SATELLITE ORBITS AND FRACTIONAL MECHANICS

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Abstract. The motion of a satellite or a spacecraft in the vicinity of the Earth is subject to numerous perturbations from a variety of sources some of which are transitory or "random". It is not feasible to incorporate all these perturbations analytically within the framework of Newton's second law. In this paper we put forward the proposition that fractional derivatives can provide a lumped sum parameter approach to incorporate all these interactions based on the "historical trajectory data" of the satellite. This may lead to longer (in time) and more accurate prediction of its trajectory. To demonstrate (theoretically) the feasibility of this conjecture we consider two test cases. The first is for the motion of a satellite under the influence of quadratic drag. The second is for the motion of a satellite in the gravitational field of oblate Earth. In both cases we show that fractional derivatives models can lead to results which are very close to the numerical solutions which incorporate the impact of these perturbations on the motion of the satellite.

Key words: oblate spheroid, quadratic drag, equatorial orbits, analytical solutions, Caputo derivative.

1. INTRODUCTION

The motion of the planets around the Sun and the stability of this motion was addressed by many mathematicians and physicists (Brouwer and Clemence [1], Poincare [13], Prussing and Conway [14], Szebehely [15], Koon et al. [9]) and is still the subject of many current research papers (Condurache and Martinusi [2, 3], Lidov [11], Humi and Carter [7], Humi [8]). At the core of this problem is the treatment of n-body interactions where n is large (Brouwer and Clemence [1], Szebehely [15]). An exception is the successful analytic treatment of two bodies and (to some extent) three-body interactions (Gurfil [5], Levi-Civita [10], Nie et al. [12]). For $n > 3$, perturbation theory is used usually to obtain approximate analytic solutions.

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From a fundamental point of view the motion of a spacecraft or satellite in the Earth vicinity is subject to corrections due to drag and Earth oblateness. Additional corrections are due to the gravitational field of the Sun, Moon and the planets. Others perturbations due to the solar wind, asteroids and other small near Earth objects are transitory and "almost random". All these nonlocal influences are "registered" in the memory of the satellite or spacecraft (viz. its historical trajectory). However it is not feasible to incorporate all these interactions analytically in the equations of motion using Newton's law of gravitation (and as far as we know nobody has done so). Usually the equations of motion for the orbit of a satellite incorporate the two most important corrections due to the Earth oblateness (J_2 effect) and drag (Humi and Carter [7], Humi [8]). Even with these simplifications the numerical solution of these problem has to address two issues. The first is due to the long time scale of the integration which may lead to large "accumulated error" (as a result of floating-point operations). The second is due to the different spatial scales which appear in these problems which lead to "numerically stiff" differential equations for the evolution of these systems.

Due to these circumstances one is motivated to look for a different approach that can produce better (and more accurate) predictions about the future satellite trajectory based on the data about its "historical trajectory".

Fractional derivatives were discussed by Leibniz and Abel. However their extensive use in applications started only in the late 20th century (Herrmann [6], El-Sayed et al. [4], Varieschi [16]). The reason can be attributed partially to the different definitions of this concept in the literature and several stumbling blocks where fractional calculus differs from ordinary calculus (e.g., the product and chain rules). Nevertheless in recent years Caputo definition of fractional derivative became "standard" and we shall adopt it throughout this paper (Herrmann [6]).

We note that fractional order derivative in time are integral operators that involve the "satellite trajectory" at (all) previous times. Based on this historical trajectory they can capture all interactions that influenced its trajectory in the past. By estimating the optimal order of the fractional derivative that yields the best representation of this trajectory in the past one can make, in principle, better and longer (in time) predictions about its future trajectory. Thus the order of the fractional derivative in this model can be viewed as a *lumped sum parameter* that represents the impact of all these interactions.

The objective of this paper is to demonstrate that fractional derivatives can actually capture the impact of perturbations on satellite trajectory. Due to lack of actual satellite data, we demonstrate this fact by considering two outstanding problems where satellite trajectory is impacted by perturbations due to drag and Earth oblateness. For these two problems we show that fractional derivatives are able to reproduce almost the same results as the well established numerical algorithms (but potentially they can do much more). *We emphasize that this comparison is not intended to show that fractional derivatives can "compete" with the well established numerical algorithms (using Newton's law) but to provide a "proof of concept" about the potential role of this new methodology in celestial mechanics.*

The plan of the paper is as follows. In Section 2, we give a brief review of the classical models for satellite motion. In Section 3, we first formulate the fractional models for the motion and then present approximate solutions of the fractional models, details of which are provided in the appendix. Section 4 provides verification of the analytic fractional model as compared with the numerical solution of the orbit equations. We end with some conclusions about the performance of fractional derivatives models within the context of celestial mechanics.

2. CLASSICAL THEORY FOR SPHERICAL AND OBLATE BODIES

In classical physics the motion of a particle of unit mass in a central force field is modeled by the following system of differential equations

$$\ddot{\mathbf{r}} = f(r)\mathbf{e}_r. \tag{2.1}$$

In this equation $\mathbf{r}$ is the radius vector of the particle from the center of attraction, $r = |\mathbf{r}|$, $\mathbf{e}_r$ is a unit vector along $\mathbf{r}$ and we use a dot above a symbol to denote differentiation with respect to time t (e.g. $\dot{r} = dr/dt$). We assume also that f is differentiable on the domain under consideration. It is easy to show that this motion is in a plane (which we take to be the $x-y$ plane). In polar coordinates (r, θ), the equations of motion are

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0, \tag{2.2}$$

$$(\ddot{r} - r\dot{\theta}^2) = f(r). \tag{2.3}$$

Equation (2.2) can be integrated to obtain that

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = h, \tag{2.4}$$

where h is a constant. Using this result to change the independent variable from t to θ in (2.3) we obtain the orbit equation

$$\frac{1}{r(\theta)} \frac{\mathrm{d}^2 r(\theta)}{\mathrm{d}\theta^2} - \frac{2}{r(\theta)^2} \left(\frac{\mathrm{d}r(\theta)}{\mathrm{d}\theta} \right)^2 = 1 + \frac{r(\theta)^3 f(r(\theta))}{h^2}. \tag{2.5}$$

Introducing the transformation $r = \frac{1}{u}$, this equation becomes

$$\frac{\mathrm{d}^2 u(\theta)}{\mathrm{d}\theta^2} + u(\theta) = -\frac{f(r)}{u(\theta)^2 h^2}. \tag{2.6}$$

When $f(r)$ represents the gravitational field due to a point particle at the origin viz. $f(r) = -\frac{\mu}{r^2}$ (where μ is a constant), the equation of motion (2.6) is reduced to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}. \tag{2.7}$$

The classical solution for the orbit of a satellite in this force field is

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos(\theta + \phi)}, \tag{2.8}$$

where e and ϕ are constants.

