GENERALIZED INTERMEDIARY POTENTIALS FOR SATELLITE ORBITS AROUND OBLATE PLANETS

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Abstract. The main problem in artificial satellite theory is modeled by an initial value problem describing the motion of a mass particle in an axially symmetric potential. In the case of Earth, this potential is truncated up to order 4 in the series that approximate the full potential, and only the even zonal harmonics terms are considered. The problem is non-integrable and even exhibits chaotic behaviour. However, approximate solutions may be given by replacing the original potential with an approximation, generally named an intermediary potential. Several solutions have been offered in this way starting from the beginning of the space era at the middle of the 20th century. The present work offers the general intermediary potential that contains all of the previous models as particular cases, opening the path in finding the most appropriate potential for various types of orbits.

Key words: main problem in artificial satellite theory, intermediary potential, radial intermediary, zonal intermediary, perturbation theory, gravitational potential.

1. INTRODUCTION

The Main Problem in Artificial Satellite Theory approaches the motion of a mass particle in the potential of an oblate planet. The gravitational potential of a planet is a function of its shape (modeled by its domain D) and its mass distribution, and it is expressed in a point in space described by the position vector $\mathbf{r}$ as:

$$U(\mathbf{r}) = \int_{Q \in D} \frac{\mathrm{d}m}{\|\mathbf{r} - \mathbf{r}_Q\|}\,. \tag{1}$$

For Earth, under the assumption of an axial symmetry, the potential in Equation (1) is expressed as a series of spherical harmonics as:

$$U(\mathbf{r}) = -\frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_e}{r}\right)^n P_n(\cos\phi)\right] \tag{2}$$

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where μ is the gravitational parameter, J_n are the zonal harmonics, r_e is the mean equatorial radius and P_n are the Legendre polynomials. Their argument is the cosine of the colatitude ϕ of the satellite, its expression being used as:

$$\cos\phi = \sin\theta \sin i \tag{3}$$

where θ is the argument of latitude the and i is the inclination of the orbit. Since the Earth dominant perturbation is given by the second zonal harmonic J_2, the other ones being three orders of magnitude smaller, i.e. $J_{3,4} = \mathscr{O}\left(J_2^2\right)$, the potential is usually truncated at the second order and has the approximate expression $U \simeq U_0 + U_{J_2}$, where

$$U_{J_2}(\mathbf{r}) = \frac{\mu J_2 r_e^2}{r^3} P_2(\cos\phi) = \frac{\mu J_2 r_e^2}{2r^3}\left(3\sin^2 i \sin^2\theta - 1\right). \tag{4}$$

The main perturbation model that is used in current applications was introduced by Brouwer [2], and its two key features are common to all subsequent perturbations theories. The first key feature is the replacement of the non-integrable Hamiltonian

$$\mathscr{H} = \mathscr{H}_0 + U_{J_2}(\mathbf{r}) \tag{5}$$

with an integrable counterpart, that contains, in the case of Brouwer, the average of the perturbation potential $U_{J_2}(\mathbf{r})$ over one period of the unperturbed motion (governed by the Hamiltonian $\mathscr{H}_0$).

The second key feature, that is more subtle, is the introduction of a near-identity canonical transformation. This is due to the fact that, in the moment when the potential is replaced, the new model represents another motion, in another phase space, being described with different canonical variables. The connection between the ”old” variables, describing the real motion, and the ”new” variables is made exactly through this near-identity canonical transformation, that in theory is an infinite series of terms depending on the natural powers of the small parameter J_2, but in practice it is truncated to a finite number of terms that give the order of the perturbation model.

Brouwer's original model suffered from a series of inconveniences, being singular at low inclinations and small eccentricities, due to the use of the Delaunay elements (which is however justified by the fact that they are the action-angle variables of the unperturbed Keplerian motion). This deficiency was later improved by Lyddane [10] through the use of the Poincaré canonical variables.

However, in the same period as Brouwer's celebrated work, some other approximate potentials were proposed. Sterne [11] proposed an integrable approximation of the J_2 potential. Incited by him, Garfinkel [7] proposes an intermediate integrable approximation and later extends it to a potential [8] that also takes into consideration the effects of the zonal harmonics $J_{3,4}$.

Several years later, Aksnes [1] finds another integrable approximation for the main problem in artificial satellite theory, and further on Cid and Lahulla [3] propose an integrable approximation that admits a closed form solution (with the help of elliptic integrals).

