Our work is dedicated to the blessed memory of L.D. Akulenko

QUASI-OPTIMAL DECELERATION OF ROTATIONS OF A GYROSTAT WITH INTERNAL DEGREE OF FREEDOM IN A RESISTIVE MEDIUM

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Abstract. The problem of the time-dependent quasi-optimal deceleration of dynamically symmetric rigid body rotations under a small control torque in the ellipsoidal domain with close unequal values of the ellipsoid's semiaxes is studied. It is assumed that the body contains a spherical cavity filled with a highly viscous fluid (assuming small Reynolds numbers). The body is assumed to have a moving mass connected to it via elastic coupling with quadratic dissipation. The moving mass is modeling the loosely attached elements in a space vehicle, which can significantly affect the vehicle's motion relative to its center of mass during a long period of time. In addition, the body is acted upon by a small medium resistance torque. The problem is solved asymptotically, based on the procedure of averaging the precession-type motion over the phase. The qualitative properties of quasi-optimal motion are analyzed and the corresponding graphs are presented.

Key words: Deceleration, Fluid, Mass, Resisting medium, Averaging.

1. INTRODUCTION

The desirable development in the field of research describing the problem of dynamics and control of rigid bodies moving about a fixed point would imply that the bodies were not absolutely rigid but rather close to ideal models. The need for the analysis of the influence of various deviations from the ideal states is caused by

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growing accuracy requirements in space exploration, gyroscopy, etc. The effect of nonideality can be analyzed by singular perturbation methods, averaging, or other asymptotic methods of nonlinear mechanics. This deviating from ideal state reduced to the presence of additional perturbation torques in Euler dynamic equations for a fictious rigid body. The motion of rigid bodies with internal degrees of freedom was investigated in [1–7].

Considerable attention was paid for analysis of the uncontrolled motion of rigid bodies with a cavity filled with a highly viscous fluid [1, 2, 8, 9]. Passive motion of rigid bodies in a resistive medium was studied in [2, 10–12].

A significant number of works were devoted to the analysis of various problems of the dynamics of space vehicles containing internal movable masses. The issues of stabilization of motions have been studied in [13–15]. The control of the rotations of quasi-rigid bodies using concentrated (applied to the body) torques, has been studied in less extent [4, 16–18].

We consider a problem of the quasi-optimal deceleration of dynamically symmetric rigid body rotations with a cavity filled with a highly viscous fluid. A moving mass is attached to a point on the body's symmetry axis. It is assumed that in relative motion the point is acted upon by the restoring elastic force and the resistance force proportional to the squared velocity (quadratic friction). In addition, the body is affected upon by the dissipative torque of the resistance of the medium.

2. STATEMENT OF THE PROBLEM

We consider controlled rotations of a dynamically symmetric rigid body with a cavity filled with a viscous fluid assuming low Reynolds numbers [1, 2]. In addition, a moving mass of relatively small linear dimensions, due to elastic relation with a quadratic dissipation [2, 3, 17], is attached to the moving body in a resistive medium. Based on the approach [4], the asymptotically approximate equations of controlled rotational motions in the coordinate system associated with the body (the Euler dynamic equations) should be presented in the form:

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = \mathbf{M}^u + \mathbf{M}^p + \mathbf{M}^v + \mathbf{M}^r \tag{1}$$

Here, $\mathbf{M}^u$ is the vector of control external reactive torque, $\mathbf{M}^p$ is the vector of internal disturbing torque caused by the presence of a viscous fluid in the cavity inside the body, $\mathbf{M}^v$ is vector of internal disturbing torque due to elasticity and quadratic friction of the damper, $\mathbf{M}^r$ is the torque of dissipation forces (resistance of medium). The vector $\mathbf{G} = \mathbf{J}\boldsymbol{\omega}$ is the angular momentum of the body, where $\mathbf{J} = \mathrm{diag}(A_1, A_1, A_3)$ is a constant symmetric inertia tensor of the unperturbed body, reduced to the main axes, $\boldsymbol{\omega} = (p, q, r)^{\mathrm{T}}$ is the vector of angular velocity represented by its projections onto the appropriate axes. The module of the angular momentum of the body has the form

$$G = |\mathbf{G}| = \left[A_1^2(p^2+q^2) + A_3^2 r^2\right]^{1/2} \equiv \left[A_1^2\omega_\perp^2 + A_3^2 r^2\right]^{1/2},$$
$$A_1 \neq A_3, \quad \omega_\perp^2 = p^2 + q^2.$$