2.1. Equatorial Orbits Around Oblate Body

In the gravitational potential of an oblate body, the most important correction term for the gravitational potential is the one containing J_2, the coefficient of the primary zonal harmonic. With this term the gravitational potential of the Earth or other oblate spheroid is approximated by

$$U = -\frac{\mu}{r}\left[1 - \frac{R^2 J_2}{r^2} P_2(\cos\phi)\right]. \tag{2.9}$$

With this equation, we associate an inertial coordinate system attached to the center of the oblate body. In this system, $\mathbf{r}$ is the radius vector where $r = |\mathbf{r}|$, ϕ is the colatitude angle, R is the radius of the oblate body in the equatorial plane, μ represents the product of the universal gravitational constant and the mass of the spheroid and P_2 is the second-order Legendre polynomial.

The force per unit mass acting on a particle at a point $\mathbf{r}$ due to the gravitational potential (2.9) is given by

$$F(\mathbf{r}) = -\nabla U(\mathbf{r}).$$

If $\mathbf{r}$ and $\dot{\mathbf{r}}$ are initially in the equatorial plane then the angular momentum is constant, both remain in this plane and $\phi = \frac{\pi}{2}$. It follows then that, in this plane, the specific gravitational force $F(\mathbf{r}) = f(r)\mathbf{e}_r$ is central where

$$f(r) = -\mu\left(\frac{1}{r^2} + \frac{3R^2 J_2}{2r^4}\right). \tag{2.10}$$

Introducing polar coordinates (r,θ) in the equatorial plane Eq. (2.5) becomes

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} - \frac{\mu k}{r^4}, \tag{2.11}$$

where $k = \frac{3}{2}R^2 J_2$. Using (2.4) to change the independent variable from t to θ and the transformation $u = 1/r$ (2.11) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}(1 + ku^2). \tag{2.12}$$

Though this nonlinear equation has no closed analytic solution,we seek a first order approximation of the solution with the ansatz $u = u_0 + J_2 u_1$, as J_2 is small for the Earth. To a first-order in J_2 the resulting solution for u_1 is

$$u_1 = \frac{\mu^3 R^2}{4h^6}(6 - e^2(\cos(2(\theta+\phi)) - 3) + 3e(2\theta\sin(\theta+\phi) + cos(\theta+\phi))). \tag{2.13}$$

For $|e| \ll 1$ we can approximate u_1 by $\frac{3\mu^3 R^2}{2h^6}$ and the (approximate) equation for the orbit of the satellite becomes

$$r = \frac{h^2}{\mu}\left(\frac{1}{1 + e\cos(\theta+\phi) + \frac{k\mu^2}{h^4}}\right). \tag{2.14}$$

3. FORMULATION USING FRACTIONAL DERIVATIVES

While the classical equations of motion in a central force field can be integrated analytically for some force fields $f(r)$, there are many other cases where the motion of the particle is subject to perturbations, which might be due to additional central force. In more general cases, these perturbations might be due to non-central forces and can be considered as "non-local". A classical example of this situation is the motion of the Earth around the Sun with perturbations due to the Moon and the other celestial objects in the solar system. In these cases, one defer (traditionally) to analytic perturbation theory or numerical methods to obtain approximate solutions to these equations.

In this work we suggest and test the feasibility of another paradigm to incorporate these perturbations by replacing (2.2)-(2.3) by

$$r\frac{\mathrm{d}^{2\alpha}\theta}{\mathrm{d}t^{2\alpha}} + \frac{\Gamma(2+\alpha)}{\Gamma(2-\alpha)}\left(\frac{\mathrm{d}^{\alpha}r}{\mathrm{d}t^{\alpha}}\right)\left(\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}}\right) = 0, \tag{3.1}$$

$$\left[\frac{\mathrm{d}^{2\alpha}r}{\mathrm{d}t^{2\alpha}} - r\left(\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}}\right)^2\right] = f(r), \tag{3.2}$$

where $\alpha = 1 \pm \varepsilon$ and $\varepsilon \approx 0$. We observe that for $\alpha = 1$ these equations revert to (2.2)-(2.3).

By (3.1) and Lemma 1 in Appendix, we have

$$\begin{aligned}
\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}(r^{1+\alpha}\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}}) &= \frac{\mathrm{d}}{\mathrm{d}t}(r^{1+\alpha}\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}}) + O(\varepsilon) \\
&= (1+\alpha)r^{\alpha}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}} + r^{1+\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}} + O(\varepsilon) \\
&= \frac{\Gamma(2+\alpha)}{\Gamma(2-\alpha)}r^{\alpha}\frac{\mathrm{d}^{\alpha}r}{\mathrm{d}t^{\alpha}}\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}} + r^{1+\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}} + O(\varepsilon) \\
&= O(\varepsilon).
\end{aligned} \tag{3.3}$$

Therefore, using Caputo definition of the derivative, we obtain

$$J = r^{1+\alpha}\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}} = h \tag{3.4}$$

is a constant (as a zero-th order approximation in ε). Therefore using the chain rule (Lemma 2), we can write

$$\frac{\mathrm{d}^{\alpha}r}{\mathrm{d}t^{\alpha}} = \frac{h}{r^{1+\alpha}}\frac{\mathrm{d}^{\alpha}r}{\mathrm{d}\theta^{\alpha}} + O(\varepsilon). \tag{3.5}$$

Using (3.5) to express $\frac{\mathrm{d}^{\alpha}\theta}{\mathrm{d}t^{\alpha}}$ and substitute in (3.2) we obtain

$$\frac{h}{r^{1+\alpha}}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}\theta^{\alpha}}\left(\frac{h}{r^{1+\alpha}}\frac{\mathrm{d}^{\alpha}r}{\mathrm{d}\theta^{\alpha}}\right) - \frac{h^2}{r^{2\alpha+1}} = f(r) + O(\varepsilon). \tag{3.6}$$

