FOUCAULT-LIKE PROPERTIES IN THE FULL-BODY RELATIVE SPACECRAFT MOTION

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Abstract. The relative orbital motion between the leader and the deputy spacecraft is a six-degree-of-freedom (6-DOF) motion, representing the coupling of the relative translational motion with the rotational one. In recent years, increasing attention has been paid to the modeling of the relative 6-DOF motion of spacecraft. Also, controlling the relative pose of satellite formation is a significant research subject. In this paper, we reveal a real and dual tensor-based procedure to obtain exact expressions for the 6-DOF relative orbital law of motion between two Keplerian confocal orbits. Orthogonal real and dual tensors play a very important role, with the representation of the solution being, to the author knowledge, the shortest approach for describing the complete state onboard solution of the 6-DOF orbital relative motion problem. A representation theorem is provided for the full-body initial value problem. Furthermore, the real and dual parts are split, and representation theorems for relative rotation and translation motions are obtained.

Key words: dual number, dual tensor, six-degree-of-freedom relative orbital motion, spacecraft motion.

Abbreviations

a = real number

$\underline{a}$ = dual number

$\mathbf{a}$ = real vector

$\underline{\mathbf{a}}$ = dual vector

$\boldsymbol{A}$ = real tensor

$\underline{\boldsymbol{A}}$ = dual tensor

$\boldsymbol{V}_3$ = real vectors set

$\underline{\boldsymbol{V}}_3$ = dual vectors set

$\boldsymbol{V}_3^{\mathbb{R}}$ = time depending real vectorial functions

$\underline{\boldsymbol{V}}_3^{\mathbb{R}}$ = time depending dual vectorial functions

$\tilde{\underline{\boldsymbol{v}}}$ = skew-symmetric dual tensor corresponding to the dual vector $\underline{\mathbf{v}}$

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f_c = true anomaly
p_c = conic parameter
$\boldsymbol{h}_c$ = specific angular momentum of the leader satellite
$\boldsymbol{L}\left(\underline{V}_3,\underline{V}_3\right)$ = dual-tensor set
$\mathbb{R}$ = real numbers set
$\underline{\mathbb{R}}$ = dual numbers set
$S\mathbb{O}_3$ = orthogonal real tensors set
$\mathrm{s}\mathbf{o}_3$ = skew-symmetric real tensor set
$\underline{S\mathbb{O}_3}$ = orthogonal dual tensor set
$\underline{\mathrm{s}\mathbf{o}}_3$ = skew-symmetric dual tensor set
$S\mathbb{O}_3^{\mathbb{R}}$ = time depending real tensorial functions
$\underline{S\mathbb{O}}_3^{\mathbb{R}}$ = time depending dual tensorial functions

1. INTRODUCTION

The relative orbital motion problem [1–4] may now be considered classic, because of so many scientific papers written on this subject in the last few decades. The model of the relative motion consists in two spacecraft flying in Keplerian orbits under the influence of the same gravitational attraction center. The main problem is to determine the state of the Deputy satellite with respect to a reference frame originated in the Chief satellite center of mass. This non-inertial reference frame, traditionally named LVLH (Local-Vertical-Local-Horizontal) is chosen as follows: the C_x axis has the same orientation as the position vector of the Chief's center of mass with respect to an inertial reference frame originated in the attraction center; the C_z axis has the same orientation as the Chief orbit angular momentum; the C_y axis completes a right-handed frame. Both, the Chief satellite and the Deputy satellite will be considered rigid bodies. Next, an analysis over the motion and the state of the mass center of the Deputy in relation with LVLH is detailed (Fig. 1).

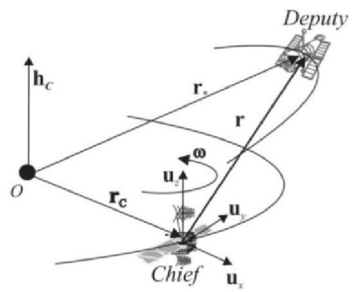

Fig. 1 – Six-degree-of-freedom spacecraft relative motion.

Consider $\mathbf{r}$ the position vector of the Deputy mass center in relation with LVLH. The initial value probem that models the motion of the Deputy satellite with respect to the LVLH reference frame is [3]:

$$\begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega}_c \times \dot{\mathbf{r}} + \boldsymbol{\omega}_c \times \left(\boldsymbol{\omega}_c \times \mathbf{r}\right) + \dot{\boldsymbol{\omega}}_c \times \mathbf{r} + \\ \quad + \dfrac{\mu}{\left|\mathbf{r_c} + \mathbf{r}\right|^3}\left(\mathbf{r_c} + \mathbf{r}\right) - \dfrac{\boldsymbol{\mu}}{r_C^3}\mathbf{r_c} = \mathbf{0}, \\ \quad \mathbf{r}\left(t_0\right) = \mathbf{r}_0,\ \dot{\mathbf{r}}\left(t_0\right) = \mathbf{v}_0, \end{cases} \tag{1}$$

where $\mu > 0$ is the gravitational parameter of the attraction center and $\mathbf{r}_0, \mathbf{v}_0$ represent the relative position and relative velocity vectors of the Deputy spacecraft with respect to LVLH at the initial moment of time $t_0 \geq 0$. In (1) vector $\boldsymbol{\omega}_c$ has the expression:

$$\boldsymbol{\omega}_c = \dot{f}_C \frac{\mathbf{h}_c}{h_c} = \frac{1}{r_c^2}\mathbf{h}_c = \left[\frac{1 + e_c \cos f_c\left(t\right)}{p_c}\right]^2 \mathbf{h}_c\,, \tag{2}$$

where vector $\mathbf{r_c}$ is expressed with respect to the LVLH frame and has the form:

$$\mathbf{r_c} = \frac{p_c}{1 + e_c \cos f_c\left(t\right)} \frac{\mathbf{r}_c^0}{r_c^0} \tag{3}$$

and p_c is the conic parameter, $\mathbf{h}_c$ is the angular momentum of the chief, $f_c\left(t\right)$ being its true anomaly.

Let $\boldsymbol{Q}$ be an element from $S\mathbb{O}_3$, which denotes the special orthogonal group of real tensors. The tensor $\boldsymbol{Q}$ gives the attitude of Deputy in relation with LVLH. The initial value problem which has a solution equal to $\boldsymbol{Q} = \boldsymbol{Q}\left(t\right)$ is [5]:

$$\begin{cases} \dot{\boldsymbol{Q}} = \tilde{\boldsymbol{\omega}}\boldsymbol{Q}, \\ \dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}}_c = \boldsymbol{Q}\boldsymbol{J}^{-1}[\boldsymbol{Q}^{\mathrm{T}}\tau - \boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{\omega}+\boldsymbol{\omega}_c)\times \\ \quad \times \boldsymbol{J}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{\omega}+\boldsymbol{\omega}_c)] + \boldsymbol{\omega}\times\boldsymbol{\omega}_c, \\ \boldsymbol{\omega}(t_0) = \boldsymbol{\omega}_0, \boldsymbol{\omega}_0 \in \boldsymbol{V}_3, \\ \boldsymbol{Q}(t_0) = \boldsymbol{Q}_0, \boldsymbol{Q}_0 \in S\mathbb{O}_3, \end{cases} \tag{4}$$

where $\boldsymbol{\omega}$ is the angular velocity of the Deputy in relation with LVLH, $\boldsymbol{\omega}_c$ is the angular velocity of LVLH, $\boldsymbol{\tau}$ is the resulting torque of the forces applied on the Deputy in relation with its mass center, $\boldsymbol{J}$ is the inertia tensor of the Deputy in relation with its mass center, $\boldsymbol{\omega}_0$ is the angular velocity of Deputy in respect to LVLH at time t_0 and $\boldsymbol{Q}_0$ is the attitude of Deputy in respect to LVLH at time t_0. The equations (1) and (4) represent the full body relative orbital motion problem. Their description is a 6-DOF motion of the Deputy in relation with the non-inertial frame LVLH.

The analysis of relative motion began in the early 1960s with the paper of Clohessy and Wiltshire [6], who obtained the equations that model the relative motion in the situation in which the chief spacecraft has a circular orbit and the attraction force is not affected by the Earth oblateness. They linearized the nonlinear initial value problem that models the relative motion by assuming that the relative distance between the two spacecraft remains small during the mission. The Clohessy - Wiltshire equations are still used today in rendezvous maneuvers, but they cannot offer a long-term accuracy because of the secular terms present in the expression of the relative position vector. Independently, Lawden [7], Tschauner and Hempel [8], and Tschauner [9] obtained the solution to the linearized equations of motion when the chief orbit is elliptic, but their solutions still involved secular terms and had singularities. The singularities in the Tschauner - Hempel equations were removed firstly by Carter [10] and by Yamanaka and Andersen [11]. Later, the formation flying concept began to be considered, and the problem of deriving equations for the relative motion with a long-term accuracy degree raised, together with the need to obtain a more accurate solution to the relative orbital motion problem [1]. Gim and Alfriend [12] used the state transition matrix in the study of the relative motion.