All the aforemementioned works make use of the classical Poincaré-von Zeipel canonical perturbation theory, that suffers a major deficiency, inherited from the classical canonical transformations in Hamiltonian mechanics: the generating function is expressed in mixed variables, namely it is expressed with the help of half of the old and half of the new variables. This makes the symbolic computations cumbersome, especially at higher orders.

Deprit [5] proposes a new perturbation paradigm, in which the near-identity canonical transformation is expressed as a Lie series, and has the enormous advantage that it is not expressed in mixed variables, allowing an explicit transformation between the new and the old variables (and vice-versa). He also proposes an integrable approximation – Deprit's radial intermediary – that has closed form solution [4].

The present paper focuses on two key aspects: (i) it presents a general perturbation framework in which all possible integrable intermediaries (including the ones listed above) may easily be deduced and (ii) it offers a method of solution (only at first order) for any of the cases. The aim to offer a novel solution to the main problem is beyond the scope of this paper, as well as proceeding to higher orders. However, all that is presented further may be extended to (i) higher order zonal harmonics in the gravitational potential and (ii) higher order solutions, all these extensions depending only on technical manipulations.

2. HISTORICAL INTERMEDIARIES

We start by setting the classical orbital elements a (semimajor axis), e (eccentricity), i (inclination), ω (argument of perigee), Ω (right ascension of the ascending node) and M (mean motion) of a Keplerian unperturbed orbit. Denote by f the true anomaly of the satellite and let $\eta = \sqrt{1-e^2}$. The semilatus rectum is $p = a\eta^2$. The argument of perigee is expressed then as $\theta = \omega + f$.

The Delaunay orbital elements (l, g, h, L, G, H) are listed here for the sake of completeness:

$$\begin{array}{lll} l = M & g = \omega & h = \Omega \\ L = \sqrt{\mu a} & G = \sqrt{\mu p} & H = \sqrt{\mu p}\cos i\,. \end{array} \tag{6}$$

We also list another set of canonical variables that are used, namely the Whittaker variables, also known as the polar-nodal ones:

$$\begin{array}{lll} r = p\,(1+e\cos f)^{-1} & \theta = g+f & \nu = h \\ R = \dot{r} = Gp^{-1}e\sin f & \Theta = G & N = H \end{array} \tag{7}$$

Also denote:

$$k_2 = \frac{\mu J_2 r_e^2}{2} \quad c = \cos i = N/\Theta \quad s = \sin i = \sqrt{1-c^2} \tag{8}$$

Since the inclination of a satellite is always betwen 0 (prograde equatorial orbit) and π (retrograde equatorial orbit), there is no sign ambiguity in the last of Eqs. (8).

With these notations, we can now list the intermediaries mentioned in the Introduction:

$$
\begin{array}{llll}
\mathscr{H}_0 + & \frac{k_2}{r^3}\left(1-3c^2-3s^2\cos 2\theta\right) & \text{full } J_2 & \\
 & \frac{3k_2}{pr^2}s^2\cos 2\theta + \frac{k_2}{r^3}\left(3c^2-1\right) & \text{Sterne, 1957} & \\
 & \frac{3k_2\eta}{p^2 r}\left(3c^2-1\right) + \frac{3k_2}{pr^2}\left(3s^2-2+s^2\cos 2\theta\right) & \text{Garfinkel, 1958} & \\
 & \frac{k_2\eta^3}{p^3}\left(3c^2-1\right) & \text{Brouwer, 1959} & (9) \\
 & \frac{2k_2}{pr^2}\left(1-3c^2-3s^2\cos 2\theta\right) & \text{Aksnes, 1965} & \\
 & \frac{k_2}{r^3}\left(3c^2-1\right) & \text{Cid-Lahulla, 1969} & \\
 & \frac{k_2}{pr^2}\left(3c^2-1\right) & \text{Deprit, 1981} &
\end{array}
$$

2.1. A General Intermediary Framework

The forms of the intermediaries in Equations (9) suggest a generalization, that is presented as follows. Assume the most general form of an intermediary, that is:

$$\mathscr{K} = \mathscr{H}_0 + \mathscr{V}_0 + \mathscr{V}_1 + \mathscr{V}_2 + \mathscr{V}_3 \tag{10}$$

where $\mathscr{H}_0$ is the unperturbed Hamiltonian and the intermediary potentials $\mathscr{V}_k$, $k \in \{0,1,2,3\}$

$$
\begin{aligned}
\mathscr{V}_0 &= \frac{k_2\eta^3}{p^3}\left(A_{00}+A_{01}\sin 2\theta + A_{02}\cos 2\theta\right) && \text{(11a)}\\
\mathscr{V}_1 &= \frac{k_2\eta}{p^2 r}\left(A_{10}+A_{11}\sin 2\theta + A_{12}\cos 2\theta\right) && \text{(11b)}\\
\mathscr{V}_2 &= \frac{k_2}{pr^2}\left(A_{20}+A_{21}\sin 2\theta + A_{22}\cos 2\theta\right) && \text{(11c)}\\
\mathscr{V}_3 &= \frac{k_2}{r^3}\left(A_{30}+A_{31}\sin 2\theta + A_{32}\cos 2\theta\right). && \text{(11d)}
\end{aligned}
$$