We introduce now the transformation $w = \frac{1}{r}$ and use

$$\frac{\mathrm{d}^{\alpha}(r^{-1})}{\mathrm{d}\theta^{\alpha}} = \frac{\mathrm{d}^{\alpha}(r^{-1})}{\mathrm{d}r^{\alpha}}\frac{\mathrm{d}^{\alpha}r}{\mathrm{d}\theta^{\alpha}} + O(\varepsilon) = -r^{-1-\alpha}\frac{\mathrm{d}^{\alpha}r}{\mathrm{d}\theta^{\alpha}} + O(\varepsilon). \tag{3.7}$$

We then have for (3.6)

$$-h^2 w^{1+\alpha}\frac{\mathrm{d}^{2\alpha}w}{\mathrm{d}\theta^{2\alpha}} - h^2 w^{2\alpha+1} = f\left(\frac{1}{w}\right) + O(\varepsilon). \tag{3.8}$$

For $f(r) = -\frac{\mu}{r^{1+\alpha}} = -\mu w^{1+\alpha}$ we then have

$$h^2 w^{1+\alpha}\left(\frac{\mathrm{d}^{2\alpha}w}{\mathrm{d}\theta^{2\alpha}} + w^{\alpha}\right) = \mu w^{1+\alpha} + O(\varepsilon). \tag{3.9}$$

So finally we obtain

$$\frac{\mathrm{d}^{2\alpha}w}{\mathrm{d}\theta^{2\alpha}} + w^{\alpha} = \frac{\mu}{h^2} + O(\varepsilon). \tag{3.10}$$

For a zero-th order approximation in ε, we have

$$\frac{\mathrm{d}^{2\alpha}w}{\mathrm{d}\theta^{2\alpha}} + w^{\alpha} = \frac{\mu}{h^2}. \tag{3.11}$$

We note that this equation is endowed with periodic boundary conditions as $w(0) = w(2\pi)$.

Remark. Under the influence of drag, the periodic boundary condition is a legitimate approximation as the orbit changes almost periodically– there are very small differences between $w(0)$ and $w(2\pi)$. Also, the periodicity will be utilized when we calculate the Caputo derivatives of exponential functions, which are still exponential functions. With other boundary conditions, the derivatives are no longer exponential functions but are Mittag-Leffler type functions.

Using a perturbation analysis in ε (see Appendix A for details), we obtain the following approximation solution

$$r = \frac{1}{E}\frac{e^{-\theta\cos\omega}}{1 + c\cos(\theta\sin\omega + \phi)}, \tag{3.12}$$

where $E = \left(\frac{\mu}{h^2}\right)^{1/\alpha} + \varepsilon\frac{\mu}{h^2}\ln\frac{\mu}{h^2}$, $\omega = \frac{2\pi}{\alpha}$ and c, ϕ, ε are constants.

3.1. Equatorial Orbits Around Oblate Body

For the motion of a satellite in equatorial orbit around an oblate body where $f(r)$ is given by (2.10), we can derive a similar formulation using fractional calculus as in (3.11). Specifically, we obtain, in place of (2.12),

$$\frac{\mathrm{d}^{2\alpha}w}{\mathrm{d}\theta^{2\alpha}} + w^{\alpha} = \frac{\mu}{h^2}(1 + kw^{1+\alpha}) + O(\varepsilon). \tag{3.13}$$

An approximate solution of this equation may be obtained as follows. Since this is a nonlinear equation we shall seek (as in Section 2.1) a first order approximate solution in J_2 that is we let $w = w_0 + J_2 w_1$ where w_0 is given by (inverse of) (3.12) . Substituting this in (3.13) leads to the following equation for w_1

$$\frac{\mathrm{d}^{2\alpha}w_1}{\mathrm{d}\theta^{2\alpha}} + \alpha w_0^{\alpha-1}w_1 = \frac{3\mu R^2}{2h^2}w_0^{1+\alpha} + O(J_2, \varepsilon). \tag{3.14}$$

Using the fact that $\alpha = 1 \pm \varepsilon$ is very close to 1, we approximate this equation by

$$\frac{\mathrm{d}^2 w_1}{\mathrm{d}\theta^2} + \alpha w_1 = \frac{3\mu R^2}{2h^2} w_0^2, \tag{3.15}$$

whose solution is

$$w_1 = \frac{3\mu}{2h^2} \frac{e^{2\theta\cos\omega}}{1+4\cos^2\omega}. \tag{3.16}$$

The final form of the approximate solution for (3.13) is

$$w = w_0 + J_2 w_1 + \varepsilon \frac{\mu}{h^2} \ln \frac{\mu}{h^2}. \tag{3.17}$$

4. MODEL VERIFICATION

In this section, we attempt to verify the fractional derivatives model that we presented in Section 3. Specifically, we compare the numerical solutions of equations with drag with solutions given in Section 3 derived from fractional calculus. As we have no *a priori* algorithm to determine α, we shall use the results of the numerical simulation (with a least square method with a quasi-Newton method) to find an optimal value of α that reproduces the orbit decay for various values of the drag coefficient. Then we compare the solutions given in Section 3 with computed α with numerical solutions.

The equation of motion of a satellite about a spherical planet under the influence of Newtonian gravitation and atmospheric drag is

$$\ddot{\mathbf{r}} = f(r)\mathbf{e}_r - D\rho(r)(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{1/2}\dot{\mathbf{r}}, \tag{4.1}$$

where $\mathbf{r}$ represents the position vector of the satellite from the center of attraction and $r = (\mathbf{r}\cdot\mathbf{r})^{1/2}$ is the magnitude of this vector. Also, $\rho(r)$ is directly proportional to the atmospheric density at a distance r from the center of attraction and D is a constant that is determined from the drag coefficient of the satellite, its geometry, and the atmospheric density at a specified altitude.

In polar coordinates (see Section 2) we obtain the following system of equations for the orbit of the satellite,

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -D\rho(r)(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{1/2} r\dot{\theta}, \tag{4.2}$$

$$\ddot{r} - r\dot{\theta}^2 = f(r) - D\rho(r)(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{1/2}\dot{r}. \tag{4.3}$$

4.1. Orbits Around Spherical Body

In this section, we consider the force in the form of

$$f(r) = -\mu/r^2, \tag{4.4}$$

where μ is the product of the universal gravitational constant and the mass of the planet. We shall compare the numerical solution of the equations with this force with the solution given by (3.12).

