta + 8\eta^2 \frac{p}{r}(2s^2 - 1) \sigma \cos 2\theta \right. \\
&\quad + 8 \left[\frac{p^3}{r^3}(7s^2 - 3) - \left(\frac{p^2}{r^2} + \frac{\eta^2}{2} \right) \eta^2 (2s^2 - 1) \right] \sin 2\theta \\
&\quad \left. - \left[\left(\frac{p^4}{r^4} + \frac{p^2}{r^2}\eta^2 + \frac{\eta^4}{8} \right) (4s^2 + 8) + \frac{p^3}{r^3}(6s^2 - 8) \right] \sin 4\theta \right\}
\end{aligned}
$$

$$
\begin{aligned}
\delta_2' R &= \chi^2 \frac{\Theta}{r} \frac{p}{r} s^2 \left\{ 2 \left[4c^2 \frac{p^3}{r^3} - \frac{p^2}{r^2}(17s^2 - 14) - 4c^2 \eta^2 \right] \sin 2\theta \right. \\
&\quad - 8c^2 \left(\frac{p^2}{r^2} - \frac{p}{r} - \frac{\eta^2}{2} \right) \sigma \cos 2\theta + \left[\frac{p}{r}(6s^2 - 4) - \frac{\eta^2}{2}(5s^2 - 4) \right] \sigma \\
&\quad \left. + \left(3\frac{p^3}{r^3} + 6\frac{p^2}{r^2} + 3\frac{p}{r}\frac{\eta^2}{2} + \frac{\eta^2}{2} \right) s^2 \sin 4\theta - 3\frac{p}{r} \left(\frac{p}{r} + 1 \right) s^2 \sigma \cos 4\theta \right\}
\end{aligned}
$$

$$
\begin{aligned}
\delta_2' \Theta &= \chi^2 \Theta s^2 \left\{ 4 \left[\frac{p^3}{r^3}(7s^2 - 6) + 2\left(\frac{p^2}{r^2} + \frac{\eta^2}{2} \right) \eta^2 c^2 \right] \cos 2\theta \right. \\
&\quad + 8\frac{p}{r} \eta^2 c^2 \sigma \sin 2\theta - \left(4\frac{p^4}{r^4} + 6\frac{p^3}{r^3} + 4\frac{p^2}{r^2}\eta^2 + \frac{\eta^4}{2} \right) s^2 \cos 4\theta \\
&\quad \left. - 4\frac{p}{r} \left(\frac{p^2}{r^2} + \frac{\eta^2}{2} \right) s^2 \sigma \sin 4\theta - 2\frac{p^3}{r^3}(11s^2 - 8) - \eta^4 (3s^2 - 2) \right\},
\end{aligned}
$$

whose right sides must be evaluated in the original variables.

4. NORMALIZATION OF THE SEMI-MAJOR AXIS

Up to the truncation order, the reduced Hamiltonian (5) with terms given by Eq. (7) is of one degree of freedom in (ℓ, L), but it is not separable. Therefore, we remove the mean anomaly by means of a new transformation $\mathscr{T}_2 : (\vec{x}'', \vec{X}''; J_2) \mapsto (\vec{x}', \vec{X}')$, to obtain $\mathscr{K} \circ \mathscr{T}_2 = \mathscr{Q}$. That is,

$$\mathscr{Q} = \mathscr{K}(\vec{x}'(\vec{x}'', \vec{X}''; J_2), \vec{X}'(\vec{x}'', \vec{X}''; J_2); J_2) \equiv \sum_{m=0}^{k} \frac{J_2^m}{m!} \mathscr{Q}_m(L'', G', H) + \mathscr{O}(J_2^{k+1}), \quad (14)$$

which, after truncation to order k, is completely reduced to a function of only the momenta.