The choice of the factors in front of the parentheses will be explained further in this paper. One may notice that the so-called fast variables are contained inside the parentheses (through the argument of latitude $\theta = \omega + f$), as well as in the negative power of the radial distance r. This power is reflected in the index of the intermediate potentials $\mathscr{V}_k$.

We choose the terms A_{km}, $k \in \{0,1,2,3\}$, $m \in \{0,1,2\}$ to be only functions of Θ and N (or G and H in Delaunay variables). What we demand from the Hamiltonian in Equation (10) is to be integrable in the first place. Following a result by Stackel, expressed in polar-nodal variables, we already know that it must be of the form:

$$\mathscr{K} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} + \mathscr{U}_1\left(r,\Theta,N\right) + \frac{\mu}{r^2}\mathscr{U}_2\left(\theta,\Theta,N\right). \tag{12}$$

The expression of the Hamiltonian in Equation (12) implies that the zonal terms (i.e. those depending on θ) to be present only in $\mathscr{V}_2$, that makes:

$$A_{km} = 0, k \in \{0,1,3\}, m \in \{1,2\}. \tag{13}$$

Equations (11) transform into:

$$\mathscr{V}_0 = \frac{k_2\eta^3}{p^3}A_{00} \tag{14a}$$

$$\mathscr{V}_1 = \frac{k_2\eta}{p^2 r}A_{10} \tag{14b}$$

$$\mathscr{V}_2 = \frac{k_2}{pr^2}\left(A_{20} + A_{21}\sin 2\theta + A_{22}\cos 2\theta\right) \tag{14c}$$

$$\mathscr{V}_3 = \frac{k_2}{r^3}A_{30}. \tag{14d}$$

With Equations (14), a general form of an intermediary has been listed. For the sake of simplicity, a different notation for the new variables has been omitted, that is related to the variables displayed in Equations from Eq. (10) onward. The connection between these variables and the old ones is given by an infinitesimal canonical transformation, or infinitesimal contact transformation, following the paradigm introduced by Deprit [5].

The potentials listed in Equations (9) are recovered through particular values given to $A_{km} = A_{km}\left(\Theta,N\right)$ as follows:

$$\begin{array}{ll}
A_{22} = 3s^2;\ A_{30} = 3c^2 - 1 & \text{Sterne, 1957} \\
A_{10} = 3\left(3c^2-1\right);\ A_{20} = 3\left(3c^2-1\right);\ A_{22} = 3s^2 & \text{Garfinkel, 1958} \\
A_{00} = 3c^2 - 1 & \text{Brouwer, 1959} \\
A_{20} = 1 - 3c^2;\ A_{22} = -3s^2 & \text{Aksnes, 1965} \\
A_{30} = 3c^2 - 1 & \text{Cid-Lahulla, 1969} \\
A_{20} = 3c^2 - 1 & \text{Deprit, 1981}
\end{array} \tag{15}$$

For each intermediary, if the values are not displayed for particular A_{km} 's it means that they are 0.

3. THE "EFFICIENCY" CONDITION

This condition was stated by Deprit [4] as a generalization of Brouwer's approach. Briefly, it states that for an intermediary to be efficient (or "natural"), it needs to have the same average (over one unperturbed period) as the full unperturbed potential, that is:

$$\int_0^T \left(-U_{J_2}(r,\theta,-,R,\Theta,N) + \sum_{k=0}^{3} \mathcal{V}_k(r,\theta,-,R,\Theta,N) \right) \mathrm{d}l = 0 \tag{16}$$

By making the change of variable

$$\mathrm{d}l = \frac{r^2}{\Theta}\,\mathrm{d}f$$

and by denoting, for an arbitrary function U :

$$< U >_l = \frac{1}{2\pi}\int_0^{2\pi} U \frac{r^2}{\Theta}\,\mathrm{d}f$$

and after some computations it follows that:

$$< \ U_{J_2} >_l = \frac{k_2\eta^3}{p^3}\left(1-3c^2\right) \tag{17a}$$

$$< \ \mathcal{V}_k >_l = \frac{k_2\eta^3}{p^3} A_{k0},\ k \in \{0,1,2,3\} \tag{17b}$$