For ρ we use a typical (exponential) model for the density of the earth's atmosphere at heights near 7000 km above the center of the Earth

$$\rho_{exp} = \rho_0 e^{-\frac{(r-r_0)}{H}}, \tag{4.5}$$

where $H = 88.667$ km and $r_0 = 7120$ km. In the following we absorb ρ_0 in D so that ρ is normalized to 1 at r_0.

To solve equations (4.2)-(4.3) numerically, we used the following parameters and initial conditions in all simulations:

$$R_0 = 7120 \text{ km}, \;\; D = 1\times 10^{-11}. \tag{4.6}$$

The initial value of $\frac{d\theta}{dt}$ was chosen so that at time $t = 0$ the gravitational and centrifugal forces balance each other

$$\frac{d\theta}{dt}(0) = \frac{R_E}{R_0}\sqrt{\frac{g}{R_0}}, \tag{4.7}$$

where $R_E = 6378$ km is the earth radius and $g = 9.807\,\text{m}/\text{sec}^2$ is the acceleration of gravity at the sea level.

The parameters α, c and ϕ in (3.12) that yield the best fit to the numerical solution are $\alpha = 1 + 0.9225\times 10^{-8}$, $c = (\alpha - 1)/4$ and $\phi = \pi/4$. The difference between the numerical and fractional solutions over ≈ 7 revolutions is shown in Fig. 1. In this figure the difference between these two orbits is less than 6 cm. This difference can be attributed in part to the accumulated numerical integration error of (4.2) and (4.3).

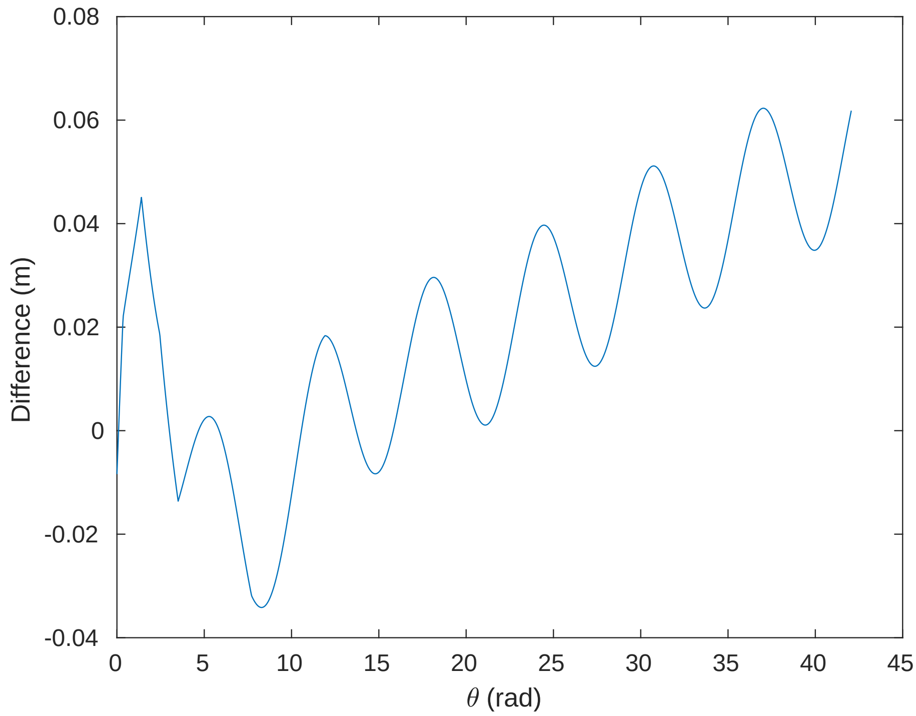

Fig. 1 – Difference between the numerical and fractional solutions, $D = 10^{-11}$.

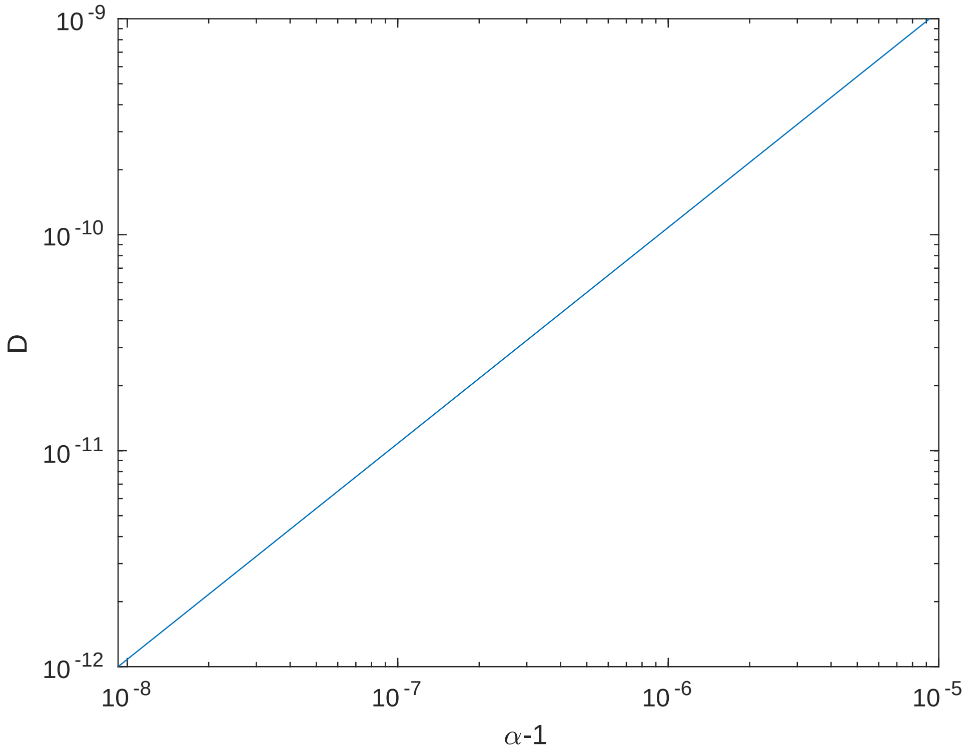

Fig. 2 – Drag coefficient *vs.* $(\alpha - 1)$.