We found that new Hamiltonian terms can be arranged in the general form

$$\mathscr{Q}_m = \frac{\mu}{2a} \chi^m \sum_{n=0}^{2(m-1)} I^{\star}_{m,n}(s) \eta^{n+1}, \quad (15)$$

where the symbols appearing in Eq. (15) are functions of the double-prime Delaunay variables, and the new inclination polynomials $I^{\star}_{m,n}$ are displayed in Tables 2 and 3. Now, Hamiltonian (14) is separable in the double prime variables, and the integration of Hamilton equations is trivial. Indeed, the actions remain constant $L'' = L''_0$, $G'' = G'_0$, and $H'' = H_0$, whereas the angles evolve linerly

$$\ell'' = \ell''_0 + \frac{\partial \mathscr{Q}}{\partial L''} t, \qquad g'' = g''_0 + \frac{\partial \mathscr{Q}}{\partial G''} t, \qquad h'' = h''_0 + \frac{\partial \mathscr{Q}}{\partial H''} t.$$

Table 2
Inclination polynomials $I^{\star}_{m,k}$, $m = 2, 3$, in Eq. (15)

$_{2,0}: -15(21s^4 - 28s^2 + 8)$	$_{3,1}: 135(3s^2 - 2)(21s^4 - 28s^2 + 8)$
$_{2,1}: -6(3s^2 - 2)^2$	$_{3,2}: -6(5667s^6 - 11124s^4 + 6576s^2 - 1088)$
$_{2,2}: 6(25s^4 - 32s^2 + 8)$	$_{3,3}: -90(3s^2 - 2)(25s^4 - 32s^2 + 8)$
$_{3,0}: 315(143s^6 - 286s^4 + 176s^2 - 32)$	$_{3,4}: 18(285s^6 - 535s^4 + 292s^2 - 40)$

Table 3
Inclination polynomials $I^{\star}_{4,k}$ in Eq. (15)

$_0: -\frac{45045}{4}(969s^8 - 2584s^6 + 2448s^4 - 960s^2 + 128)$
$_1: -\frac{135}{2}(30639s^8 - 81704s^6 + 78400s^4 - 31808s^2 + 4544)$
$_2: 135(83573s^8 - 220192s^6 + 204272s^4 - 77184s^2 + 9600)$
$_3: 30(92601s^8 - 243156s^6 + 227376s^4 - 88416s^2 + 11776)$
$_4: -\frac{45}{2}(134037s^8 - 345408s^6 + 308864s^4 - 109312s^2 + 11904)$
$_5: -126(5295s^8 - 13500s^6 + 12056s^4 - 4352s^2 + 512)$
$_6: 3(75675s^8 - 186976s^6 + 157392s^4 - 50496s^2 + 4480)$

The analytical solution is completed with the computation of the periodic corrections of the second transformation $\mathscr{T}_2$. Since there is not risk of confusion, we use the same notation as we used in the first transformation. Thus, the first order corrections are

$$\delta_1 r = \chi r\left[2\eta+\beta\frac{p}{r}\left(\eta+\frac{p}{r}\right)\right](3s^2-2) \quad (16)$$

$$\delta_1\theta = -\chi\left\{\left[\frac{p}{r}(3s^2-2)+2\eta(6s^2-5)\right]\beta\sigma+3(5s^2-4)(\beta\sigma+\phi)\right\} \quad (17)$$

$$\delta_1\nu = -6\chi c(\sigma+\phi) \quad (18)$$

$$\Delta_1 R = -\chi\frac{\Theta}{p}(3s^2-2)\left(\eta+\frac{p^2}{r^2}\beta\right) \quad (19)$$

where $\beta = 1/(1+\eta)$ and $\phi = f-\ell$ is the equation of the center. Analogously to the first normalization, the first order corrections (16)–(19) are evaluated in double-prime variables for direct corrections, and with opposite sign and prime variables for inverse corrections.

At the second order, the direct corrections take the form

$$\delta_2 r = \chi^2 r\left(r_{0,6}\frac{p}{r}\sigma\phi+\beta^2\sum_{i=0}^{5}\frac{p^i}{r^i}\sum_{j=0}^{5}r_{i,j}\eta^j\right) \quad (20)$$

$$\delta_2\theta = \chi^2\left[\left(\theta_{0,0}+\theta_{5,0}\eta^2+2\theta_{1,3}\frac{p^2}{r^2}\right)\phi+\beta^2\sigma\sum_{i=0}^{3}\frac{p^i}{r^i}\sum_{j=0}^{4}\theta_{i,j}\eta^j\right] \quad (21)$$

$$\delta_2\nu = \chi^2 c\left[\left(\nu_{0,0}+\nu_{3,0}\eta^2\right)\phi+\beta\sigma\sum_{i=0}^{2}\frac{p^i}{r^i}\sum_{j=0}^{3}\nu_{i,j}\eta^j\right] \quad (22)$$