We describe the structure of the control action. The magnitude of the control torque $\mathbf{M}^u$ is assumed to be small of order ε, where $\varepsilon \ll 1$ – is the small dimensionless parameter. Components of control torques should be presented in the form of products $\varepsilon b_i u_i$, $i = 1,2,3$ [4]:

$$M_i^u = \varepsilon b_i u_i, \; u_i = -G_i G^{-1}, \; i = 1,2,3, \; |\mathbf{u}| \leq 1. \tag{2}$$

Here, the constant expressions b_i all have the dimension of the torque of force and are quite close to each other, u_i are dimensionless control functions, which we should determine.

To simplify the solving of the optimal control problem, a structural constraint is introduced into the system (1). It is believed that the torque of resistance forces of the medium is small and proportional to the angular momentum of the body [2, 11, 16–18]

$$\mathbf{M}^r = -\varepsilon\lambda\mathbf{J}\boldsymbol{\omega}, \tag{3}$$

where λ is some constant coefficient of proportionality, determined mainly by the properties of the medium and the shape of the body and having the dimension of angular velocity.

Taking into account (2), (3), the approximate system of equations of controlled motion (1) in projections onto the main axes of inertia of the body has the form [2–4, 10, 16–18]:

$$\begin{aligned}
&A_1\dot{p} + (A_3 - A_1)qr = -\varepsilon b_1 A_1 p G^{-1} + Lpr^2 + FG^2 qr + Spr^6\omega_\perp - \varepsilon\lambda A_1 p \\
&A_1\dot{q} + (A_1 - A_3)pr = -\varepsilon b_2 A_1 q G^{-1} + Lqr^2 - FG^2 pr + Sqr^6\omega_\perp - \varepsilon\lambda A_1 q \\
&A_3\dot{r} = -\varepsilon b_3 A_3 r G^{-1} + H(p^2+q^2)r - A_1 A_3^{-1} S r^5 \omega_\perp^3 - \varepsilon\lambda A_3 r \\
&\qquad 0 < A_3 \leq 2A_1, \; A_3 \neq A_1.
\end{aligned} \tag{4}$$

It is worth to note that when the coefficients are equal $b_1 = b_2 = b_3 = b$ $(b > 0)$ where parameter b can be suggested to be time-dependent, control (2) is optimal. This property explains the assumption of proximity b_i and the introduction of the term «quasi-optimal control» [4].

The coefficients F, S, H, L in (4) should be expressed through parameters of the system in this way [2–4]:

$$F = m\rho^2\Omega^{-2}A_1^{-3}A_3,\ \ S = m\rho^3\Lambda\Omega^{-3}d\left|d\right|A_1^{-4}A_3^4,\ d = 1 - A_3A_1^{-1}$$
$$H = \beta P_0\nu^{-1}A_1^{-1}\left(A_3 - A_1\right),\ L = \beta P_0\nu^{-1}A_1^{-2}A_3\left(A_1 - A_3\right),\ \omega_\perp^2 = p^2 + q^2. \tag{5}$$

We introduce the notations in (5) which characterizes the torques of forces (caused by the presence of an elastic element). Here, m is the mass of the moving point, $\boldsymbol{\rho}$ is the radius-vector of the point of attachment of the moving mass which is located on the axis of dynamic symmetry of the given body, $\rho = |\boldsymbol{\rho}|$. The constants $\Omega^2 = c/m$, $\lambda_1 = \mu/m = \Lambda\Omega^3$, $\Omega >> \omega_0$ determine the frequency of oscillations and the velocity of their attenuation, respectively; c is the stiffness, μ is the quadratic friction coefficient, ω_0 is the modulus of the initial value for the angular velocity vector.