The main goal was to express the linearized equations of motion with respect to the initial conditions, with applications in formation initialization and reconfiguration. Attempts to offer more accurate equations of motion starting from the nonlinear initial value problem that models the motion were made. Gurfil and Kasdin [13] derived closed-form expression of the relative position vector, but only when the reference trajectory is circular. Similar expressions for the law of relative motion starting from the nonlinear model are presented in [1, 14–16]. The relative orbital motion problem was also studied from the point of view of the associated differential manifold. Gurfil and Kholshevnikov [17] introduced a metric which helps to study the relative distance between Keplerian orbits. Gronchi [18, 19] also introduced a metric between two confocal Keplerian orbits and used this instrument in problems of asteroid and comet collisions.

In 2007, Condurache and Martinusi [2, 3] offered the closed-form solution to the nonlinear unperturbed model of the relative orbital motion. The method led to closed form vectorial coordinate free expressions for the relative law of motion and relative velocity and it was based on an approach first introduced in 1995 [20]. It involves the Lie group of proper orthogonal tensor functions and its associated Lie algebra of skew-symmetric tensor functions. Then, the solution was generalized to the problem of the relative motion in a central force field [4, 21, 22]. An inedite solution to the Kepler problem by using the algebra of hypercomplex numbers was offered in [23]. Based on this solution and by using the hypercomplex eccentric anomaly, a unified closed-form solution to the relative orbital motion was determined [24]. The relative motion between the leader and the deputy is a six-degrees-of-freedom (6-DOF) motion which represents the coupling of the relative translational motion with the rotational one. In recent years, an increasing attention has been paid to the modeling of the 6-DOF motion of spacecraft [25–27]. Also, controlling the relative pose of satellite formation is a very important research subject [5, 28]. The common approach is to consider the relative translational and rotational dynamics for the chief-deputy spacecraft formation to be modeled using vector and tensor formalism.

The present approach offers a tensor procedure to obtain exact expressions for the relative law of motion and the relative velocity between two Keplerian confocal orbits. The solution is obtained by pure analytical methods and it holds for any chief and deputy trajectories, without involving any secular terms or singularities. The relative orbital motion is reduced, by an adequate change of variables, into the classic Kepler problem. It is proved that the relative orbital motion problem is super integrable. The tensor plays only a catalyst role, the final solution being expressed in a vectorial form.

To obtain this solution, one must know only the inertial motion of the chief spacecraft and the initial conditions of the deputy satellite in the local-vertical-local-horizontal (LVLH) frame. Both the relative law of motion and the relative velocity of the deputy are obtained, by using the tensor instrument that is developed in the first part of the paper. Another contribution is the expression of the solution to the relative orbital motion by using universal functions, in a compact and unified form. Also, a representation theorem is presented, this theorem allows the problem of finding the attitude of the Deputy in relation to LVLH to be solved as Euler fixed point classical problem.

2. MATHEMATICAL PRELIMINARIES

The key notions that are studied in this Section are proper orthogonal tensorial maps and a Sundman-like vectorial regularization, the latter introduced via a vectorial change of variable. The proper orthogonal tensorial maps are related with the skew-symmetric tensorial maps via the Poisson-Darboux equation. The results presented in this section appeared for the first time in [20]. The section related to orthogonal tensorial maps over a powerful instrument in the study of the motion with respect to a non-inertial reference frame.

We denote $S\mathbb{O}_3^{\mathbb{R}}$ the set of maps defined on the set of real numbers $\mathbb{R}$ with values in the set of proper orthogonal tensors $S\mathbb{O}_3^{\mathbb{R}}$

$$S\mathbb{O}_3^{\mathbb{R}} = \left\{ \boldsymbol{R} : \mathbb{R} \to S\mathbb{O}_3^{\mathbb{R}} \middle| \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}_3, \det \boldsymbol{R} = 1 \right\}. \tag{5}$$

We denote $s\mathbf{o}_3^R$ the set of maps defined on the set of real numbers $\mathbb{R}$ with values in the set of skew-symmetric tensors $s\mathbf{o}_3^R$:

$$s\mathbf{o}_3^R = \left\{ \tilde{\boldsymbol{\omega}} : \mathbb{R} \to s\mathbf{o}_3^R \middle| \tilde{\boldsymbol{\omega}}^T = -\tilde{\boldsymbol{\omega}} \right\}. \tag{6}$$

We denote $\boldsymbol{V}_3^{\mathbb{R}}$ to be the set of applications that can be on $\mathbb{R}$ with values in the free vectors set with dimension 3, namely $\boldsymbol{V}_3$.

Theorem 1: *The initial value problem:*

$$\dot{\boldsymbol{Q}} + \tilde{\boldsymbol{\omega}}\boldsymbol{Q} = \mathbf{0}, \boldsymbol{Q}(t_0) = \boldsymbol{I}_3 \tag{7}$$

has a unique solution $\boldsymbol{Q} \in S\mathbb{O}_3^{\mathbb{R}}$ *for any continuous map* $\tilde{\boldsymbol{\omega}} \in s\mathbf{o}_3^{\mathbb{R}}$.

Proof: Let $\boldsymbol{Q}$ be the solution of (7) and denote by $\boldsymbol{Q}^T$ its transpose. Computing

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}\boldsymbol{Q}^T\right) = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{Q}}^T = \boldsymbol{Q}\tilde{\boldsymbol{\omega}}\boldsymbol{Q}^T - \boldsymbol{Q}\tilde{\boldsymbol{\omega}}\boldsymbol{Q}^T = \mathbf{0}_3 \tag{8}$$

it follows that:

$$\boldsymbol{Q}\boldsymbol{Q}^{\boldsymbol{T}} = \boldsymbol{Q}\boldsymbol{Q}^{\boldsymbol{T}}(t_0) = \boldsymbol{I}_3. \tag{9}$$

Since $\boldsymbol{Q} = \boldsymbol{Q}(t)$ is a continuous map, $t \geq t_0$, it follows that $\det(\boldsymbol{Q})$ is a continuous map too. From Eq. (8) it results $\det(\boldsymbol{Q}) \in [-1,1]$. Since $\det(\boldsymbol{Q}(t_0)) = \det \boldsymbol{I}_3 = 1$, it follows that

$$\begin{cases} \boldsymbol{Q}\boldsymbol{Q}^{\boldsymbol{T}} = \boldsymbol{I}_3, \\ \det(\boldsymbol{Q}) = 1, \end{cases} \tag{10}$$

therefore $\boldsymbol{Q} \in S\mathbb{O}_3^{\mathbb{R}}$ is a proper orthogonal tensor map.

Equation (7) represents the tensor form of the Poisson-Darboux equation [29], [30]. Its solution will be denoted $\boldsymbol{R}_{-\boldsymbol{\omega}}$. It models the rotation with instantaneous angular velocity $-\boldsymbol{\omega}(\boldsymbol{\omega}$ is the vector map associated to the skew-symmetric tensor $\tilde{\boldsymbol{\omega}}$). The link between them is given by: $\tilde{\boldsymbol{\omega}}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$, $\forall \mathbf{x} \in \boldsymbol{V}_3^{\mathbb{R}}$; where $\boldsymbol{V}_3^{\mathbb{R}}$ is the three-dimensional linear space of free vectors and ”$\times$” denotes the cross product.

The inverse (in this case the transpose) of tensor $\boldsymbol{R}_{-\boldsymbol{\omega}}$ is denoted:

$$\boldsymbol{R}^{\boldsymbol{T}}{}_{-\boldsymbol{\omega}} = \mathbf{F}_{\boldsymbol{\omega}} . \tag{11}$$

Theorem 2. *The tensor map* $\mathbf{F}_{\boldsymbol{\omega}}$ *satisfies:*

1. $\mathbf{F}_{\boldsymbol{\omega}}$ *is invertible and* $\mathbf{F}_{\boldsymbol{\omega}}^{-1} = \mathbf{F}_{\boldsymbol{\omega}}^{T}$;
2. $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} \cdot \mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} = \mathbf{u} \cdot \mathbf{v}, \forall \mathbf{u}, \mathbf{v} \in \boldsymbol{V}_3^{\mathbb{R}}$;
3. $|\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u}| = |\mathbf{u}|, \forall \mathbf{u} \in \boldsymbol{V}_3^{\mathbb{R}}$;
4. $\mathbf{F}_{\boldsymbol{\omega}}(\mathbf{u} \times \mathbf{u}) = \mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} \times \mathbf{F}_{\boldsymbol{\omega}}\mathbf{v}, \forall \mathbf{u}, \mathbf{v} \in \boldsymbol{V}_3^{\mathbb{R}}$;
5. $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}_{\boldsymbol{\omega}} = \mathbf{F}_{\boldsymbol{\omega}}(\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}), \forall \mathbf{u} \in \boldsymbol{V}_3^{\mathbb{R}}$, *differentiable*;
6. $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{F}_{\boldsymbol{\omega}} = \mathbf{F}_{\boldsymbol{\omega}}(\ddot{\mathbf{u}} + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{u}) + \dot{\boldsymbol{\omega}} \times \mathbf{u}), \forall \mathbf{u} \in \boldsymbol{V}_3^{\mathbb{R}}$.