In Fig. 2 we plotted on a logarithmic scale the functional relationship between the value of the drag coefficient D and the optimal value of α. This relationship turns out to be linear over a wide range of values of D.

4.2. Equatorial Orbits Around Oblate Body

We consider the motion (4.1) with a force of the form (2.10):

$$f(r) = -\mu\left(\frac{1}{r^2} + \frac{3R^2J_2}{2r^4}\right).$$

With the form (2.10), Eq. (4.1) in the polar coordinate (r,θ) (in the equatorial plane) becomes

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -D\rho(r)(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{1/2}r\dot{\theta}, \tag{4.8}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} + \frac{\mu k}{r^4} - D\rho(r)(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{1/2}r\dot{\theta}, \tag{4.9}$$

where $k = \frac{3}{2}R^2J_2$. We use the same initial conditions as in 4.1.

We first obtain numerical solution of (4.8)-(4.9) with the standard Runge-Kutta

4 in time to the orbit of a satellite using (3.13). The best fit between the numerical solution of and the approximate analytic solution (3.17) was obtained with

$$\alpha = 1+9.102\times 10^{-7},\ c = 0.0.4112(\alpha-1),\ \phi = \frac{37.5\pi}{180}, \varepsilon = 2.35\times 10^{-4}.$$

The difference between the numerical solution of (4.8)-(4.9) and the approximate analytic solution of (3.17) for the orbit over ≈ 7 revolutions is less than 8.5 m as shown in Fig. 3. This represents a relative error of 10^{-6} between the two orbits.

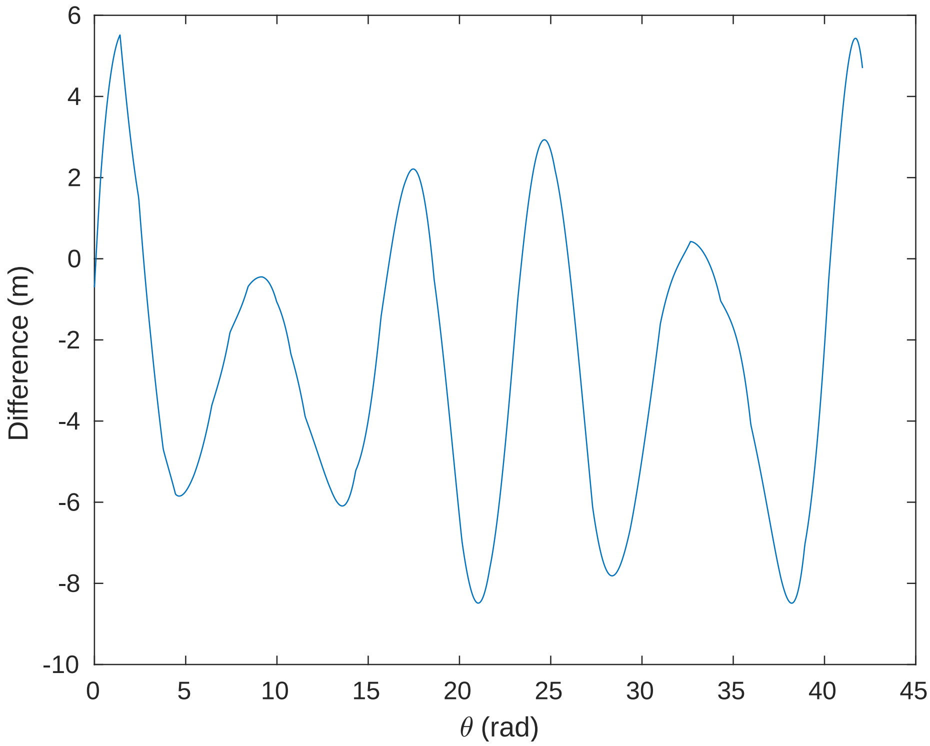

Fig. 3 – Difference between the numerical and fractional solutions, $D = 10^{-10}$.

5. CONCLUSION

The motion a spacecraft or a satellite in the Earth vicinity is subject to large numbers of “perturbations” from a variety of sources some of which might be considered as random. Due to the fact that it is not feasible to account for all these interactions using Newton’s law of gravitation a different paradigm is needed to predict the trajectory of the satellite accurately and for long(er) period of time. The present paper examined a lumped-parameter formulation of this problem using fractional derivatives. It was demonstrated that fractional derivatives can capture perturbations due to quadratic drag and Earth oblateness to an accuracy comparable to the numerical solutions. These two examples demonstrate that the paradigm of fractional derivatives might provide an alternative to the traditional perturbation approach for the inclusion of nonlocal effects in celestial mechanics. It is clear that these examples do not provide “a general proof” for the utility of this paradigm. However they demonestrate its possible potential in celestial mechanics.

For a practical application of this method, data about the historical trajectory of the satellite (or spacecraft) is needed. This data can be used then to derive the optimal value for the order of the fractional derivative that describes best this historical data. The model can be applied then to make accurate and superior projections about the future trajectory of the satellite or the spacecraft.

APPENDIXES

A. APPROXIMATE ANALYTICAL SOLUTIONS TO (3.11)

In this section, we apply perturbation analysis and superposition principle to obtain an approximate solution to Equation (3.11). We also show that this solution satisfies Equation (2.1) approximately.

To solve for the homogeneous part of (3.11) we substitute

$$w_h(\theta) = e^{\nu\theta}. \tag{A.1}$$

This yields the algebraic equation, up to a first-order perturbation in ε

$$\nu^{2\alpha} + 1 = 0, \tag{A.2}$$

which has two complex conjugate solutions for ν

$$\nu_{1,2} = \cos\left(\frac{\pi}{2\alpha}\right) \pm i \sin\left(\frac{\pi}{2\alpha}\right). \tag{A.3}$$

It follows then that we have two solutions

$$w_1 = e^{\theta\cos(\omega)}\exp(i\theta\sin(\omega)), \;\; w_2 = e^{\theta\cos(\omega)}\exp(-i\theta\sin(\omega)), \tag{A.4}$$

where $i = \sqrt{-1}$, $\omega = \frac{\pi}{2\alpha}$.