If we assume that the coupling coefficients λ_1 and Ω are such that the "free" point motions caused by the initial deviations decay much faster than the body makes a revolution, then in this case the body's motion should be close to the Euler-Poinsot motion, and the relative oscillations caused by this motion will be small.

Inequality $\Omega >> \omega_0$ allows us to introduce a small parameter into expressions F, S (5) and we can consider the corresponding perturbation torques as small in order to apply the averaging method outside the possible initial transient process.

The coefficients H, L in (5) determine the torque of forces due to the motions of a highly viscous fluid in the body cavity, β is a fluid density, ν is a kinematic viscosity coefficient, P_0 is the coefficient, which depends on the shape of the cavity and characterizes energy dissipation due to fluid viscosity. In the case of a spherical cavity of radius d it equals $P_0 = 8\pi d^7/525$ [1, 2]. The main admission is the assumption that the Reynolds number is small $\mathrm{Re} << \varepsilon << 1$.

3. ASYMPTOTIC APPROACH TO THE PROBLEM OF QUASI-OPTIMAL DECELERATION

First, let us make the problem to be dimensionless. For definiteness, let us select the moment of inertia of the rigid body with respect to the axis x_1 – $A_1 = A_2$ and the variable ω_0 of the order of its initial velocity as characteristic parameters for the problem. Introducing the dimensionless time $\tau = \omega_0 t$ and the dimensionless inertia coefficients $\tilde{A}_i = A_i/A_1$, system (4) will take the form:

$$\frac{d\tilde{p}}{d\tau} = -\left(\tilde{A}_3 - 1\right)\tilde{q}\tilde{r} - \varepsilon\tilde{b}_1\tilde{p}\tilde{G}^{-1} + \varepsilon\tilde{L}\tilde{p}\tilde{r}^2 + \varepsilon\tilde{F}\tilde{G}^2\tilde{q}\tilde{r} + \varepsilon\tilde{S}\tilde{p}\tilde{r}^6(\tilde{p}^2 + \tilde{q}^2)^{1/2} - \varepsilon\tilde{\lambda}\tilde{p}, \tag{6}$$

$$\frac{d\tilde{q}}{d\tau} = \left(\tilde{A}_3 - 1\right)\tilde{p}\tilde{r} - \varepsilon\tilde{b}_2\tilde{q}\tilde{G}^{-1} + \varepsilon\tilde{L}\tilde{q}\tilde{r}^2 - \varepsilon\tilde{F}\tilde{G}^2\tilde{p}\tilde{r} + \varepsilon\tilde{S}\tilde{q}\tilde{r}^6(\tilde{p}^2 + \tilde{q}^2)^{1/2} - \varepsilon\tilde{\lambda}\tilde{q},$$

$$\frac{d\tilde{r}}{d\tau} = -\varepsilon\tilde{b}_3\tilde{r}\tilde{G}^{-1} + \varepsilon\tilde{H}\left(\tilde{p}^2 + \tilde{q}^2\right)\tilde{r} - \varepsilon\tilde{A}_3^{-2}\tilde{S}\tilde{r}^5(\tilde{p}^2 + \tilde{q}^2)^{3/2} - \varepsilon\tilde{\lambda}\tilde{r}.$$

Here, taking into account the assumptions made, the following notations are also introduced:

$$\begin{gathered}\varepsilon\tilde{F} = m\rho^2\Omega^{-2}A_1^{-1}\tilde{A}_3\omega_0^2,\ \ \varepsilon\tilde{S} = m\rho^3\Lambda\Omega^{-3}(1-\tilde{A}_3)\,|\,1-\tilde{A}_3\,|\,A_1^{-1}\tilde{A}_3^4\omega_0^6,\\ \varepsilon\tilde{L} = \beta P_0\nu^{-1}A_1^{-1}\tilde{A}_3(1-\tilde{A}_3)\omega_0,\ \ \varepsilon\tilde{H} = \beta P_0\nu^{-1}\tilde{A}_3^{-1}(\tilde{A}_3-1)A_1^{-1}\omega_0,\\ \tilde{b}_i = b_i/A_1\omega_0^2,\ \tilde{\lambda} = \lambda/\omega_0,\ \tilde{G} = G/A_1\omega_0,\ \tilde{A}_1 = \tilde{A}_2 = 1.\end{gathered} \tag{7}$$