If vector $\boldsymbol{\omega}$ has fixed direction, given by the unit vector $\mathbf{u}$; $\boldsymbol{\omega} = \omega(t)\mathbf{u}$ with ω a continuous real valued map, the Poisson-Darboux equation (6) has the explicit solution:

$$\boldsymbol{R}_{-\boldsymbol{\omega}} = \mathbf{I}_3 - (\sin\varphi)\tilde{\mathbf{u}} + (1 - \cos\varphi)\tilde{\mathbf{u}}^2, \tag{12}$$

where $\varphi(t) = \int_{t_0}^{t} \omega(s)\mathrm{d}s$. Following from Eq ((11), if vector $\boldsymbol{\omega}$ is constant and nonzero, the solution to the Poisson-Darboux equation is written as:

$$\boldsymbol{R}_{-\boldsymbol{\omega}} = \mathbf{I}_3 - [\sin\omega(t - t_0)]\frac{\tilde{\boldsymbol{\omega}}}{\omega} + [1 - \cos\omega(t - t_0)]\frac{\tilde{\boldsymbol{\omega}}^2}{\omega} . \tag{13}$$

We introduce a vectorial operator which is related to the angular velocity $\boldsymbol{\omega}$ of the reference frame to whom an arbitrary vector is related. It is a derivation-like operator and its use will be revealed further.

We define operator $\left(\ \right)^{'} : \boldsymbol{V}_3^{\mathbb{R}} \to \boldsymbol{V}_3^{\mathbb{R}}$ by

$$\left(\ \right)^{'} = \dot{\left(\ \right)} + \boldsymbol{\omega} \times \left(\ \right). \tag{14}$$

For an arbitrary vectorial map $\boldsymbol{u} : \mathbb{R} \to \boldsymbol{V}_3^{\mathbb{R}}$, it will hold:

$$\boldsymbol{u}' = \dot{\boldsymbol{u}} + \boldsymbol{\omega} \times \boldsymbol{u}. \tag{15}$$

The next results present the properties of this operator, together with the link between $\left(\ \right)^{'}$ and $\boldsymbol{F}_{\boldsymbol{\omega}}$.

Lemma 1. *The following affirmations hold true:*

1. $\boldsymbol{\omega}' = \dot{\boldsymbol{\omega}}$;

2. $\left(\mathbf{u} + \mathbf{v}\right)^{'} = \mathbf{u}' + \mathbf{v}', \left(\forall\right)\mathbf{u}, \mathbf{v} \in C^2\left(\boldsymbol{V}_3^{\mathbb{R}}\right)$;

3. $\left(\lambda\mathbf{u}\right)^{'} = \dot{\lambda}\mathbf{u} + \boldsymbol{\lambda}\left(\mathbf{u}\right)^{\prime}, \left(\forall\right)\mathbf{u} \in C^2\left(\boldsymbol{V}_3^{\mathbb{R}}\right), \left(\forall\right)\lambda : \mathbb{R} \to \mathbb{R}$, *differentiable*;

4. $\left(\mathbf{u} + \mathbf{v}\right)^{'} = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}^{'}, \left(\forall\right)\mathbf{u}, \mathbf{v} \in C^2\left(\boldsymbol{V}_3^{\mathbb{R}}\right)$;

5. $\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}^{'} = \dot{\mathbf{u}} \cdot \mathbf{v} + \mathbf{u} \cdot \dot{\mathbf{v}} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{u} \cdot \mathbf{v}\right), \left(\forall\right)\mathbf{u}, \mathbf{v} \in C^2\left(\boldsymbol{V}_3^{\mathbb{R}}\right)$;

6. $\mathbf{u}'' = \ddot{\mathbf{u}} + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{u}\right) + \dot{\boldsymbol{\omega}} \times \mathbf{u}, \left(\forall\right)\mathbf{u} \in C^2\left(\boldsymbol{V}_3^{\mathbb{R}}\right)$;

7. $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u}\right) = \mathbf{F}_{\boldsymbol{\omega}}\left(\mathbf{u}'\right), \left(\forall\right)\mathbf{u} \in C^2\left(\boldsymbol{V}_3^{\mathbb{R}}\right)$;

8. $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u}\big|_{t=t_0} = \mathbf{u}\left(t_0\right); \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u}\right)\Big|_{t=t_0} = \dot{\mathbf{u}}\left(t_0\right) + \boldsymbol{\omega}\left(t_0\right) \times \mathbf{u}\left(t_0\right)$.

Lemma 2. *Let* $\boldsymbol{u} : \mathbb{R}_+ \to \boldsymbol{V}_3^{\mathbb{R}}$ *be a differential vectorial valued map such as:*

$$\boldsymbol{u}' = \mathbf{0}, \boldsymbol{u}\left(t_0\right) = \boldsymbol{u}_0. \tag{16}$$

Then

$$\boldsymbol{u} = \boldsymbol{R}_{-\omega}\boldsymbol{u}_0, \tag{17}$$

where $\boldsymbol{R}_{-\boldsymbol{\omega}}$ *is the solution of the initial value problem* (7).

Proof. From $\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{R}_{-\omega}\boldsymbol{u}_0\right)=\dot{\boldsymbol{R}}_{-\omega}\boldsymbol{u}_0=-\tilde{\boldsymbol{\omega}}\boldsymbol{R}_{-\omega}\boldsymbol{u}_0$ results $\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{R}_{-\omega}\boldsymbol{u}_0\right)+\boldsymbol{\omega}\times\boldsymbol{R}_{-\omega}\boldsymbol{u}_0=\mathbf{0}$. The solution of the Cauchy problem (13) is $\boldsymbol{u}=\boldsymbol{R}_{-\omega}\boldsymbol{u}_0$.

Remark 1. From Lemma 2 it results that if a vectorial map $\boldsymbol{u}:\mathbb{R}_+\to\boldsymbol{V}_3^{\mathbb{R}}$ satisfies $\boldsymbol{u}'=\mathbf{0}$, then vector $\boldsymbol{u}$ is the rotation with the angular velocity $-\boldsymbol{\omega}$ of a constant $\boldsymbol{u}_0=\boldsymbol{u}\left(t_0\right)$. It will be useful in giving a geometrical interpretation for the prime integrals that occur in the two-body problem in the non-inertial reference frames.

3. CLOSED-FORM SOLUTION TO THE RELATIVE ORBITAL MOTION PROBLEM – TRANSLATION PART

In this section we present the closed-form, coordinate-free exact solution to equation (1). In the initial value problem (1), we make the change of variable:

$$\mathbf{r}_*=\mathbf{F}_{\boldsymbol{\omega}_c}\left(\mathbf{r}+\mathbf{r}_c\right), \tag{18}$$

where $\mathbf{r}_c$ is the solution of the initial value problem:

$$\begin{cases}\ddot{\mathbf{r}}_c+2\boldsymbol{\omega}_c\times\dot{\mathbf{r}}_c+\boldsymbol{\omega}_c\times\left(\boldsymbol{\omega}_c\times\mathbf{r}_c\right)+\\ \qquad+\dot{\boldsymbol{\omega}}_c\times\mathbf{r}_c-\dfrac{\mu}{r_c^3}\mathbf{r}_c=\mathbf{0},\\ \qquad\mathbf{r}_c\left(t_0\right)=\mathbf{r}_c^0,\dot{\mathbf{r}}_c\left(t_0\right)=\dot{\mathbf{r}}_c^0.\end{cases} \tag{19}$$

After some algebra, it follows that:

$$\ddot{\mathbf{r}}_c=\mathbf{F}_{\boldsymbol{\omega}_c}\left\{\left(\ddot{\mathbf{r}}+\ddot{\mathbf{r}}_c\right)+2\boldsymbol{\omega}_c\times\left(\dot{\mathbf{r}}+\mathbf{r}_c\right)+\boldsymbol{\omega}_c\times\left(\boldsymbol{\omega}_c\times\left(\mathbf{r}+\mathbf{r}_c\right)\right)+\dot{\boldsymbol{\omega}}_c\times\left(\mathbf{r}+\mathbf{r}_c\right)\right\} \tag{20}$$

and furthermore:

$$\begin{aligned}\ddot{\mathbf{r}}_*=\mathbf{F}_{\boldsymbol{\omega}_c}\{\ddot{\mathbf{r}}+2\boldsymbol{\omega}_c\times\dot{\mathbf{r}}+\boldsymbol{\omega}_c\times\left(\boldsymbol{\omega}_c\times\mathbf{r}\right)+\dot{\boldsymbol{\omega}}_c\times\mathbf{r}\}+\\ +\mathbf{F}_{\boldsymbol{\omega}_c}\{\ddot{\mathbf{r}}+2\boldsymbol{\omega}_c\times\dot{\mathbf{r}}_c+\boldsymbol{\omega}_c\times\left(\boldsymbol{\omega}_c\times\mathbf{r}\right)+\dot{\boldsymbol{\omega}}_c\times\mathbf{r}_c\}.\end{aligned} \tag{21}$$

Using equations (1) and (19) we obtain:

$$\ddot{\mathbf{r}}_*=\mathbf{F}_{\boldsymbol{\omega}_C}\left[\frac{\mu}{r_C^3}\mathbf{r}_C-\frac{\mu}{\left|\mathbf{r}+\mathbf{r}_C\right|^3}\left(\mathbf{r}+\mathbf{r}_C\right)-\frac{\mu}{r_C^3}\mathbf{r}_C\right]=-\frac{\mu}{\left|\mathbf{r}+\mathbf{r}_C\right|^3}\mathbf{F}_{\boldsymbol{\omega}_C}\left(\mathbf{r}+\mathbf{r}_C\right) \tag{22}$$

which leads to:

$$\ddot{\mathbf{r}}_c + \frac{\mu}{r_*^3}\mathbf{r}_* = \mathbf{0} \tag{23}$$

The initial conditions for equation (23) are deduced by taking into account $\mathbf{F}_{\omega_c}\left(t_0\right) = \mathbf{I}_3$, equation (5) and Theorem 2:

$$\mathbf{r}_*\left(t_0\right) = \mathbf{r}_c^0 + \mathbf{r}_0 \overset{\text{def}}{=} \mathbf{r}_*^0, \tag{24}$$

$$\dot{\mathbf{r}}_*\left(t_0\right) = \mathbf{v}_c^0 + \mathbf{v}_0 + \boldsymbol{\omega}_c\left(t_0\right) \times \mathbf{r}_0 \overset{\text{def}}{=} \dot{\mathbf{r}}_*^0, \tag{25}$$

where $\mathbf{r}_c^0 = \mathbf{r}_c\left(t_0\right); \mathbf{v}_c^0 = \dot{\mathbf{r}}_c\left(t_0\right) + \boldsymbol{\omega}_c\left(t_0\right) \times \mathbf{r}_c^0$.