$$\Delta_2 R = \chi^2\frac{\Theta}{p}\left[2R_{0,5}\frac{p^2}{r^2}\left(\frac{p}{r}-1\right)\phi+\beta^2\sigma\sum_{i=0}^{4}\frac{p^i}{r^i}\sum_{j=0}^{5}R_{i,j}\eta^j\right] \quad (23)$$

where the right sides of these corrections must be evaluated in double-prime variables, and the inclination polynomials $r_{i,j}$, $\theta_{i,j}$, $\nu_{i,j}$, and $R_{i,j}$ are given in Table 4. For the inverse corrections, we obtain

$$\delta_2' r = \chi^2 r\left(-r_{0,6}'\frac{p}{r}\sigma\phi+\frac{\beta^2}{\eta}\sum_{i=0}^{5}\frac{p^i}{r^i}\sum_{j=0}^{5}r_{i,j}'\eta^j\right) \quad (24)$$

$$\delta_2'\theta = \chi^2\left[\left(\theta_{0,1}'+\theta_{5,0}'\eta^2+2\theta_{1,4}'\frac{p^2}{r^2}\right)\phi+\frac{\beta^2}{\eta}\sigma\sum_{i=0}^{4}\frac{p^i}{r^i}\sum_{j=0}^{5}\theta_{i,j}'\eta^j\right] \quad (25)$$

$$\Delta_2' R = \chi^2\frac{\Theta}{p}\left[-R_{2,1}'\frac{p^2}{r^2}\left(\frac{p}{r}-1\right)\phi+\frac{\beta^2}{\eta}\sigma\sum_{i=0}^{5}\frac{p^i}{r^i}\sum_{j=0}^{6}R_{i,j}'\eta^j\right] \quad (26)$$

where the second-order correction to ν is just the opposite of Eq. (22) and is not printed. The right sides of these corrections are now evaluated in prime variables, and the needed inclination polynomials $r_{i,j}$, $\theta_{i,j}$, and $R_{i,j}$, are listed in Table 5.

It is worth recalling that the most relevant corrections in a perturbation theory are the inverse corrections, since they allow for the accurate initialization of the constants

Table 4

Inclination polynomials $r_{i,j}$, $\theta_{i,j}$, $\nu_{i,j}$ and $R_{i,j}$ in Eqs. (20)–(23)

r:	$_{0,1}: -15(21s^4-28s^2+8)$	$_{1,2}: -4(39s^4-54s^2+17)$	$_{2,3}: \frac{1}{2}r_{0,5}$
	$_{0,2}: -4(153s^4-204s^2+58)$	$_{1,3}: 6(19s^4-22s^2+4)$	$_{3,0}: 4(15s^4-18s^2+4)$
	$_{0,3}: -129s^4+180s^2-56$	$_{1,4}: \frac{3}{2}(63s^4-76s^2+16)$	$_{3,1}: 6(17s^4-20s^2+4)$
	$_{0,4}: 2(159s^4-204s^2+52)$	$_{2,0}: -2r_{0,5}$	$_{3,2}: \frac{1}{2}r_{3,1}$
	$_{0,5}: 6(25s^4-32s^2+8)$	$_{2,1}: -\frac{1}{2}(957s^4-1212s^2+296)$	$_{4,0}: -(3s^2-2)^2$
	$_{1,1}: \frac{1}{2}r_{0,1}$	$_{2,2}: \frac{1}{2}(-171s^4+204s^2-40)$	$_{0,6}: -r_{0,5}$
θ:	$_{0,0}: \frac{105}{2}(33s^4-48s^2+16)$	$_{1,0}: \frac{1}{3}\theta_{0,0}$	$_{2,0}: (9s^2-8)^2$
	$_{0,1}: \frac{15}{2}(441s^4-644s^2+216)$	$_{1,1}: \frac{1}{2}(1923s^4-2820s^2+952)$	$_{2,1}: 2(45s^4-90s^2+44)$
	$_{0,2}: 2(651s^4-972s^2+340)$	$_{1,2}: 291s^4-450s^2+164$	$_{2,2}: 3(3s^4-12s^2+8)$
	$_{0,3}: -507s^4+696s^2-200$	$_{1,3}: -\frac{1}{2}r_{0,5}$	$_{3,0}: -2r_{4,0}$
	$_{0,4}: -6(42s^4-59s^2+18)$	$_{5,0}: -3(225s^4-324s^2+104)$	
ν:	$_{0,0}: 210(3s^2-2)$	$_{0,3} = -2(63s^2-38)$	$_{2,0} = \frac{2}{15}\nu_{0,0}$
	$_{0,1}: \nu_{0,0}$	$_{1,0} = \frac{1}{3}\nu_{0,0}$	$_{2,1} = \frac{8}{105}\nu_{0,0}$
	$_{0,2} = -2(45s^2-26)$	$_{1,1} = \frac{29}{105}\nu_{0,0}$	$_{3,0} = -12(25s^2-16)$
R:	$_{0,1}: \frac{15}{2}(21s^4-28s^2+8)$	$_{0,5}: -\frac{1}{2}r_{0,5}$	$_{3,0}: -105s^4+132s^2-32$
	$_{0,2}: 333s^4-444s^2+128$	$_{2,0}: -2R_{0,5}$	$_{3,1}: -6(29s^4-36s^2+8)$
	$_{0,3}: \frac{1}{2}(237s^4-324s^2+104)$	$_{2,1}: \frac{1}{2}(465s^4-588s^2+152)$	$_{3,2}: \frac{1}{2}R_{3,1}$
	$_{0,4}: -4(33s^4-42s^2+10)$	$_{2,2}: \frac{1}{2}(-21s^4+36s^2-8)$	$_{4,0}: -3r_{4,0}$