Here and below, we omit the sign «~» at using the dimensionless variables.
Then we use the common generating solution of the system (6) for $\varepsilon = 0$:

$$p = a\cos\psi,\quad q = a\sin\psi,\quad a > 0,\quad r = \text{const} \neq 0. \tag{8}$$

Here, $\psi = \left(A_3 - 1\right)r\tau + \psi_0$ is the phase of oscillating the equatorial component of the angular velocity vector.

We substitute (8) into the third equation of system (6). For the first two equations of (6), the expressions are true: $a^2 = p^2 + q^2$ and $\dot{a} = \dot{p}\cos\psi + \dot{q}\sin\psi$. We average the obtained system of equations for a and r over the phase ψ. Introducing the slow argument $\theta = \varepsilon\tau$ after averaging, we obtain that the system takes the form (here, $' = d/d\theta$):

$$\begin{aligned} a' &= -a\left[G^{-1}\left(b_1 + b_2\right)/2 - Lr^2 - Sr^6a + \lambda\right],\\ r' &= -r\left(b_3G^{-1} - Ha^2 + A_3^{-2}Sr^4a^3 + \lambda\right).\end{aligned} \tag{9}$$

The averaging of expressions containing factor F equals zero. With additional demands $b_1 = b_2 = b_3 = b$ equations for a and r can be fully integrated and this optimal control problem is thus solved analytically.

4. APPROXIMATE SOLUTION

Let us consider a special case

$$0.5(b_1 + b_2) = b_3 = b. \tag{10}$$

Multiplying the first equation in (6) by p, the second equation by q, and the third one by A_3^2r then, we should then sum them all. After averaging, we obtain next equation:

$$G' = -b - \lambda G. \tag{11}$$

Taking into account the initial $G(\theta_0) = G^0$ and final conditions $G\left(T, \theta_0, G^0\right) = 0,\ T = T\left(\theta_0, G^0\right)$, we obtain after the appropriate integrating (11)

$$G(\theta) = -\frac{b}{\lambda} + \left(G^0 + \frac{b}{\lambda}\right)\exp(-\lambda\theta),\ \Theta = \frac{1}{\lambda}\ln\left(G^0\frac{\lambda}{b} + 1\right). \tag{12}$$

Let us note that in (12) the quantity $\Theta \to \infty$ while $G^0/b \to \infty$ for different λ; in turn, $\Theta \to 0$ while $G^0\lambda/b \to 0$ (λ is arbitrary) or $\lambda \to \infty$.

We consider below the change of variables for system (9) subjected to the condition (10); besides, we consider the change of variables: $r = \eta G,\ a = \alpha G$. The equations of system (9) should be then written in the form:

$$\frac{d\alpha}{d\theta} = \alpha\eta^2 G^2\left(L + SG^5\alpha\eta^4\right),\quad \frac{d\eta}{d\theta} = \alpha^2\eta G^2(H - A_3^{-2}SG^5\alpha\eta^4). \tag{13}$$

We divide the first equation of the system (13) by the second and obtain:

$$\frac{d\alpha}{d\eta} = \frac{\eta\left(L + SG^5\alpha\eta^4\right)}{\alpha(H - A_3^{-2}SG^5\alpha\eta^4)}. \tag{14}$$

This equation (14) above can be solved numerically.