Although the superposition principle does not hold for (3.11), we shall combine these two solutions to obtain

$$w_h = e^{\theta\cos\omega}[C\cos(\theta\sin\omega) + D\sin(\theta\sin\omega)],$$

where C, D are constants. This can be rewritten more succinctly in the form

$$w_h = Ae^{\theta\cos\omega}\cos(\theta\sin\omega + \phi), \tag{A.5}$$

where A, ϕ are constants.

A particular solution of (3.11) is

$$w_p = \left(\frac{\mu}{h^2}\right)^{1/\alpha}. \tag{A.6}$$

A.1. Superposition of the solutions

Since the superposition principle does **NOT** hold for(3.11) it follows that $w_h + w_p$ is not a solution in of this equation in the strict sense:

$$\frac{d^{2\alpha}(w_h + w_p)}{d\theta^{2\alpha}} + (w_h + w_p)^{\alpha} - \frac{\mu}{h^2} \neq 0. \tag{A.7}$$

However, since $\varepsilon = |1-\alpha| \ll 1$, we shall compute the first order correction term in ε to this expression to obtain an approximate solution of (3.11) (viz. up to terms of order ε^2). To modify this solution so that (A.7) holds up to terms of order ε^2 we consider a solution of the form

$$w = w_h + w_p + \varepsilon w_s,$$

where w_s is the first order correction term.

Substituting the expression for w (using (A.5) and (A.6) in (3.11) we find that this equation is satisfied to up to zero order in ε. To have an order 1 in ε we obtain the following equation for w_s

$$\frac{d^{2\alpha}(w_s)}{d\theta^{2\alpha}} + w_s = -\left(w_h \ln\, w_h - \frac{\mu}{h^2}\ln\frac{\mu}{h^2}\right).$$

Since we can not solve this equation analytically, we use an approximation which ignores the first term on the right hand side of this equation and let

$$w_s = \frac{\mu}{h^2}\ln\frac{\mu}{h^2}.$$

The (approximate) solution of (3.11) is

$$w = w_h + w_p + \varepsilon w_s = Ae^{\theta\cos\omega}\cos(\theta\sin\omega + \phi) + \left(\frac{\mu}{h^2}\right)^{1/\alpha} + \varepsilon\frac{\mu}{h^2}\ln\frac{\mu}{h^2}. \tag{A.8}$$

Then we obtain the equation (3.12) for $r = \frac{1}{w}$, where $c = \frac{A}{E}$ and $E = \left(\frac{\mu}{h^2}\right)^{1/\alpha} + \varepsilon\frac{\mu}{h^2}\ln\frac{\mu}{h^2}$.

Now we verify that $w = w_h + w_p + \varepsilon w_s$ is an approximate solution to (3.11) with an order one in ε. Denote that $w_h = A_1 w_1 + A_2 w_2$, where $w_i = e^{\nu_i\theta}$, $i = 1, 2$. Then $\frac{d^{2\alpha}(w_h)}{d\theta^{2\alpha}} = (A_1\nu_1^{2\alpha}e^{\nu_1\theta} + A_2\nu_2^{2\alpha}e^{\nu_2\theta})$.

$$\begin{aligned}
\frac{d^{2\alpha}(w)}{d\theta^{2\alpha}} + w^{\alpha} &= \frac{d^{2\alpha}(w_h + w_p + \varepsilon w_s)}{d\theta^{2\alpha}} + (w_h + w_p + \varepsilon w_s)^{1\pm\varepsilon} \\
&= \frac{d^{2\alpha}w_h}{d\theta^{2\alpha}} + (w_h + w_p + \varepsilon w_s)^{1\pm\varepsilon} \\
&= (A_1\nu_1^{2\alpha}e^{\nu_1\theta} + A_2\nu_2^{2\alpha}e^{\nu_2\theta}) + (w_h + w_p + \varepsilon w_s) \\
&\pm \varepsilon(w_h + w_p)\ln(w_h + w_p) + o(\varepsilon) \\
&= (A_1(O(\varepsilon) - 1)e^{\nu_1\theta} + A_2(O(\varepsilon) - 1)e^{\nu_2\theta}) \\
&\quad + (w_h + w_p + \varepsilon w_s) - \varepsilon(w_h + w_p)\ln(w_h + w_p) + o(\varepsilon) \\
&= -w_h + O(\varepsilon) + (w_h + w_p + \varepsilon w_s) + O(\varepsilon) \\
&= w_p + O(\varepsilon) = w_p^{\alpha} + O(\varepsilon) = \frac{\mu}{h^2} + O(\varepsilon).
\end{aligned}$$

A.2. The solution (3.12) is an approximate solution to equation (2.1)

We next show that the solution (3.12) satisfies the following equation up to a first-order approximation in ε:

$$\ddot{\mathbf{r}} = f(r)\mathbf{e}_r, \quad f(r) = -\frac{\mu}{r^2}. \tag{A.9}$$

We only need to show that

$$\ddot{r} - r\dot{\theta}^2 = f(r) + O(\varepsilon), \tag{A.10}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = O(\varepsilon). \tag{A.11}$$

By (3.12), we then have

$$\begin{aligned} \dot{r} &= \frac{-\cos\omega e^{-\theta\cos\omega}\dot{\theta}}{E + A\cos(\theta\sin\omega+\phi)} - r\frac{-A\sin(\theta\sin\omega+\phi)\sin\omega}{E+A\cos(\theta\sin\omega+\phi)}\dot{\theta} && \text{(A.12)}\\ &= -r\dot{\theta}\cos\omega + r\dot{\theta}\frac{A\sin(\theta\sin\omega+\phi)\sin\omega}{E+A\cos(\theta\sin\omega+\phi)} && \text{(A.13)}\\ &= -r\dot{\theta}\cos\omega + r^2\dot{\theta}e^{\theta\cos\omega}A\sin(\theta\sin\omega+\phi)\sin\omega. \end{aligned}$$