From (18) and (11) we deduce:

$$\mathbf{r} = \boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \mathbf{r}_c. \tag{26}$$

The solution to the relative orbital motion problem, described by the initial value problem (1) is:

$$\mathbf{r} = \boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \frac{p_C}{1 + e_C \cos f_C\left(t\right)} \frac{\mathbf{r}_C^0}{r_C^0}, \tag{27}$$

where $\boldsymbol{R}_{-\omega_C} = \boldsymbol{I}_3 - \sin f_C^0 \dfrac{\tilde{\mathbf{h}}_C}{\boldsymbol{h}_C} + \left(1 - \cos f_C^0\right) \dfrac{\tilde{\mathbf{h}}_C^2}{\boldsymbol{h}_C^2}$ with $f_C^0 = f_C\left(t\right) - f_C\left(t_0\right)$, is the solution of equation (7) where $\mathbf{r}_*$ is the solution to the initial value problem:

$$\begin{gathered} \ddot{\mathbf{r}}_* + \frac{\mu}{r_*^3}\mathbf{r}_* = 0; \\ \mathbf{r}_*\left(t_0\right) = \mathbf{r}_*^0; \dot{\mathbf{r}}_*\left(t_0\right) = \dot{\mathbf{r}}_*^0 \end{gathered} \tag{28}$$

and the relative velocity may be computed as:

$$\mathbf{v} = \boldsymbol{R}_{-\omega_C}\dot{\mathbf{r}}_* - \tilde{\boldsymbol{\omega}}\boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \frac{e_C\left|\tilde{\mathbf{h}}_C\right| \sin f_C\left(t\right)}{p_C} \frac{\mathbf{r}_C^0}{r_C^0}. \tag{29}$$

This result shows a very interesting property of the relative orbital motion problem (1). We have proven that this problem is super-integrable, by reducing it to the classic Kepler problem (28). The solution of the relative orbital motion problem is expressed thus:

$$\begin{aligned}\mathbf{r} &= \mathbf{r}\left(t, t_0, \mathbf{r}_0, \mathbf{v}_0\right),\\ \mathbf{v} &= \mathbf{v}\left(t, t_0, \mathbf{r}_0, \mathbf{v}_0\right).\end{aligned} \tag{30}$$

Next we present the explicit solution to the relative orbital motion by Battin universal functions. Let $U_k, k = \{0,1,2,3\}$, $U_k = U_k(\chi, \alpha)$ be the universal functions defined in [31] with:

$$\alpha = \frac{2}{\left|\mathbf{r}_*^0\right|} - \frac{\left|\dot{\mathbf{r}}_*^0\right|^2}{\mu} = -\mu\xi \tag{31}$$

and χ a Sudman-like independent universal variable that satisfies:

$$\frac{\mathrm{d}t}{\mathrm{d}\chi} = \frac{1}{\sqrt{\mu}} \mathrm{r}_* . \tag{32}$$

Then, the solution to the initial value problem (28) may be expressed as:

$$\mathbf{r}_* = \left[U_0 - \frac{1}{\left|\mathbf{r}_*^0\right|} U_2\right] \mathbf{r}_*^0 + \left(U_1 \frac{\left|\mathbf{r}_*^0\right|}{\sqrt{\mu}} + U_2 \frac{\mathbf{r}_*^0 \cdot \dot{\mathbf{r}}_*^0}{\mu}\right) \dot{\mathbf{r}}_*^0 , \tag{33}$$

and the magnitude of the solution is:

$$r_* = \left|\mathbf{r}_*^0\right| U_0 + \frac{\mathbf{r}_*^0 \cdot \dot{\mathbf{r}}_*^0}{\mu} U_1 + U_2 . \tag{34}$$

The velocity of the motion governed by equation (28) is:

$$\dot{\mathbf{r}}_* = -\frac{\sqrt{\mu}}{r_*} U_1 \frac{\mathbf{r}_*^0}{\left|\mathbf{r}_*^0\right|} + \left(1 - \frac{1}{r_*} U_2\right) \dot{\mathbf{r}}_*^0 \tag{35}$$

Then, using (26) and (29) together with (33) and (35), the solution to the initial value problem (1) may be written as:

$$\begin{aligned}\mathbf{r} = \boldsymbol{R}_{-\omega_C} &\left\{\left[U_0 - \frac{1}{\left|\mathbf{r}_*^0\right|} U_2\right] \mathbf{r}_*^0 + \left(U_1 \frac{\left|\mathbf{r}_*^0\right|}{\sqrt{\mu}} + U_2 \frac{\mathbf{r}_*^0 \cdot \dot{\mathbf{r}}_*^0}{\mu}\right) \dot{\mathbf{r}}_*^0 \right\} - \\ &- \frac{p_C}{1 + e_C \cos f_C(t)} \frac{\mathbf{r}_C^0}{r_C^0} ,\end{aligned} \tag{36}$$

while the velocity vector is

$$\mathbf{v}=\boldsymbol{R}_{-\omega_C}\left[-\frac{\sqrt{\mu}}{r_*}U_1\frac{\mathbf{r}_*^0}{\left|\mathbf{r}_*^0\right|}+\left(1-\frac{1}{r_*}U_2\right)\dot{\mathbf{r}}_*^0\right]-$$
$$-\tilde{\boldsymbol{\omega}}_C\mathbf{R}_{-\omega_C}\left\{\left[U_0-\frac{1}{\left|\mathbf{r}_*^0\right|}U_2\right]\mathbf{r}_*^0+\left(U_1\frac{\left|\mathbf{r}_*^0\right|}{\sqrt{\mu}}+U_2\frac{\mathbf{r}_*^0\cdot\dot{\mathbf{r}}_*^0}{\mu}\right)\dot{\mathbf{r}}_*^0\right\}- \tag{37}$$
$$-\frac{e_C\left|\mathbf{h}_C\right|\sin f_C(t)}{p_C}\frac{\mathbf{r}_C^0}{r_C^0},$$

where f_C is the true anomaly of the chief spacecraft and $\boldsymbol{R}_{-\omega_C}=\boldsymbol{I}-\sin f_c^0\frac{\tilde{\boldsymbol{h}}_C}{\mathrm{h_c}}+\left(1-\cos f_c^0\right)\frac{\tilde{\boldsymbol{h}}_C^{\,2}}{\mathrm{h_c}^2}$ with $f_C^0=f_C(t)-f_C(t_0)$.

The universal functions U_k are linked by a Kepler-like equation [31]:

$$\sqrt{\mu}\left(t-t_0\right)=U_1(\chi,\alpha)\left|\mathbf{r}_*^0\right|+U_2(\chi,\alpha)\frac{\mathbf{r}_*^0\cdot\dot{\mathbf{r}}_*^0}{\sqrt{\mu}}+U_3(\chi,\alpha). \tag{38}$$

Equation (37) and (38) offer the closed-form compact solution to the relative orbital motion problem. They hold for all types of reference trajectories of the chief (elliptic, parabolic, hyperbolic) and deputy (elliptic, parabolic, hyperbolic, rectilinear).