5. NUMERICAL INVESTIGATIONS

With the aim of solving the system (9), we have performed numerical investigations for various initial conditions and parameters of the problem. The renormalized quantities at the initial time are equal to values $\omega_0 = 1$, $A_3 = 1.2$, $A_3 = 1.5$ respectively, as well as control torque coefficients $b_1 = 0.1625$, $b_2 = 0.1$, $b_3 = 0.15$ (moreover, $0.5(b_1 + b_2) \neq b_3$). In this paper, we have considered two cases which are corresponding to the set of initial data as below:

$$a_0 = 0.35, \tag{15}$$

$$a_0 = 0.626. \tag{16}$$

We determine the angular velocity of rotation about the axis of dynamically symmetry by the formula $r_0 = \left(\omega_0 - a_0^2\right)^{1/2}$ at $\omega_0 = 1$. The calculations were performed at two values of the renormalized resistance coefficient $\lambda = 0.2,\ 0.5$ as well as the coefficients $S = 0.5,\ 0.8$, $H = 0.5,\ 0.8$, $L = -0.72,\ -1.152$. We select the parameters in such a way as to satisfy the conditions $A_3 \leq 2$, $a_0 < r_0$. The expression $G = |\mathbf{G}| = \left(a^2 + A_3^2 r^2\right)^{1/2}$ was used to plot the angular momentum absolute value. We present on the Figs. 1–4 very similar type of plots for the changing of functions a, r and G, which we obtain as a result of numerical integration.

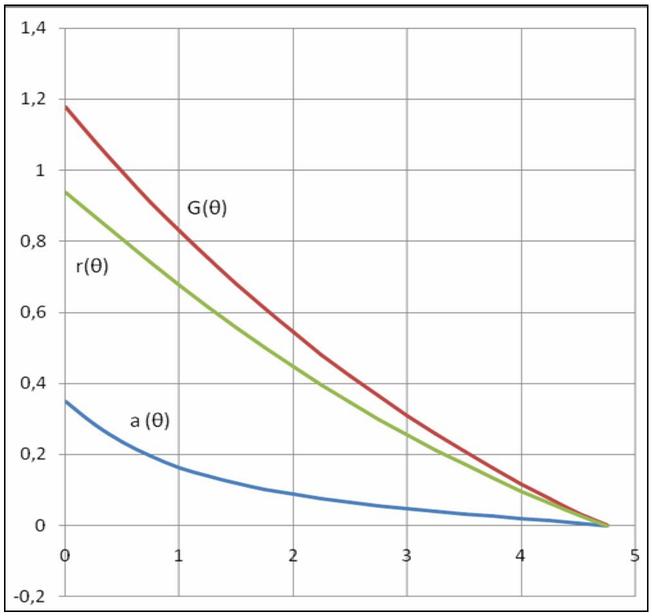

Fig. 1 – $\lambda = 0.2$; $S = 0.5$; $H = 0.5$.

Plots of functions a, r, G at values $\lambda = 0.2$ and initial data (15) is presented on the Figs. 1 and 2. The deceleration time equals $T \simeq 4.75$ for different values of the coefficients S and H, but at the increasing in the coefficients S and H the plot of function $a(\theta)$ is more curved (Fig. 2).

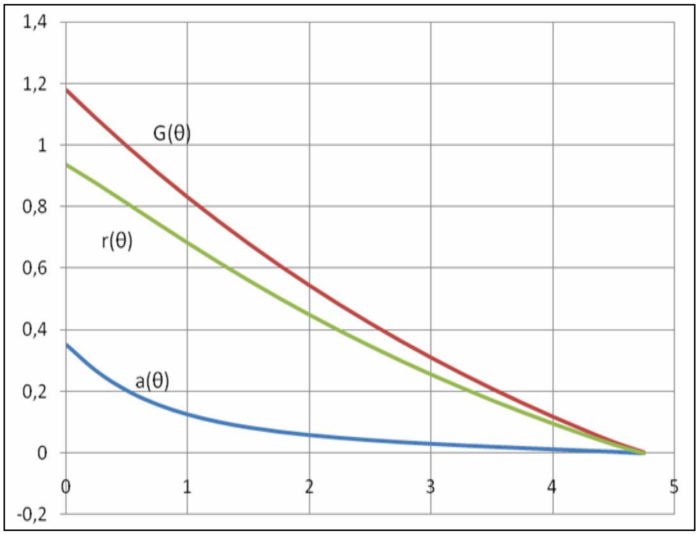

Fig. 2 – $\lambda = 0.2$; $S = 0.8$; $H = 0.8$.