Table 5

Inclination polynomials $r'_{i,j}$, $\theta'_{i,j}$, and $R'_{i,j}$ in Eqs. (24)–(26)

r':	$_{0,2}: -r_{0,1}$	$_{1,4}: -2\left(39s^4-42s^2+4\right)$	$_{3,1}: 4\left(3s^4-6s^2+4\right)$
	$_{0,3}: 8\left(81s^4-108s^2+31\right)$	$_{1,5}: -r_{1,4}$	$_{3,2}: -4\left(21s^4-24s^2+4\right)$
	$_{0,4}: 201s^4-276s^2+88$	$_{2,1}: -2r'_{0,6}$	$_{3,3}: -\frac{1}{2}r_{3,1}$
	$_{0,5}: -2\left(141s^4-180s^2+44\right)$	$_{2,2}: -r_{2,1}$	$_{4,0}: \frac{1}{2}r'_{3,0}$
	$_{0,6}: -r_{0,5}$	$_{2,3}: \frac{1}{2}\left(207s^4-252s^2+56\right)$	$_{4,1}: -\frac{1}{4}r'_{3,0}$
	$_{1,2}: \frac{1}{2}r'_{0,2}$	$_{2,4}: \frac{1}{2}r'_{0,6}$	$_{5,0}: -\frac{1}{2}r'_{3,0}$
	$_{1,3}: 2\left(87s^4-120s^2+38\right)$	$_{3,0}: -4r_{4,0}$	
θ':	$_{0,1}: -\theta_{0,0}$	$_{1,1}: \frac{1}{3}\theta'_{0,1}$	$_{2,2}: -\theta_{2,1}$
	$_{0,2}: -\theta_{0,1}$	$_{1,2}: \frac{1}{2}\left(-1815s^4+2676s^2-904\right)$	$_{2,3}: -\theta_{2,2}$
	$_{0,3}: -6\left(223s^4-332s^2+116\right)$	$_{1,3}: -255s^4+402s^2-148$	$_{3,0}: -2r_{4,0}$
	$_{0,4}: 3\left(163s^4-224s^2+64\right)$	$_{1,4}: \frac{1}{2}r_{0,5}$	$_{3,1}: \theta'_{3,0}$
	$_{0,5}: -\theta_{0,4}$	$_{2,1}: -\theta_{2,0}$	$_{4,0}: \theta'_{3,0}$
	$_{5,0}: -\theta_{5,0}$		
R':	$_{0,2}: -\frac{1}{2}r'_{0,2}$	$_{2,1}: -r_{0,5}$	$_{3,1}: 69s^4-84s^2+16$
	$_{0,3}: -297s^4+396s^2-112$	$_{2,2}: \frac{1}{2}\left(-249s^4+300s^2-56\right)$	$_{3,2}: 8\left(24s^4-30s^2+7\right)$
	$_{0,4}: \frac{1}{2}\left(-93s^4+132s^2-40\right)$	$_{2,3}: \frac{1}{2}r'_{0,4}$	$_{3,3}: -\frac{1}{2}R_{3,1}$
	$_{0,5}: 8\left(21s^4-27s^2+7\right)$	$_{2,4}: R'_{0,6}$	$_{4,1}: -\frac{3}{2}R'_{3,0}$
	$_{0,6}: \frac{1}{2}r_{0,5}$	$_{3,0}: 2r_{4,0}$	$_{5,0}: -R'_{3,0}$