4. EXACT SOLUTION TO THE RELATIVE ORBITAL MOTION PROBLEM - ROTATIONAL PART

In this section we give a representation theorem for the tensor $\boldsymbol{Q}\in S\mathbb{O}_3^{\mathbb{R}}$ which parametrizes the rotation of the Deputy around its mass center, motion that is recovered from the initial value problem (4).
For (4), consider the following change of variable:

$$\boldsymbol{\omega}_*=\boldsymbol{Q}^T\left(\boldsymbol{\omega}+\boldsymbol{\omega}_c\right). \tag{39}$$

This change of variable leads to $\dot{\boldsymbol{\omega}}_*=\dot{\boldsymbol{Q}}^T\left(\boldsymbol{\omega}+\boldsymbol{\omega}_C\right)+\boldsymbol{Q}^T\left(\dot{\boldsymbol{\omega}}+\dot{\boldsymbol{\omega}}_C\right)=-\boldsymbol{Q}^T\tilde{\boldsymbol{\omega}}\left(\boldsymbol{\omega}+\boldsymbol{\omega}_C\right)+\boldsymbol{Q}^T\left(\dot{\boldsymbol{\omega}}+\dot{\boldsymbol{\omega}}_C\right)$. The result is equivalent with $\dot{\boldsymbol{\omega}}_*=\boldsymbol{Q}^T\left(\boldsymbol{\omega}_C\times\boldsymbol{\omega}+\dot{\boldsymbol{\omega}}+\dot{\boldsymbol{\omega}}_C\right)$ or

$$\boldsymbol{\omega}_C\times\boldsymbol{\omega}+\dot{\boldsymbol{\omega}}+\dot{\boldsymbol{\omega}}_C=\boldsymbol{Q}\dot{\boldsymbol{\omega}}_*. \tag{40}$$

After some steps of algebraic calculus, from (39), (40) and (4), results that

$$\begin{cases} \boldsymbol{J}\dot{\boldsymbol{\omega}}_* + \boldsymbol{\omega}_* \times \boldsymbol{J}\boldsymbol{\omega}_* = \boldsymbol{\tau}_*, \\ \boldsymbol{\omega}_*(t_0) = \boldsymbol{Q}_0^T\left(\boldsymbol{\omega}_0 + \boldsymbol{\omega}_C(t_0)\right), \end{cases} \tag{41}$$

where $\boldsymbol{\tau}_* = \boldsymbol{Q}^T\boldsymbol{\tau}$ is the in-body torque related to the mass center in the body frame of Deputy. Equation (41) is a Euler fixed point classic problem. If $\boldsymbol{Q}$ is the solution of (4) then:

$$\begin{cases} \dot{\boldsymbol{Q}} = \tilde{\boldsymbol{\omega}}\boldsymbol{Q}, \\ \boldsymbol{Q}(t_0) = \boldsymbol{Q}_0. \end{cases} \tag{42}$$

Making use of (39), results that $\boldsymbol{Q}\boldsymbol{\omega}_* = \boldsymbol{\omega} + \boldsymbol{\omega}_C$. If the $\tilde{}$ operator is used the previous calculus is transformed into $\widetilde{\boldsymbol{Q}\boldsymbol{\omega}_*} = \tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}_C \iff \boldsymbol{Q}\tilde{\boldsymbol{\omega}}_*\boldsymbol{Q}^T = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \tilde{\boldsymbol{\omega}}_C$.
After multiplying the last expression by $\boldsymbol{Q}$, we obtain the initial value problem:

$$\begin{cases} \dot{\boldsymbol{Q}} = \boldsymbol{Q}\tilde{\boldsymbol{\omega}}_* - \tilde{\boldsymbol{\omega}}_C\boldsymbol{Q}, \\ \boldsymbol{Q}(t_0) = \boldsymbol{Q}_0. \end{cases} \tag{43}$$

Using the variable change (39), the initial value problem (4) has been decoupled into two distinct initial value problems (41) and (43). Considering $\boldsymbol{Q} = \boldsymbol{R}_{-\omega_C}\boldsymbol{Q}_*$, a representation theorem is valid.

Theorem 3. *The solution of* (4) *results from the application of* $\boldsymbol{R}_{-\omega_C}$ *to the solution of the classical Euler fixed point problem:*

$$\begin{cases} \dot{\boldsymbol{Q}}_* = \boldsymbol{Q}_*\tilde{\boldsymbol{\omega}}_*, \\ \boldsymbol{J}\dot{\boldsymbol{\omega}}_* + \boldsymbol{\omega}_* \times \boldsymbol{J}\boldsymbol{\omega}_* = \boldsymbol{\tau}_*, \\ \boldsymbol{\omega}_*(t_0) = \boldsymbol{Q}_0^T\left(\boldsymbol{\omega}_0 + \boldsymbol{\omega}_C(t_0)\right), \\ \boldsymbol{Q}_*(t_0) = \boldsymbol{Q}_0. \end{cases} \tag{44}$$

5. RIGID BODY MOTION PARAMETERIZATION USING DUAL LIE ALGEBRA

The key notions that will be presented in this section are dual tensorial and vectorial parameterizations that can be used to properly describe the rigid-body motion. We discuss the properties of proper orthogonal dual tensorial maps. Orthogonal dual tensorial maps are a powerful instrument in the study of the rigid motion with respect to an inertial and non-inertial reference frames. More on dual numbers, dual vectors and dual tensors can be found in [32–39].
Let the orthogonal dual tensor set be denoted by:

$$\underline{S\mathbb{O}}_3 = \left\{\underline{\boldsymbol{R}} \in \mathbf{L}\left(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3\right) \middle| \underline{\boldsymbol{R}}\underline{\boldsymbol{R}}^T = \underline{\boldsymbol{I}},\ \det\underline{\boldsymbol{R}} = 1\right\}, \tag{45}$$

where $\underline{S\mathbb{O}}_3$ is the set of special orthogonal dual tensors and $\underline{\boldsymbol{I}}$ is the unit orthogonal dual tensor.

The internal structure of any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$ is illustrated in a series of results which were detailed in our previous work [34, 35, 40].

Theorem 4. (Structure Theorem). *For any* $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$ *a unique decomposition is viable:*

$$\underline{\boldsymbol{R}} = \left(\boldsymbol{I} + \varepsilon\tilde{\boldsymbol{\rho}}\right)\boldsymbol{Q}\,, \tag{46}$$

where $\boldsymbol{Q} \in S\mathbb{O}_3$ *and* $\boldsymbol{\rho} \in V_3$ *are called* ***structural invariants.***

Taking into account the Lie group structure of $\underline{S\mathbb{O}}_3$ and the result presented in previous theorem, it can be concluded that any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$ can be used globally parameterize displacements of rigid bodies.

Theorem 5 *For any orthogonal dual tensor* $\underline{\boldsymbol{R}}$ *defined as in Eq.* (46)*, a dual number* $\underline{\alpha} = \alpha + \varepsilon d$ *and a dual unit vector* $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{u}_0$ *can be computed to have the following equation* [34, 35]:

$$\underline{\boldsymbol{R}}\left(\underline{\alpha},\underline{\mathbf{u}}\right) = \boldsymbol{I} + \sin\underline{\alpha}\,\underline{\tilde{\boldsymbol{u}}} + \left(1-\cos\underline{\alpha}\right)\underline{\tilde{\boldsymbol{u}}}^2 = \exp\left(\underline{\alpha}\,\underline{\tilde{\boldsymbol{u}}}\right). \tag{47}$$

The parameters $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are called the **natural invariants** of $\underline{\boldsymbol{R}}$**.** The unit dual vector $\underline{\mathbf{u}}$ gives the Plücker representation of the Mozzi-Chalses axis [33], [41]. while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translated distance d .

The Lie algebra of the Lie group $\underline{S\mathbb{O}}_3$ is the skew-symmetric dual tensor set denoted by $\underline{\mathbf{so}}_3 = \left\{\underline{\tilde{\boldsymbol{\alpha}}} \in \mathbf{L}\left(\underline{\mathbf{V}}_3,\underline{\mathbf{V}}_3\right)\middle|\,\underline{\tilde{\boldsymbol{\alpha}}} = -\underline{\tilde{\boldsymbol{\alpha}}}^{\mathrm{T}}\right\}$, where the internal mapping is $\left[\underline{\tilde{\boldsymbol{\alpha}}}_1,\underline{\tilde{\boldsymbol{\alpha}}}_2\right] = \widetilde{\underline{\tilde{\boldsymbol{\alpha}}}_1\underline{\boldsymbol{\alpha}}_2}$.

The link between the Lie algebra $\underline{\mathbf{so}}_3$, the Lie group $\underline{S\mathbb{O}}_3$, and the exponential map is given by the following.

Theorem 6. *The mapping*

$$\begin{gathered}\exp : \underline{\mathbf{so}}_3 \to \underline{\mathrm{S}\mathbb{O}}_3,\\ \exp\left(\underline{\tilde{\boldsymbol{\alpha}}}\right) = e^{\underline{\tilde{\boldsymbol{\alpha}}}} = \sum_{\mathrm{k}=0}^{\infty}\frac{\underline{\tilde{\boldsymbol{\alpha}}}^{\mathrm{k}}}{\mathrm{k}!}\end{gathered} \tag{48}$$

is well defined and surjective.

Any screw axis that embeds a rigid displacement can be parameterized by a unit dual vector, whereas the screw parameters (angle of rotation about the screw and the translation along the screw axis) can be structured as a dual angle. The computation of the screw axis is linked with the problem of finding the logarithm of an orthogonal dual tensor $\underline{\boldsymbol{R}}$, which is a multifunction defined by:

$$\begin{aligned} &\log : \underline{\mathrm{S}\mathbb{O}}_3 \to \underline{\mathrm{s}\mathbf{o}}_3, \\ &\log \underline{\boldsymbol{R}} = \left\{ \underline{\tilde{\boldsymbol{\psi}}} \in \underline{\mathrm{s}\mathbf{o}}_3 \,\middle|\, \exp\left(\underline{\tilde{\boldsymbol{\psi}}}\right) = \underline{\boldsymbol{R}} \right\} \end{aligned} \tag{49}$$

and is the inverse of Eq. (44).