The result of numerical integrations for the system (9) at different initial data (16) and value $\lambda = 0.5$ is presented on the Figs 3, 4. In this case of calculations, $T \simeq 3.51$. As we can see from the Fig. 2, the increasing of the coefficients of forces torques is due to the presence of the elastic element and viscous fluid in the cavity which causes a change in the decreasing of $a(\theta)$.

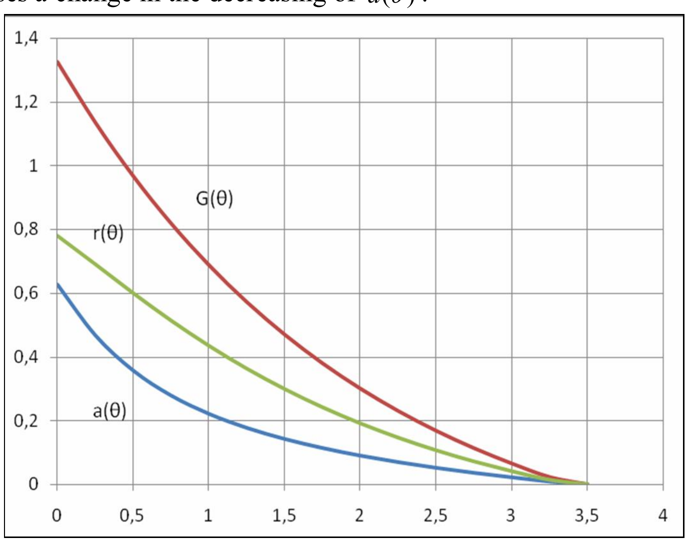

Fig. 3 – $\lambda = 0.5$; $S = 0.5$; $H = 0.5$.

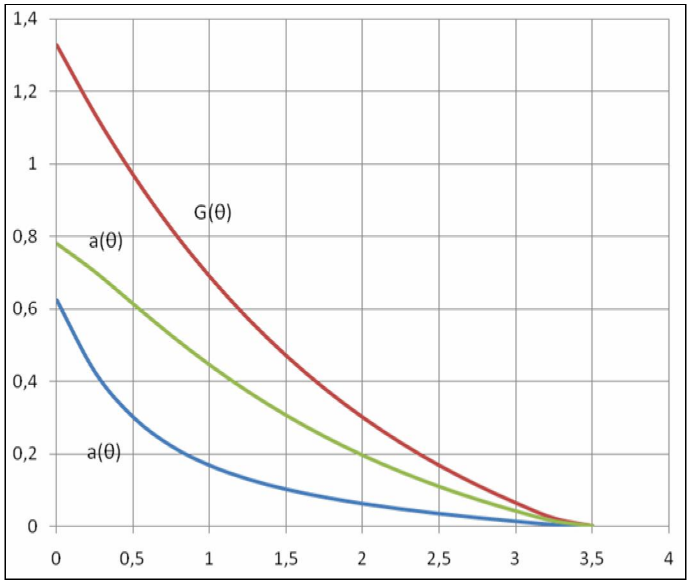

Fig. 4 – $\lambda = 0.5$; $S = 0.8$; $H = 0.8$.

We can see from the plots that the value of the time of deceleration T affect values of resistance coefficient λ, with an increase in which the deceleration of the rigid body occurs faster, as well as the same is valid for the coefficient A_3. The influence of the initial values a and r on the time of deceleration is insignificant. Variation of variables a, r, G is monotonous. Thus, the quasi-optimal deceleration problem has been solved.

6. CONCLUSION

The problem of the time-dependent quasi-optimal deceleration for the dynamically symmetric rigid body with a cavity filled with a fluid of high viscosity and a moving mass connected with the body by a quadratic dissipation damper in a medium with resistance was investigated. In the framework of the asymptotic approach, the averaged system of equations has been obtained, the deceleration time has been determined ($T \simeq 4.75$ and $T \simeq 3.51$) for the chosen numerical values of dimensionless parameters. The plots of the angular momentum G of the body and the values of a and r (i.e., the equatorial, the axial component of the angular velocity, vector of a quasi-rigid body) are constructed.

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