of the theory. On the other hand, since the most sensitive element to initialize is the mean motion, the inverse corrections of higher-order are sometimes simplified to include only the corrections of the elements related to the semi-major axis [53, 54].

5. NUMERICAL TESTS

Vinti's dynamical model only mimics the real dynamics of an artificial satellite of the Earth during short time intervals. Beyond that, comparisons with a full zonal model disclose important growths of the error. However, a significant part of the error is due to the long-period effects caused by the odd zonal harmonics, which cannot be approximated with Vinti's dynamical model. While Vinti later improved his model to take the full effect of J_3 into account, this refinement has not been tackled here for simplicity, yet it could be approached analogously. Therefore, to make our comparisons meaningful, we take a zonal model that only includes the contribution of the even zonal harmonics of the Earth like the true reference model.

The different tests carried out have shown that, in general, Vinti's solution starts with very small errors, but they notably grow in a one-day interval, reaching position errors in the range from about 100 m for LEO to the km level in the case of GTO. A similar behavior is, of course, obtained with the perturbation solution, which can be as close as desired to Vinti's exact solution for much longer intervals. However, to be competitive with the exact solution, the perturbation solution needs to achieve analogous errors with respect to the true orbit. Of course, this accuracy of the perturbation solution should be reached with less computational burden than the exact solution, which is not at all the case of the higher orders of the perturbation theory. We illustrate the performance of the lower orders of the perturbation approach with two different orbits, yet the test have been extended to a variety of orbits.

5.1. The Prisma Orbit

The first case corresponds to the orbital parameters of the Prisma mission [55]. Namely, $a = 6878.137$ km, $e = 0.001$, $I = 97.42°$, $\Omega = 168.162°$, $\omega = 20°$, and $M = 30°$, for a given initial epoch. The propagation of the root sum square (RSS) error in Cartesian coordinates of Vinti's solution $(x_\mathrm{v}, y_\mathrm{v}, y_\mathrm{v})$ with respect to the true orbit (x,y,z), $\varepsilon = \sqrt{(x-x_\mathrm{v})^2+(y-y_\mathrm{v})^2+(z-z_\mathrm{v})^2}$ is shown in Fig. 1 for an interval of one day. The RSS error grows approximately at a linear rate of 3.5 m times hour, with periodic oscillations of ~ 2 m amplitude over the linear trend, and reaches about 85 m at the end of the interval.

The lower-order perturbation solution is computed in the style of Brouwer [12]. That is, computing the initializations constants of the theory only with the first order inverse corrections, evaluating the secular frequencies up to second order effects, and recovering the osculating orbit through first order periodic corrections. We denote this approximation as 1-2-1 for identification purposes. If we evaluate this solution in the

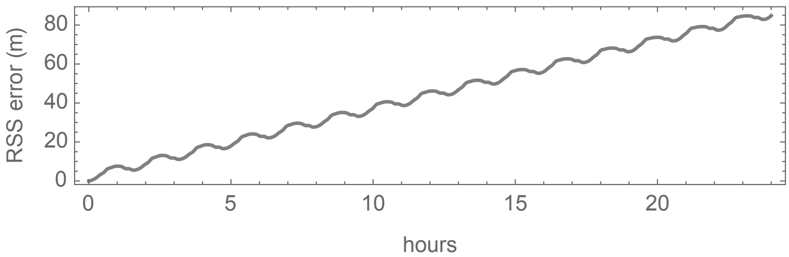

Fig. 1 – Prisma orbit: RSS errors of the Vinti dynamical model with respect to the true orbit.

same points as we previously did with Vinti's exact solution, and compute the RSS errors with respect to the true orbit, we obtain the errors displayed in Fig. 2, which are superimposed to the previous case for illustration purposes. As shown in the figure, the errors of the 1-2-1 solution grow at a much higher rate, of ~ 9.9 m times hour, than Vinti's solution, only remaining comparable at the beginning of the propagation interval. However, the things notably improve when the constants of the perturbation solution are initialized taking second order corrections into account, a case that we term the 2-2-1 case. The RSS errors of the 2-2-1 case are shown in Fig. 3, where, to better appreciate the detail, the linear trend of the errors of Vinti's solution are subtracted to the RSS errors of both solutions. Since the initialization process is made once for all, this improvement of the perturbation solution only increases slightly its computational burden.