The choice of the coefficients in front of the parantheses in Equations (14) becomes now obvious. The efficiency condition (16) becomes:

$$< U_{J_2} >_l = < \mathcal{V}_0 + \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 >_l$$

that gives :

$$S_0 = A_{00} + A_{10} + A_{20} + A_{30} = 1 - 3c^2. \tag{18}$$

It is important to notice that the only potential in Equations (9), namely that of Garfinkel, it does not obey this efficiency condition (although it obeys the form of the Hamiltonian listed in Equation (12) that guarantees integrability). In the case of Garfinkel's potential, the sum S_0 defined in Equation (18) is:

$$S_0^{Garfinkel} = A_{10} + A_{20} = 6\left(3c^2 - 1\right) \neq \left(3c^2 - 1\right). \tag{19}$$

One interesting fact to be mentioned is that by this procedure, we have embedded into our general approach all the known historical intermediaries, unlike the attempt to find their general form proposed by Ferrandiz et al. [6]. This is due to the fact that we have first searched for the intermediary potential, and only afterwards we have proceeded to build the perturbation theory based on Lie series.

4. THE INFINITESIMAL CONTACT TRANSFORMATION

The transformation is generated by following the rules of the Lie triangle [5], and the procedure (developed here only for the first order) is as follows. We choose to seek for the generating function by using the Delaunay canonical variables because of their simplicity and because the very simple form of the unperturbed Hamiltonian, keeping in mind the the Lie derivative is invariant to canonical changes of variable. The infinitesimal contact transformation, which is computed from the partial derivatives of the generating function, can be expressed with the help of the polar-nodal (Whittaker) canonical variables, that do not exhibit any singularity for small eccentricities, while the singularity that exists for small inclinations is practically removed by the axially-symmetric nature of the potential field, h being a cyclic variable.

In what follows, to be consistent with Deprit's notations (and also with the very clear perturbation approach presented in [9]), we will denote by $\mathscr{H}_{0,0}$ the unperturbed Keplerian Hamiltonian:

$$\mathscr{H}_{0,0} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} = -\frac{\mu^2}{2L^2} \tag{20}$$

by $\mathscr{H}_{1,0}$ the potential is the J_2 perturbation part of the full J_2 Hamiltonian,

$$\mathscr{H}_{1,0} = \frac{k_2}{r^3}\left(1 - 3c^2 - 3s^2\cos 2\theta\right) \tag{21}$$

and by $\mathscr{H}_{0,1}$ the intermediary potential that is the J_2 part of the approximate Hamiltonian:

$$\mathscr{H}_{0,1} = \sum_{k=0}^{3} \mathscr{V}_k\left(r, \theta, -, R, \Theta, N\right). \tag{22}$$

Note that the canonical variable R appears in $\mathscr{H}_{0,1}$ only if one chooses A_{00} or A_{10} to be non-zero (we are in the integrable setting defined in the general form by the expressions of the $\mathscr{V}_k$ potentials from Equations (14)).

The first term of the generating function W, denoted by W_1, satisfies the omological equation:

$$\{W_1, \mathscr{H}_{0,0}\} = \mathscr{H}_{1,0} - \mathscr{H}_{0,1} \tag{23}$$

where $\{\cdot,\cdot\}$ denotes the Lie-Poisson bracket, defined for any set $\mathbf{X} = \mathbf{X}(\mathbf{q},\mathbf{p})$ of canonical variables and two scalar valued functions $f = f(\mathbf{X})$, $g = g(\mathbf{X})$ as:

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{X}} \mathscr{J} \left(\frac{\partial g}{\partial \mathbf{X}}\right)^T \tag{24}$$

where $\partial f/\partial \mathbf{X}$ denotes the gradient of f with respect to the vector field $\mathbf{X}$,

$$\frac{\partial f}{\partial \mathbf{X}} = \left[\begin{array}{cccccc} \frac{\partial f}{\partial q_1} & \frac{\partial f}{\partial q_2} & \frac{\partial f}{\partial q_3} & \frac{\partial f}{\partial p_1} & \frac{\partial f}{\partial p_2} & \frac{\partial f}{\partial p_3} \end{array}\right]$$

and $\mathscr{J}$ is the symplectic matrix:

$$\mathscr{J} = \left[\begin{array}{cc} \mathbf{O}_3 & -\mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{O}_3 \end{array}\right].$$