Based on Theorem 5 and Theorem 6, for any orthogonal dual tensor $\underline{\boldsymbol{R}}$, a dual vector $\underline{\boldsymbol{\psi}} = \underline{\alpha}\underline{\mathbf{u}} = \boldsymbol{\omega} + \varepsilon \mathbf{v}$ can be computed and it represents the Euler dual vector, which embeds the screw axis and screw parameters. The form of $\underline{\boldsymbol{\psi}}$ implies that $\underline{\tilde{\boldsymbol{\psi}}} \in \log \underline{\boldsymbol{R}}$.

Also, if $\left\|\boldsymbol{\omega}\right\| < 2\pi$, Theorem 5 and Theorem 6 can be used to uniquely recover the Euler dual vector $\underline{\boldsymbol{\psi}}$, which is equivalent with computing $\log \underline{\boldsymbol{R}}$.

Next, we'll introduce the isomorphism between the Lie group $S\mathbb{E}_3$ and the Lie group $\underline{S\mathbb{O}}_3$ [42].

Theorem 7. (Isomorphism Theorem): *The special Euclidean group* $\left(S\mathbb{E}_3, \cdot\right)$ *and* $\left(\underline{S\mathbb{O}}_3, \cdot\right)$ *are connected via the isomorphism of the Lie groups:*

$$\begin{aligned} &\Phi : S\mathbb{E}_3 \to \underline{S\mathbb{O}}_3, \\ &\Phi\left(g\right) = \left(\boldsymbol{I} + \varepsilon \tilde{\boldsymbol{\rho}}\right) \boldsymbol{Q}, \end{aligned} \tag{50}$$

where $g = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$, $\boldsymbol{\Phi} \in S\mathbb{O}_3$, $\boldsymbol{\rho} \in \boldsymbol{V}_3$.

The Lie algebra $\mathfrak{se}\left(3\right)$ *and* $\underline{\boldsymbol{V}}_3$ *are connected via the isomorphism of the Lie algebras:*

$$\begin{aligned} &\varphi : \mathfrak{se}\left(3\right) \to \underline{\boldsymbol{V}}_3, \\ &\varphi\left(\hat{\xi}\right) = \boldsymbol{\omega} + \varepsilon \mathbf{v}, \end{aligned} \tag{51}$$

where $\hat{\xi} = \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix}$, $\tilde{\boldsymbol{\omega}} \in \mathrm{s}\mathbf{o}_3$, $\mathbf{v} \in \boldsymbol{V}_3$.

Proof: For any $g_1, g_2 \in S\mathbb{E}_3$, the map defined in Eq. (51) yields

$$\Phi\left(g_1 \cdot g_2\right) = \Phi\left(g_1\right) \cdot \Phi\left(g_2\right). \tag{52}$$

Let $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3$. Based on Theorem 4, which ensures a unique decomposition, we can conclude that the only choice for g, such that $\Phi\left(g\right) = \underline{\boldsymbol{R}}$ is $g = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$. This underlines that Φ is a bijection and keeps all the internal operations, where $\boldsymbol{Q}$ and $\boldsymbol{\rho}$ are denoted ast structural invariant of the orthogonal tensor $\boldsymbol{Q}$.

For any $\hat{\xi}_1, \hat{\xi}_2 \in \mathrm{se}(3)$, the mapping defined by Eq.(51) verifies the identity

$$\varphi\left(\left[\hat{\xi}_1, \hat{\xi}_2\right]\right) = \varphi\left(\hat{\xi}_1\right) \times \varphi\left(\hat{\xi}_2\right). \tag{53}$$

Also, for any $\underline{\boldsymbol{\omega}} \in \underline{V}_3$, $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon \mathbf{v}$, there is only determined $\hat{\xi} = \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$ such that $\varphi\left(\hat{\xi}\right) = \underline{\boldsymbol{\omega}}$ .Thus, φ is a bijective mapping.

Remark 2: The inverse of Φ is

$$\Phi^{-1} : \underline{S\mathbb{O}}_3 \leftrightarrow S\mathbb{E}_3; \Phi^{-1}\left(\underline{\boldsymbol{R}}\right) = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{54}$$

where $\boldsymbol{Q} = \mathrm{Re}\left(\underline{\boldsymbol{R}}\right)$, $\boldsymbol{\rho} = \mathrm{vect}\left(Du\left(\underline{\boldsymbol{R}}\right) \cdot \boldsymbol{Q}^{\mathrm{T}}\right)$.

6. FOUCAULT-LIKE PROPERTIES OF RIGID BODY MOTION IN ARBITRARY NON-INERTIAL FRAME

To the authors' knowledge, in the field of astrodynamics there aren't many reports on how the motion of rigid body can be studied in arbitrary non-inertial frames. Next, we proposed a dual tensors-based model for the motion of the rigid body in an arbitrary non-inertial frame. The proposed method eludes the calculus of inertia forces that contributes to the rigid body relative state. So, the coordinate-free state equation of the rigid body motion in an arbitrary non-inertial frame will be obtained.

Let $\underline{\boldsymbol{R}}_D$ and $\underline{\boldsymbol{R}}_C$ be the dual orthogonal tensors which describe the motion of two rigid bodies relative to the inertial frame.

If $\underline{\boldsymbol{R}}$ is the orthogonal dual tensor which embeds the six degree of freedom relative orbital motion of rigid body D relative to rigid body C, then:

$$\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}_C^T \underline{\boldsymbol{R}}_D . \tag{55}$$

Let $\underline{\boldsymbol{\omega}}_{\mathrm{C}}$ denote the dual angular velocity of the rigid body C and $\underline{\boldsymbol{\omega}}_{\mathrm{D}}$ the dual angular velocity of the rigid body D, both being related to inertial reference frame. In the following, the inertial motion of the rigid body C is considered to be known. If $\underline{\boldsymbol{\omega}}$ is the dual angular velocity of the rigid body D relative to the rigid body C, then, conforming with (55):

$$\underline{\boldsymbol{\omega}} = \underline{\boldsymbol{\omega}}_D - \underline{\boldsymbol{\omega}}_C . \tag{56}$$

Considering $\underline{\boldsymbol{\omega}}_D^B$ being the dual angular velocity vector of the rigid body D in the body frame, the dual form of the Euler equation given in [43] results that:

$$\underline{\boldsymbol{M}}\,\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_D^B = \underline{\boldsymbol{\tau}}^B . \tag{57}$$

In (57) $\underline{\boldsymbol{\tau}}^B = \mathbf{F}^B + \varepsilon \boldsymbol{\tau}^{\mathrm{B}}$, where $\mathbf{F}^B$ the force applied in the mass centre and $\boldsymbol{\tau}^B$ is the torque. Also in (57), $\underline{\boldsymbol{M}}$ represents the inertia dual operator, which is given by $\underline{\boldsymbol{M}} = m_D \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\boldsymbol{I} + \varepsilon \boldsymbol{J}$, where $\boldsymbol{J}$ is the inertia tensor of the rigid body D related to its mass center and m_D is the mass of the rigid body D. Combining $\underline{\boldsymbol{M}}^{-1} = \boldsymbol{J}^{-1}\dfrac{\mathrm{d}}{\mathrm{d}\varepsilon} + \varepsilon \dfrac{1}{m_D}\boldsymbol{I}$ with (57) results:

$$\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{M}}^{-1}\left(\underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_D^B\right) = \underline{\boldsymbol{M}}^{-1}\underline{\boldsymbol{\tau}}^{\mathrm{B}} . \tag{58}$$

Considering that $\underline{\boldsymbol{\omega}}_D = \underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B$, the dual angular velocity vector can be computed from

$$\underline{\boldsymbol{\omega}} = \underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B - \underline{\boldsymbol{\omega}}_C , \tag{59}$$

this through differentiation gives:

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{\omega}}_D^B + \underline{\boldsymbol{R}}\dot{\underline{\boldsymbol{\omega}}}_D^B . \tag{60}$$

If the previous equation is multiplied by $\underline{\boldsymbol{R}}^T$, then

$$\underline{\boldsymbol{R}}^T\left(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C\right) = \underline{\boldsymbol{R}}^T\dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{\omega}}_D^B + \dot{\underline{\boldsymbol{\omega}}}_D^B , \tag{61}$$

which combined with $\dot{\underline{\boldsymbol{R}}} = \tilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}$ generates:

$$\underline{\boldsymbol{R}}^T\left(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C\right) = \underline{\boldsymbol{R}}^T\tilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_D^B + \dot{\underline{\boldsymbol{\omega}}}_D^B . \tag{62}$$

After a few steps, equation (62) is transformed into

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\dot{\underline{\boldsymbol{\omega}}}_D^B + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C , \tag{63}$$

which combined with (58) gives:

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}\boldsymbol{M}}^{-1}\underline{\boldsymbol{\tau}}^B - \underline{\boldsymbol{R}\boldsymbol{M}}^{-1}\left(\underline{\boldsymbol{\omega}}_D^B \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_D^B\right) + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C . \tag{64}$$

Because $\underline{\boldsymbol{\omega}}_D^B = \underline{\boldsymbol{R}}^T\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right)$, the final equation is:

$$\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}\boldsymbol{M}}^{-1}\left[\underline{\boldsymbol{\tau}}^{\mathrm{B}} - \underline{\boldsymbol{R}}^T\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right) \times \underline{\boldsymbol{M}\boldsymbol{R}}^T\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right)\right] + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C . \tag{65}$$

The system:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \tilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}, \\ \dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_{\mathrm{C}} = \underline{\boldsymbol{R}\boldsymbol{M}}^{-1}[\underline{\boldsymbol{R}^T\boldsymbol{\tau}} - \underline{\boldsymbol{R}}^T\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right) \times \\ \times \underline{\boldsymbol{M}\boldsymbol{R}}^T\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right)] + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{\omega}}_C, \\ \underline{\boldsymbol{\omega}}(\mathrm{t}_0) = \underline{\boldsymbol{\omega}}_0, \underline{\boldsymbol{\omega}}_0 \in \underline{\boldsymbol{V}}_3, \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0, \underline{\boldsymbol{R}}_0 \in \underline{S\mathbb{O}}_3 \end{cases} \tag{66}$$

is a compact form which can be used to model the 6-DOF relative motion problem. In the previous equation the state of the rigid body D in relation with the rigid body C is modelled by the dual tensor $\underline{\boldsymbol{R}}$ and the dual angular velocities field $\underline{\boldsymbol{\omega}}$. This initial value problem can be used to study the behavior of the rigid body D in relation with the reference frame attached to the rigid body C. In (66), all the vectors are represented in the body frame of C, which shows that the proposed solution is onboard and has the property of being coupled in $\underline{\boldsymbol{R}}$ and $\underline{\boldsymbol{\omega}}$.