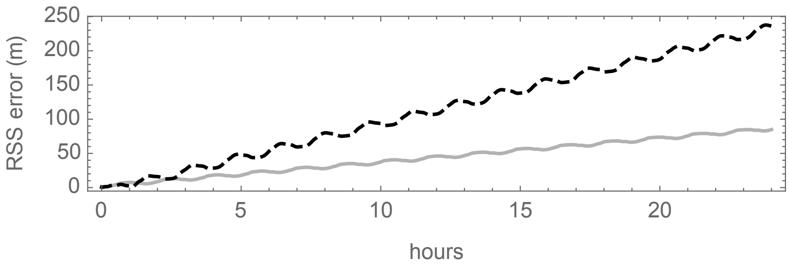

Fig. 2 – Prisma orbit: RSS errors of the 1-2-1 perturbation solution with respect to the true orbit (black-dashed line) superimposed to the previous case (gray line).

Therefore, the accuracy of the 2-2-1 solution is generally equivalent to that of Vinti's exact model for a real propagation, except for orbit propagation intervals in the range of just a few minutes. This was expected from the nature of perturbation solutions, which always yield an initial error due to the truncation of the solution, as is clearly noticed in Fig. 4, where the truncation of the direct corrections to first order effects yield an initial error of the order of about 3 meters. The issue, however, is easily amended by recovering the direct corrections up to second order effects (2-2-2 solution), as illustrated in Fig. 5. Still, the increase in precision of the perturbation solution is now at the cost of a non-negligible increase of the computational burden

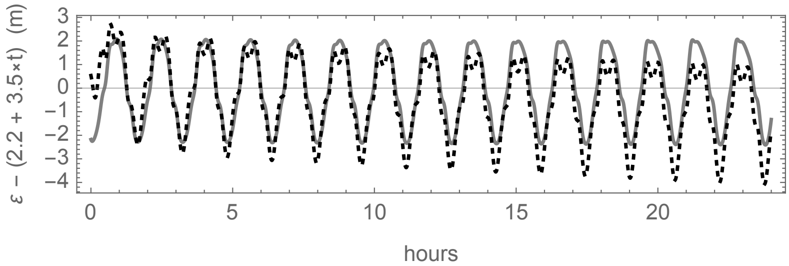

Fig. 3 – Prisma orbit: RSS errors ε of the 2-2-1 perturbation solution (black-dashed line) superimposed to those of the Vinti's case (gray line).

due to the length of the series comprising the direct corrections.

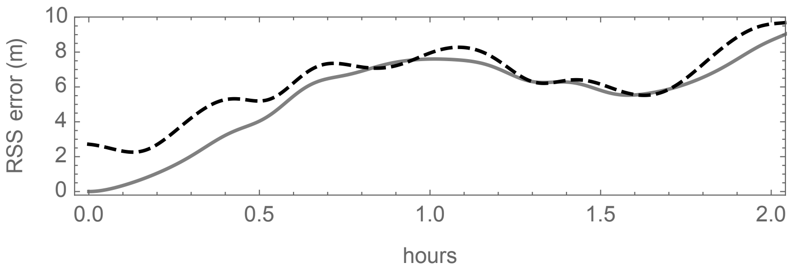

Fig. 4 – Prisma orbit: errors of the 2-2-1 perturbation solution with respect to the true orbit (black-dashed line) superimposed to those of the Vinti's case (gray line).