Observe that the first-order derivative of $\tilde{r}^2\dot{\theta}$ is zero where $\tilde{r} = \frac{1}{\mu/h^2 + A\cos(\theta+\phi)}$. Then the formula (A.11) holds $r^{-1} - \tilde{r}^{-1} = O(\varepsilon)$. Thus by the fact that $\omega - \frac{\pi}{2} = O(\varepsilon)$ and (A.11),

$$\begin{aligned} \ddot{r} &= -\cos\omega(r\ddot{\theta} + \dot{r}\dot{\theta}) + (2r\dot{r}\dot{\theta} + r^2\ddot{\theta})e^{\theta\cos\omega}A\sin(\theta\sin\omega+\phi)\sin\omega\\ &\quad + r^2\dot{\theta}^2 e^{\theta\cos\omega}A\sin(\theta\sin\omega+\phi)\sin\omega\cos(\omega)\\ &\quad + r^2\dot{\theta}^2 e^{\theta\cos\omega}A\cos(\theta\sin\omega+\phi)\sin^2\omega\\ &= r^2\dot{\theta}^2 e^{\theta\cos\omega}A\cos(\theta\sin\omega+\phi)\sin^2\omega + O(\varepsilon)\\ &= r^2\dot{\theta}^2 e^{\theta\cos\omega}\left(\frac{1}{re^{\theta\cos\omega}} - E\right) + O(\varepsilon) = r\dot{\theta}^2 - r^2\dot{\theta}^2 E + O(\varepsilon). \end{aligned}$$

This can be written as $\ddot{r} - r\dot{\theta}^2 = -r^2\dot{\theta}^2 E + O(\varepsilon)$. As $r^2\dot{\theta} = h + O(\varepsilon)$ by (A.11) and $E = \mu/h^2 + O(\varepsilon)$, we obtain

$$-r^2\dot{\theta}^2 E = -(h+O(\varepsilon))^2\frac{1}{r^2}E = -(h+O(\varepsilon))^2\frac{1}{r^2}(\mu/h^2 + O(\varepsilon)) = -\frac{\mu + O(\varepsilon)}{r^2} = f(r) + O(\varepsilon).$$

We have reached the formulation (A.10).

B. PERTURBATIONS IN THE FRACTIONAL ORDER

We recall the left Caputo fractional derivative ${}_0^C\mathscr{D}_t^\mu$, which is by

$${}_0^C\mathscr{D}_t^\mu u(t) = {}_0\mathscr{I}_t^{n-\mu}\frac{d^n}{dt^n}u, \quad {}_0\mathscr{I}_t^{1-\mu}v = \frac{1}{\Gamma(1-\mu)}\int_0^t \frac{v(s)}{(t-s)^\mu}\,\mathrm{d}s, \quad n-1 < \mu \le n. \tag{B.1}$$

Here $n \in \mathbb{N}$ is a natural number, i.e., $n = 1, 2, 3, \ldots$.

LEMMA 1. *For $n-1<\beta\le n$, let $n-\beta=\varepsilon$ and then*

$$
\begin{aligned}
{}_0^C D_t^\beta f &= f^{(n)}(0)\frac{t^\varepsilon}{\Gamma(1+\varepsilon)}+\int_0^t f^{(n+1)}(\tau)\frac{(t-\tau)^\varepsilon}{\Gamma(1+\varepsilon)}\,\mathrm{d}\tau \\
&= f^{(n)}(t)+\varepsilon\gamma f^{(n)}(t)+\varepsilon f^{(n)}(0)\ln(t)+\varepsilon\int_0^t f^{(n+1)}(\tau)\ln(t-\tau)\,\mathrm{d}\tau+o(\varepsilon).
\end{aligned}
$$

Proof. When $\beta=n$, the conclusion is straightforward. We prove in the following the case $n-1<\beta<n$. By the definition of the Caputo derivative and integration by parts, we have

$$
\begin{aligned}
{}_0^C D_t^\beta f &= \frac{f^{(n)}(0)(t-0)^{n-\beta}}{\Gamma(n+1-\beta)}+\frac{1}{\Gamma(n+1-\beta)}\int_0^t f^{(n+1)}(\tau)(t-\tau)^{n-\beta}\,\mathrm{d}\tau \\
&= f^{(n)}(0)+\int_0^t f^{(n+1)}(\tau)\,\mathrm{d}\tau \\
&\quad +f^{(n)}(0)[\frac{t^\varepsilon}{\Gamma(1+\varepsilon)}-1]+\int_0^t f^{(n+1)}(\tau)[\frac{(t-\tau)^\varepsilon}{\Gamma(n+1-\beta)}-1]\,\mathrm{d}\tau.
\end{aligned}
$$

This can be simplified as

$$
{}_0^C D_t^\beta f-f^{(n)}(t)=f^{(n)}(0)[\frac{t^\varepsilon}{\Gamma(1+\varepsilon)}-1]+\int_0^t f^{(n+1)}(\tau)[\frac{(t-\tau)^\varepsilon}{\Gamma(n+1-\beta)}-1]\,\mathrm{d}\tau. \quad \text{(B.2)}
$$

By the Taylor's expansion for the log-gamma function, it holds that for $|\varepsilon|<1$,

$$
\ln\Gamma(1+\varepsilon)=-\gamma\varepsilon+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-\varepsilon)^k,
$$

where $\zeta(k)$ is the zeta function and $\gamma=\int_0^\infty\frac{e^{-t}-1+t}{t(e^t-1)}\,\mathrm{d}t$. By Taylor's expansion in ε, we have

$$
(t-b)^\varepsilon=1+\sum_{k=1}^{K}\frac{(\ln(t-b))^k}{k!}\varepsilon^k+o(\varepsilon^K), \quad t>b.
$$

Then

$$
\begin{aligned}
\frac{(t-b)^\varepsilon}{\Gamma(1+\varepsilon)}-1 &= (t-b)^\varepsilon\exp(\gamma\varepsilon-\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-\varepsilon)^k)-1 \\
&= (1+\sum_{k=1}^{K}\frac{(\ln(t-b))^k}{k!}\varepsilon^k+o(\varepsilon^{K+1}))(1+\gamma\varepsilon-\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(-\varepsilon)^k+o(\varepsilon))-1 \\
&= \varepsilon(\gamma+\ln(t-b))+o(\varepsilon). \qquad \text{(B.3)}
\end{aligned}
$$