Since in Delaunay variables the unperturbed Hamiltonian $\mathscr{H}_{0,0}$ depends only on the action variable L, most partial derivatives in the Lie bracket vanish, leaving the homological equation (23) to be:

$$\{W_1, \mathscr{H}_{0,0}\} = n\frac{\partial W_1}{\partial l} \tag{25}$$

where $n = \Theta\eta^3 p^{-3}$ is the angular frequency of the Keplerian unperturbed orbit. The partial derivatives equation (23) may be solved through simple integration, giving:

$$
\begin{aligned}
W_1 \quad = \quad & \frac{k_2\eta^3}{p^3}\left\{\left[A_{20}+A_{30}+\frac{1}{2}\left(3c^2-1\right)\right](f-l)\right. \\
& +\sigma\left[\frac{\eta r}{p}A_{10}+A_{30}+\frac{1}{2}\left(3c^2-1\right)\right] \\
& \left.+\left[\frac{1}{2}A_{22}+s^2\left(\kappa+\frac{3}{4}\right)\right]\sin 2\theta-\frac{1}{2}\left(A_{21}+es^2\right)\cos 2\theta\right\}
\end{aligned}
\tag{26}
$$

where the functions κ and σ are defined as:

$$
\kappa = e\cos f = \frac{p}{r}-1 \quad \sigma = e\sin f = \frac{pR}{\Theta} \quad . \tag{27}
$$

The quantity $\phi = f - l$ is called the equation of the center. The partial derivatives of W_1 with respect of the polar-nodal variables (r,θ,h,R,Θ,N) can be computed by taking into account [9]:

$$
\begin{aligned}
&\frac{\partial\eta}{\partial r}=\frac{\kappa p}{\eta r^3} && \frac{\partial\eta}{\partial R}=-\frac{\sigma p}{\eta\Theta} && \frac{\partial\eta}{\partial\Theta}=\frac{\eta^2-(1+\kappa)^2}{\eta\Theta} \\
&\frac{\partial\phi}{\partial r}=\frac{\sigma}{r}\left(\frac{1+\kappa}{1+\eta}+\frac{\eta}{1+\kappa}\right) && \frac{\partial\phi}{\partial R}=\frac{\sigma}{R}\left(\frac{\kappa}{1+\eta}+\frac{2\eta}{1+\kappa}\right) && \frac{\partial\phi}{\partial\Theta}=-\frac{\sigma}{R}\frac{2+\kappa}{1+\eta}
\end{aligned}
\tag{28}
$$

The partial derivatives of the eccentricity may be easily computed by taking into account that $e^2+\eta^2=1$, that gives

$$
\frac{\partial e}{\partial u} = -\frac{\eta}{e}\frac{\partial\eta}{\partial u} \tag{29}
$$

for any variable u among the polar-nodal ones.

One has also to take into account that the (unspecified) factors A_{km} depend solely on Θ and N. Their choice may be made at will, under the restriction that they should satisfy the efficiency condition (18).

The infinitesimal contact transformation (at first order), direct and inverse, may be deduced from:

$$
\mathbf{X} = \mathbf{X}' + \left\{W_1, \mathbf{X}'\right\} \tag{30a}
$$

$$
\mathbf{X}' = \mathbf{X} - \left\{W_1, \mathbf{X}\right\}. \tag{30b}
$$

Since the aim of this paper is not to develop a full perturbation theory based on the general form of an intermediary, we will not display here the expanded form of Equations (30). These derivations, together with the tuning of the unspecified terms $A_{km} = A_{km}(\Theta, N)$, will be subject to future work. Note that in this framework, Equations (30), togehter with the explicit expression of the generating function W_1 in Equation (26), contain all the contact transformations derived for all the "historical" intermediaries listed within this paper.

5. CONCLUSIONS

We offer a general framework to seek for intermediary solutions for the main problem in artificial satellite theory, more exactly the motion around an oblate spheroidal planet. By using simple algebraic manipulations, we deduce the general form for intermediary potentials, the ”natural” ones being those whose average is the same as the average of the full perturbing potential (they satisfy the efficiency condition). After imposing the necessary condition of integrability, we offer a general perturbation framework with unspecified parameters that may be tuned arbitrarily in the given efficiency condition. Some particular value sets of these parameters lead to the known ”historical” intermediaries that started being used at the beginning of the space era. In this way, we have shown that there exists a general intermediary that embeds all possible situations. This approach leads the way to build new perturbation theories in the main problem of artificial satellite theory. The same approach may be applied for higher-order zonal harmonics and, of course, higher-order perturbation models.

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