Next, we present a procedure that allows the decoupling of the proposed solution.

Theorem 8. (Representation Theorem). *The solution of* (66) *results from the application of the tensor* $\underline{\boldsymbol{R}}_{-\underline{\boldsymbol{\omega}}_C}$ *to the solution of the classical dual Euler fixed point problem:*

$$\begin{cases} \dot{\underline{\boldsymbol{R}}}_* = \underline{\boldsymbol{R}}_* \tilde{\underline{\boldsymbol{\omega}}}_*, \\ \underline{\boldsymbol{M}}\,\dot{\underline{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_*, \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_{*0}, \\ \underline{\boldsymbol{R}}_*(t_0) = \underline{\boldsymbol{R}}_{*0}, \end{cases} \tag{67}$$

where $\underline{\boldsymbol{\omega}}_{*0} = \underline{\boldsymbol{R}}_0^T\left(\underline{\boldsymbol{\omega}}_0 + \underline{\boldsymbol{\omega}}_C(t_0)\right), \underline{\boldsymbol{R}}_{*0} = \left(\boldsymbol{I} + \varepsilon\tilde{\boldsymbol{r}}_C(t_0)\right)\underline{\boldsymbol{R}}_0$, $\underline{\boldsymbol{\tau}}_* = \underline{\boldsymbol{R}}^T\underline{\boldsymbol{\tau}}$, *and* $\underline{\boldsymbol{R}}_{-\underline{\boldsymbol{\omega}}_C}$ *it's the unique solution of the dual replica of Poisson-Darboux equation*:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} + \tilde{\underline{\boldsymbol{\omega}}}_C \underline{\boldsymbol{R}} = \mathbf{0}, \\ \underline{\boldsymbol{R}}(t_0) = \boldsymbol{I} - \varepsilon\tilde{\mathbf{r}}_C(t_0). \end{cases}$$

Proof: In order to describe the solution to (66), we consider the following change of variable:

$$\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{R}}^T\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right). \tag{68}$$

This change of variable leads to $\dot{\underline{\boldsymbol{\omega}}}_* = \dot{\underline{\boldsymbol{R}}}^T\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right) + \underline{\boldsymbol{R}}^T\left(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C\right) = -\underline{\boldsymbol{R}}^T\tilde{\underline{\boldsymbol{\omega}}}\left(\underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C\right) + \underline{\boldsymbol{R}}^T\left(\dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C\right)$. The result is equivalent with $\dot{\underline{\boldsymbol{\omega}}}_* = \underline{\boldsymbol{R}}^T\left(\underline{\boldsymbol{\omega}}_C \times \underline{\boldsymbol{\omega}} + \dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C\right)$ or

$$\underline{\boldsymbol{\omega}}_C \times \underline{\boldsymbol{\omega}} + \dot{\underline{\boldsymbol{\omega}}} + \dot{\underline{\boldsymbol{\omega}}}_C = \underline{\boldsymbol{R}}\dot{\underline{\boldsymbol{\omega}}}_*. \tag{69}$$

After some steps of algebraic calculus, from (68), (69) and (65), results that:

$$\begin{cases} \underline{\boldsymbol{M}}\,\dot{\underline{\boldsymbol{\omega}}}_* + \underline{\boldsymbol{\omega}}_* \times \underline{\boldsymbol{M}}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\tau}}_*, \\ \underline{\boldsymbol{\omega}}_*(t_0) = \underline{\boldsymbol{\omega}}_*^0, \end{cases} \tag{70}$$

where $\underline{\boldsymbol{\tau}}_* = \underline{\boldsymbol{R}}^T\underline{\boldsymbol{\tau}}$ is the dual torque related to the mass center in the body frame of the rigid body D and $\underline{\boldsymbol{\omega}}_*^0 = \underline{\boldsymbol{R}}_0^T\left(\underline{\boldsymbol{\omega}}_0 + \underline{\boldsymbol{\omega}}_C(t_0)\right)$. Equation (70) is a dual Euler fixed point classic problem.

For any $\underline{\boldsymbol{R}} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$, the solution of (66) emerges from

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \tilde{\underline{\boldsymbol{\omega}}}\underline{\boldsymbol{R}}, \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0. \end{cases} \tag{71}$$

Making use of (68) results that $\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_* = \underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C$. If $\tilde{}$ operator used the previous calculus is transformed into $\widetilde{\underline{\boldsymbol{R}}\underline{\boldsymbol{\omega}}_*} = \tilde{\underline{\boldsymbol{\omega}}} + \tilde{\underline{\boldsymbol{\omega}}}_C \Leftrightarrow \underline{\boldsymbol{R}}\tilde{\underline{\boldsymbol{\omega}}}_*\underline{\boldsymbol{R}}^T = \dot{\underline{\boldsymbol{R}}}\underline{\boldsymbol{R}}^T + \tilde{\underline{\boldsymbol{\omega}}}_C$. After multiplying the last expression by $\underline{\boldsymbol{R}}$, we obtain the initial value problem:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\boldsymbol{R}}\tilde{\underline{\boldsymbol{\omega}}}_* - \tilde{\underline{\boldsymbol{\omega}}}_C\underline{\boldsymbol{R}}, \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0. \end{cases} \tag{72}$$

Using the variable change (68), the initial value problem (66) has been decoupled into two distinct initial value problems (70) and (72).

Let $\underline{\boldsymbol{R}}_{-\underline{\boldsymbol{\omega}}_C} \in \underline{S\mathbb{O}}_3^{\mathbb{R}}$ be the unique solution of:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} + \tilde{\underline{\boldsymbol{\omega}}}_C\underline{\boldsymbol{R}} = \mathbf{0}, \\ \underline{\boldsymbol{R}}(t_0) = \boldsymbol{I} - \varepsilon\tilde{\mathbf{r}}_C(t_0). \end{cases} \tag{73}$$

Considering $\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}_{-\underline{\boldsymbol{\omega}}_C}\underline{\boldsymbol{R}}_*$, a representation theorem of the solution of Eq. (66) can be formulated.

7. A DUAL TENSOR FORMULATION OF THE SIX DEGREE OF FREEDOM RELATIVE ORBITAL MOTION PROBLEM

The results from the previous chapter will be used to study the six degrees of freedom relative orbital motion problem using dual algebra.

This problem is also quite important, due to its numerous applications: spacecraft formation flying, rendezvous operations, distributed spacecraft missions [1, 5, 10–12, 26, 44].

The model of the full-body relative motion consists in two spacecraft flying in Keplerian orbits due to the influence of the same gravitational attraction center (Fig. 1). The main problem is to determine the pose of the Deputy satellite with respect to a reference frame originated in the Leader satellite center of mass. This non-inertial reference frame, traditionally named LVLH (Local-Vertical-Local-Horizontal). The angular velocity of the LVLH is given by the vector $\boldsymbol{\omega}_C$, which has the expression:

$$\boldsymbol{\omega}_C = \dot{f}_C \frac{\mathbf{h}_C}{h_C} = \frac{1}{r_C^2}\mathbf{h}_C = \left[\frac{1 + e_C \cos f_C(t)}{p_C}\right]^2 \mathbf{h}_C, \tag{74}$$

where vector $\mathbf{r}_C$ is:

$$\mathbf{r}_C = \frac{p_C}{1 + e_C \cos f_C(t)} \frac{\mathbf{r}_C^0}{r_C^0}, \tag{75}$$

where p_C is the conic parameter, $\mathbf{h}_C$ is the angular momentum of the Chief, $f_C(t)$ being the true anomaly and e_C is the eccentricity of the Chief.
We propose dual tensors-based model for the motion and the pose for the mass center of the Deputy in relation to LVLH. Both the Chief satellite and the Deputy satellite can be considered rigid bodies.