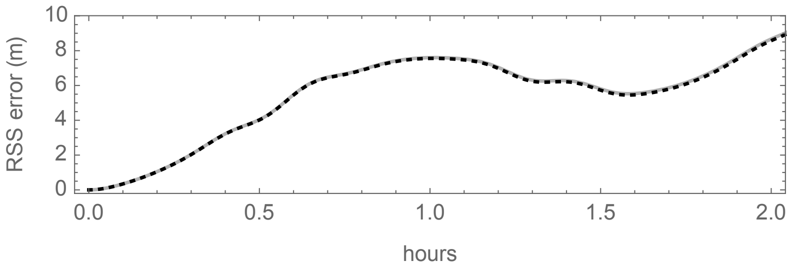

Fig. 5 – Prisma orbit: RSS errors of the 2-2-2 perturbation solution (black-dashed line) closely superimposed to those of the Vinti's case (gray line).

5.2. The Molnya Orbit

The second test deals with the typical Molnya orbit, which is highly eccentric and critically inclined. While classical perturbation theories fail in propagating orbits close to the critical inclination, and even yield overflow for exactly critically inclined orbits, neither Vinti's integrable solution nor its perturbation approximations suffer these kinds of problems, yet, of course, they cannot deal properly with the perigee libration behavior of orbits close to the critical inclination. We propagate the initial conditions corresponding to the orbital parameters $a = 26554$ km, $e = 0.72$, $i = 63.4°$, $\Omega = 0.1$, $\omega = 280°$, and $M = 0$ [56]. The RSS position errors of Vinti's solution along one day are displayed jointly with the 1-2-1 approximation in Fig. 6, where it is shown that the errors clearly peak at perigee passages.

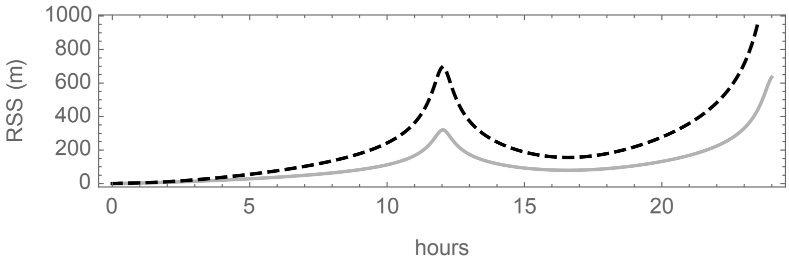

Fig. 6 – Molnya critically inclined orbit: RSS errors of the 1-2-1 perturbation solution with respect to the true orbit (black-dashed line) superimposed to the errors of Vinti's corresponding solution (gray line).

Like in the case of the Prisma orbit, the errors of the lower order perturbation solution clearly exceed those yield by Vinti's orbit. However, the use of the 2-2-1 approximation notably improves the performance of perturbation solution. As shown in Fig. 7, the difference between the errors of both Vinti's solution and the 2-2-1 case with respect to the true orbit remain of the same order. We checked that this difference is further reduced to the cm level when the direct corrections are included in the perturbation model up to second order effects (case 2-2-2).

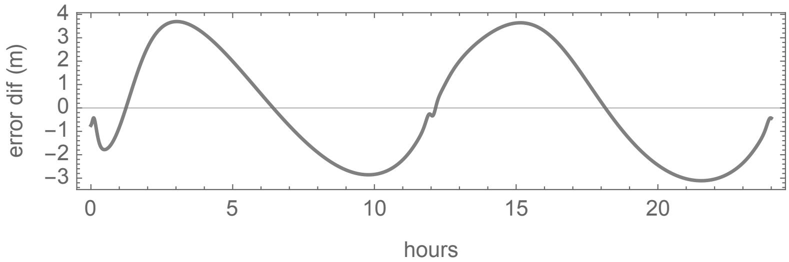

Fig. 7 – Molnya orbit: difference between the RSS errors of the 2-2-1 perturbation solution and those of Vinti's dynamical model with respect to the true orbit.

6. CONCLUSIONS

In those cases in which Vinti's oblate spheroidal model is suitable to replace the true Geopotential, the intricacies of Vinti's exact solution can be replaced by a straightforward approximation based on perturbation methods. Even for the lower order truncations of the perturbation approach, the latter results in errors with respect to the true Earth orbit of the same order of magnitude as those obtained with Vinti's solution, while involving only arithmetic operations and the evaluation of basic trigonometric functions, contrary to the special functions in which the exact solution depends upon.

For simplicity, we constrained here to the case of equatorial symmetry of the dynamical model. But the ideas in this paper can be applied analogously to the more elaborated Vinti's oblate spheroidal model that also includes the whole effect of the third-order zonal harmonic in addition to the second. Results for this later case are in progress and will be published elsewhere.

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