Then by (B.2) and (B.3), we conclude that

$$
\begin{aligned}
{}_0^C D_t^\beta f &= f^{(n)}(t)+f^{(n)}(0)[\frac{t^\varepsilon}{\Gamma(1+\varepsilon)}-1]+\int_0^t f^{(n+1)}(\tau)[\frac{(t-\tau)^\varepsilon}{\Gamma(1+\varepsilon)}-1]\,\mathrm{d}\tau \\
&= f^{(n)}(t)+\varepsilon f^{(n)}(0)(\gamma+\ln(t))+\varepsilon\int_0^t f^{(n+1)}(\tau)(\gamma+\ln(t-\tau))\,\mathrm{d}t+o(\varepsilon) \\
&= f^{(n)}(t)+\varepsilon\gamma f^{(n)}(t)+\varepsilon f^{(n)}(0)\ln(t)+\varepsilon\int_0^t f^{(n+1)}(\tau)\ln(t-\tau)\,\mathrm{d}t+o(\varepsilon). \qquad \blacksquare
\end{aligned}
$$

LEMMA 2 (Approximation of chain rule). *For $n-1<\beta\le n$ ($n=1$), let $n-\beta=\varepsilon$ and then there exists a constant C depending on f,g that*

$$ {}_0^C D_t^{\beta} f(g(t)) = {}_0\mathscr{I}_t^{1-\beta} f'(g) {}_0^C D_t^{\beta} g + C\varepsilon. \tag{B.4}$$

Proof. By Lemma 1, we have

$$\begin{aligned}
{}_0^C D_t^{\beta} f(g(t)) &= [f(g)]^{(n)}(t) + \varepsilon\gamma[f(g)]^{(n)}(t) + \varepsilon[f(g)]^{(n)}(0)\ln(t)\\
&+ \varepsilon\int_0^t [f(g)]^{(n+1)}(\tau)\ln(t-\tau)\,\mathrm{d}\tau + o(\varepsilon)\\
&= f'(g)g'(t)(1+\varepsilon\gamma) + \varepsilon f'(g(0))g'(0)\ln(t)\\
&+ \varepsilon\int_0^t [f(g)]^{(n+1)}(\tau)\ln(t-\tau)\,\mathrm{d}\tau + o(\varepsilon)\\
&= ({}_0\mathscr{I}_t^{1-\beta} f'(g) - C_f\varepsilon)({}_0^C D_t^{\beta} g - C_g\varepsilon)(1+\varepsilon\gamma) + \varepsilon[C_{f(g)} - \gamma f'(g)g'(t)]\\
&= {}_0\mathscr{I}_t^{1-\beta} f'(g) {}_0^C D_t^{\beta} g + C\varepsilon.
\end{aligned}$$

Here C is a constant depending on f, g and β. This conclusion implies that the chain rule holds up to a first-order approximation in ε.

LEMMA 3 (Approximation of product rule). *For $n-1<\beta\le n$ ($n\in\mathbb{N}$), let $n-\beta=\varepsilon$ and then there exists a constant C depending on f,g that*

$$ {}_0^C D_t^{\beta} fg = {}_0^C D_t^{\beta} f {}_0^C D_t^{\beta} g + C\varepsilon. \tag{B.5}$$

The *proof* of this lemma is based on Lemma 1 as in the *proof* of Lemma 2 and is thus omitted.

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REFERENCES

1. BROUWER, D., CLEMENCE, G.M., *Celestial mechanics* (Chapter 3), Academic Press, New York, 1961.
2. CONDURACHE, D., MARTINUSI, V., *A closed form solution of the two body problem in non-inertial reference frames*, Advances in the Astronautical Sciences, **143**, pp. 1649–1668, 2012.
3. CONDURACHE, D., MARTINUSI, V., *Relative spacecraft motion in a central force field*, Journal of Guidance, Control, and Dynamics, **30**, *3*, pp. 873–876, 2007, DOI: 10.2514/1.26361.
4. EL-SAYED, A.M.A., NOUR, H.M., RASLAN, W.E., EL-SHAZLY, E.S., *A study of projectile motion in a quadratic resistant medium via fractional differential transform method*, Applied Mathematical Modelling, **39**, pp. 2829–2835, 2015, DOI: 10.1016/j.apm.2014.10.018.
5. GURFIL, P., *Generalized solutions for relative spacecraft orbits under arbitrary perturbations*, Acta Astronautica, **60**, pp. 61–78, 2007, DOI: 10.1016/j.actaastro.2006.07.013.
6. HERRMANN, R., *Fractional calculus: An introduction for physicists* (Chapters 3, 5 and 6), World Scientific, 2011.
7. HUMI, M., CARTER, T., *Orbits and relative motion in the gravitational field of an oblate body*, AIAA J. Guidance and Control and Dynamics, **31**, *3*, pp. 522–532, 2008, DOI: 10.2514/1.32413.
8. HUMI, M., *Semi-equatorial orbits aaround an oblate body*, Journal of Guidance, Control, and Dynamics, **35**, *1*, pp. 316–321, 2012, DOI: 10.2514/1.55408.

9. KOON, W.S., LO, M.W., MARSDEN, J., ROSS, S.H., *Dynamical systems, the three-body problem and space mission design*, Caltech, California, 2006.
10. LEVI-CIVITA, T., *Sur la régularisation du problème des trois corps*, Acta Mathematica, **42**, pp. 99–144, 1920.
11. LIDOV, M.L., *Evolution of the orbits of artificial satellites of planets as affected by gravitational perturbation from external bodies*, Artificial Satellite Earth, **8**, pp. 5–45, 1961, DOI: 10.2514/3.1983.
12. NIE, T., GURFIL, P., ZHANG, S., *Bounded lunar relative orbits*, Acta Astronautica, **157**, pp. 500-516, 2019, DOI: 10.1016/j.actaastro.2019.01.018.
13. POINCARE, H., *New methods of celestial mechanics* (Chapter 6), Springer, New York, 1992.
14. PRUSSING, J.E., CONWAY, B.A., *Orbital mechanics*, Oxford University Press, New-York, 1993, pp. 139–169.
15. SZEBEHELY, V., *Theory of orbits: The restricted problem of three bodies* (Chapters 1 and 8), Academic Press, New York, 1967.
16. VARIESCHI, G.U., *Applications of Fractional Calculus to Newtonian mechanics*, Journal of Applied Mathematics and Physics, **6**, pp. 1247–1257, 2018, DOI: 10.4236/jamp.2018.66105.