Furthermore, the time derivative of $\mathbf{r}_C$ is:

$$\dot{\mathbf{r}}_C = \frac{e_C|\mathbf{h}_C|\sin f_C(t)}{p_C}\frac{\mathbf{r}_C^0}{r_C^0}. \tag{76}$$

In order to a easier to read the list of notations, for $t = t_0$ there will be used the followings:

$$\boldsymbol{\omega}_C^0 = \left[\frac{1+e_C\cos f_C(t_0)}{p_C}\right]^2 \mathbf{h}_C, \tag{77}$$

$$\dot{\mathbf{r}}_C^0 = \frac{e_C|\mathbf{h}_C|\sin f_C(t_0)}{p_C}\frac{\mathbf{r}_C^0}{r_C^0}, \tag{78}$$

where $\frac{\mathbf{r}_C^0}{r_C^0}$ is the unity vector of the X-axis from LVLH.

The full-body relative orbital motion is described by the Eq.(66) where the dual angular velocity of the Chief satellite is:

$$\underline{\boldsymbol{\omega}}_C = \boldsymbol{\omega}_C + \varepsilon(\dot{\mathbf{r}}_C + \boldsymbol{\omega}_C \times \mathbf{r}_C) \tag{79}$$

and the dual torque related to the mass center of Deputy satellite is:

$$\underline{\boldsymbol{\tau}} = -\frac{\mu}{|\mathbf{r}_c+\mathbf{r}|^3}(\mathbf{r}_c+\mathbf{r}) + \varepsilon\boldsymbol{\tau}. \tag{80}$$

The representation theorem (Theorem 8) is applied in this case using the conditions (76)–(79), the solution of the Poisson-Darboux problem (73) is:

$$\underline{\boldsymbol{R}}_{-\underline{\omega}_C} = \left(\boldsymbol{I} - \varepsilon\tilde{\boldsymbol{r}}_C(t)\right)\left(\boldsymbol{I} - \sin f_c^0\frac{\tilde{\boldsymbol{h}}_C}{h_c} + \left(1-\cos f_c^0\right)\frac{\tilde{\boldsymbol{h}}_C^2}{h_c^2}\right). \tag{81}$$

In (81), we've noted $h_c = \|\mathbf{h_c}\|$ and $f_c^0 = f_c(t) - f_c(t_0)$.

Theorem 9. (Representation Theorem of the full body relative orbital motion). *The solution of (66) results from the application of the tensor* $\underline{\boldsymbol{R}}_{-\underline{\omega}_C}$ *from (81) to the solution of the classical dual Euler fixed point problem (67).*

7.1. The rotational and translational parts of the relative orbital motion

Consider first the real part of (66). This leads to an initial value problem:

$$\begin{cases} \dot{\boldsymbol{Q}} = \tilde{\boldsymbol{\omega}}\boldsymbol{Q}, \\ \dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}}_{\mathrm{c}} = \boldsymbol{Q}\boldsymbol{J}^{-1}[\boldsymbol{Q}^{\mathrm{T}}\tau - \boldsymbol{Q}^{\mathrm{T}}\left(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathrm{c}}\right)\times \\ \times \boldsymbol{J}\boldsymbol{Q}^{\mathrm{T}}\left(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathrm{c}}\right)] + \boldsymbol{\omega}\times\boldsymbol{\omega}_{\mathrm{c}}, \\ \boldsymbol{\omega}\left(\mathrm{t}_0\right) = \boldsymbol{\omega}_0, \boldsymbol{\omega}_0 \in \boldsymbol{V}_3, \\ \boldsymbol{Q}\left(\mathrm{t}_0\right) = \boldsymbol{Q}_0, \boldsymbol{Q}_0 \in \mathrm{S}\mathbb{O}_3, \end{cases} \tag{82}$$

which has the solution $\boldsymbol{Q} = \boldsymbol{Q}(\mathrm{t})$, the real tensor $\boldsymbol{Q}$ being the attitude of Deputy in relation to LVLH. In (82), $\boldsymbol{\omega}$ is the angular velocity of the Deputy in relation to LVLH, $\boldsymbol{\omega_c}$ is the angular velocity of LVLH, $\boldsymbol{\tau}$ is the resulting torque of the forces applied on the Deputy in relation to its mass center, $\boldsymbol{J}$ is the inertia tensor of the Deputy in relation to its mass center, $\boldsymbol{\omega}_0$ is the angular velocity of Deputy in respect to LVLH at time t_0 and $\boldsymbol{Q}_0$ is the orientation of Deputy in respect to LVLH at time t_0.

Consider now the dual part of Eq. (66). Taking into account the internal structure of $\underline{\boldsymbol{R}}$, which is given by (47), after some basic algebraic calculus we obtain a second initial value problem that models the translation of the Deputy satellite mass center with respect to the LVLH reference frame:

$$\begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega}_c \times \dot{\mathbf{r}} + \boldsymbol{\omega}_c \times \left(\boldsymbol{\omega}_{\mathrm{c}} \times \mathbf{r}\right) + \dot{\boldsymbol{\omega}}_c \times \mathbf{r} + \\ + \dfrac{\mu}{\left|\mathbf{r}_c + \mathbf{r}\right|^3}\left(\mathbf{r}_c + \mathbf{r}\right) - \dfrac{\mu}{r_c^3}\mathbf{r}_c = \mathbf{0}, \\ \mathbf{r}\left(t_0\right) = \mathbf{r}_0, \dot{\mathbf{r}}\left(t_0\right) = \mathbf{v}_0, \end{cases} \tag{83}$$

where $\mu > 0$ is the gravitational parameter of the attraction center and $\mathbf{r}_0, \mathbf{v}_0$ represent the relative position and relative velocity vectors of the mass center of the Deputy spacecraft with respect to LVLH at the initial moment of time $\mathrm{t}_0 \geq 0$.
Based on the *representation of* Theorem 9, the following theorem results.

Theorem 10. *The solutions to problems* (82) *and* (83) *are given by*

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{R}_{-\omega_C}\boldsymbol{Q}_*, \\ \mathbf{r} &= \boldsymbol{R}_{-\omega_C}\mathbf{r}_* - \mathbf{r}_{\mathrm{c}}, \end{aligned} \tag{84}$$

where $\boldsymbol{Q}_*$ *and* $\mathbf{r}_*$ *are the solutions of the the classical Euler fixed point problem and, respectively, Kepler's problem*:

$$\begin{cases} \dot{\boldsymbol{Q}}_* = \boldsymbol{Q}_*\tilde{\boldsymbol{\omega}}_*, \\ \boldsymbol{J}\dot{\boldsymbol{\omega}}_* + \boldsymbol{\omega}_* \times \boldsymbol{J}\boldsymbol{\omega}_* = \boldsymbol{\tau}_*, \\ \boldsymbol{\omega}_*\left(t_0\right) = \boldsymbol{Q}_0^{\mathrm{T}}\left(\boldsymbol{\omega}_0 + \boldsymbol{\omega}_c\left(t_0\right)\right), \\ \boldsymbol{Q}_*\left(t_0\right) = \boldsymbol{Q}_0, \end{cases} \tag{85}$$

and

$$\begin{cases} \ddot{\mathbf{r}}_* + \dfrac{\mu}{r_*^3}\mathbf{r}_* = \mathbf{0}, \\ \mathbf{r}_*(t_0) = \mathbf{r}_c^0 + \mathbf{r}_0, \\ \dot{\mathbf{r}}_*(t_0) = \dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times \left(\mathbf{r}_C^0 + \mathbf{r}_0\right), \end{cases} \tag{86}$$

where

$$\boldsymbol{R}_{-\omega_C} = \boldsymbol{I} - \sin f_c^0 \frac{\tilde{\boldsymbol{h}}_C}{|\mathbf{h}_c|} + \left(1 - \cos f_c^0\right)\frac{\tilde{\boldsymbol{h}}_C^{\,2}}{|\mathbf{h}_c|^2} \tag{87}$$

and $\mathbf{r}_c$ *is given by Eq.* (75).

Remark 3. The result displayed in Theorem 10 gives a very meaningful insight on the motion of any rigid body with respect to a non-inertial frame. A straightforward method to approach its motion is revealed as follows: (*i*) The problem is solved in an inertial frame, that is our non-inertial frame "frozen" at the initial moment of time; (*ii*) The solution to the non-inertial problem is obtained by applying tensor $\underline{\boldsymbol{R}}_{-\omega_C}$ to the solution obtained at the previous step (*i*).

This insight reveals that in fact any rigid body motion with respect to a non-inertial frame is a Foucault pendulum-like motion, the same type that is comprehensively studied in [22].

Remark 4. The problems (82) and (83) are coupled because, in general case, the torque $\boldsymbol{\tau}$ depends of the position vector $\mathbf{r}$.

The exact closed form, coordinate-free, solution of the translational motion can be found in [2–4, 22, 45].

8. CONCLUSION

The present paper develops new methods for recovering a solution property to the full body relative orbital motion problem in a Keplerian field. A representation theorem is provided for the full body initial value problem, using dual Lie algebra of dual vectors. For this, the isomorphism between the Lie group of the rigid body displacements and the Lie group of the orthogonal dual tensors is used. Furthermore, the representation theorems for the rotation part and translation part of the six-degree-of-freedom relative motion in a non-inertial reference frame are obtained. The core result of the paper offers a meaningful insight and a natural geometrical interpretation of the motion, namely that it is in fact derived from the motion in a well-defined inertial frame which is seen through a transform that depends solely on an orthogonal tensor that models the behavior of the non-inertial frame. The obtained results interest the domains of the spacecraft formation flying, rendezvous operation, autonomous mission and control theory.

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