TIME OPTIMAL DE-ORBITING OF SOLAR SAILS

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Abstract. Increasing interest in optimal low-thrust orbital transfers was triggered in the last decade by technological progress in electric propulsion and by the ambition of efficiently leveraging on orbital perturbations to enhance the maneuverability of small satellites. This paper focuses on time optimal disposal maneuvers using solar radiation as a propulsive means. After formulating the necessary conditions for optimality, the numerical solution of the two-point boundary value problem is facilitated by averaging the Hamiltonian with respect to both satellite and Sun longitudes. Initial conditions for the osculating trajectory are finally inferred via a near-identity transformation that carefully approximate the quasi-periodic oscillations of both state and adjoint variables. Classical approaches for the assessment of this transformation are shown to be inadequate to the problem at hand because of the particular structure of the Hamiltonian.

Key words: solar radiation pressure, solar sail de-orbiting, propellantless maneuvers, optimal control.

1. INTRODUCTION

The development of miniaturized satellite systems and the availability of low-cost launchers facilitated the access to space. Hence, the number of objects orbiting around the Earth experienced exponential growth during the last decade. Because of the modest maneuverability capabilities of small satellites, innovative techniques exploiting orbital perturbations caught on to fulfill the 25 year requirement for the end-of-life disposal of satellites whose trajectories cross the low-Earth orbit (LEO) region [1].

The plain deployment of a sail leveraging on solar radiation pressure (SRP) to lower the perigee of the orbit is among them. This passive strategy was mainly studied from a dynamical systems point of view, and its main drawback is the arguably long time required to achieve the disposal [2]. In addition, conditions on the minimum area-to-mass ratio and on the initial orbital elements need to be satisfied to exploit these strategies [3]. Controlling the orientation of the sail with respect to the Sun direction may overcome these issues. Hence, the problem is tackled from a control perspective in this paper in order to gain insight into how the sail should be oriented to minimize the maneuvering time.

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Necessary conditions for optimality of these disposal maneuvers are detailed in Section 2. Solving the resulting two-point boundary value problem (TPBVP) is challenging because of the *fast-oscillating* nature of orbital dynamics and of the *bang-bang* structure of the optimal control action.

A simplification of the problem can be achieved by averaging the Hamiltonian of the extremal flow with respect to both satellite and Sun longitudes. Although this possibility is appealing, the Hamiltonian at hand is not in the classical form of fast-oscillating systems. In addition, time optimal disposal trajectories are shown to be characterized by two bang events per orbit, so that discontinuities of the vector field occur at very fast rate. These two considerations make questionable the rigorous exploitation of averaging theory in this problem. Nonetheless, trajectories of the original system can be well approximated by their averaged counterpart if boundary conditions of the adjoint variables are adequately transformed [4]. We discuss this transformation in detail in Section 3, and we emphasize fundamental differences with respect to well-known mean-to-osculating transformations of uncontrolled motion [5, 6, 7]. In particular, these classical approaches are inadequate when applied to the reconstruction of short-period variations of the adjoints to slow variables because of the peculiar form of their equations of motion.

The methodology is finally applied to the disposal of a Satellite in a highly eccentric orbit. Trajectories of the doubly-averaged system are drastically streamlined, so that the averaged counterpart of the TPBVP is reasonably easy to solve.

2. TIME OPTIMAL SOLAR-SAIL ASSISTED DE-ORBITING

2.1. Formulation of the problem

We consider the optimal control problem

$$\min_{u(t)} t_f \qquad \text{subject to} \tag{1}$$

$$I(0) = I_0 \tag{2}$$

$$\varphi(0) = \varphi_0 \tag{3}$$

$$r_p\left(I(t_f)\right) = r_f \tag{4}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon\left[f_0\left(I,\varphi\right) + f_u\left(I,\varphi\right)u\right] \tag{5}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega\left(I\right) + \varepsilon\left[g_0\left(I,\varphi\right) + g_u\left(I,\varphi\right)u\right] \tag{6}$$

$$0 \le u(t) \le 1 \qquad \forall t \in [0, t_f] \tag{7}$$

Here, t_f denotes the maneuvering time, I are the slowly-varying equinoctial elements [8] of the orbit, namely

$$I = \begin{bmatrix} a\left(1-e^2\right) \\ e\cos\left(\omega+\Omega\right) \\ e\sin\left(\omega+\Omega\right) \\ \tan\dfrac{i}{2}\cos\Omega \\ \tan\dfrac{i}{2}\sin\Omega \end{bmatrix},$$

where a, e, i, ω, and Ω denote the orbital semi-major axis, eccentricity, inclination, argument of perigee (AoP), and right ascension of the ascending node (RAAN), respectively. The vector φ defined on the two-dimensional torus $\mathbb{T}^2$ denotes satellite and Sun mean longitudes (φ_1 and φ_2, respectively), which are referred to as fast variables in the reminder.

Initial conditions are expressed by Eqs. (2) and (3), whereas disposal conditions at the end of the maneuver are modeled by Eq. (4). Specifically, de-orbiting is achieved when the perigee radius,

$$r_p(I) = \frac{I_1}{1+\sqrt{I_2^2+I_3^2}},$$

is lowered down the prescribed value r_f, which guarantees the imminent re-entry in the atmosphere of the satellite.

The motion of I and φ is governed by Eqs. (5) and (6). Here, all functions are periodic with respect to φ, and ε is a formal small parameter that is introduced to distinguish between slow and fast dynamics. The components of the frequency vector, $\omega(I)$, are the mean motion of the satellite,

$$\omega_1 = \sqrt{\mu\left(\frac{1-I_2^2-I_3^2}{I_1}\right)^3},$$

where μ is the gravitational parameter of the Earth, and the mean motion of the Sun, ω_2. The orbit of the Sun is assumed to be Keplerian, so that ω_2 is constant. The modeling of f_0, f_1, g_0, and g_1 is based on the following additional assumptions:

- Eclipses are ignored;
- SRP is toward the opposite direction of the Sun position vector, $s(\varphi_2)$, and proportional to the cross-sectional area projected in the direction of the Sun (*cannonball* model), as shown in Figure 1.
- The satellite is modeled as a flat surface (sail) with area-to-mass ratio $\frac{A}{m}$, so that the force per mass unit due to the radiation pressure is

$$f_{SRP}(I,\varphi,u) = \underbrace{-\frac{c}{\|s-r\|^2}\frac{A}{m}}_{f_{SRP}^{max}}\hat{s}\ u,$$

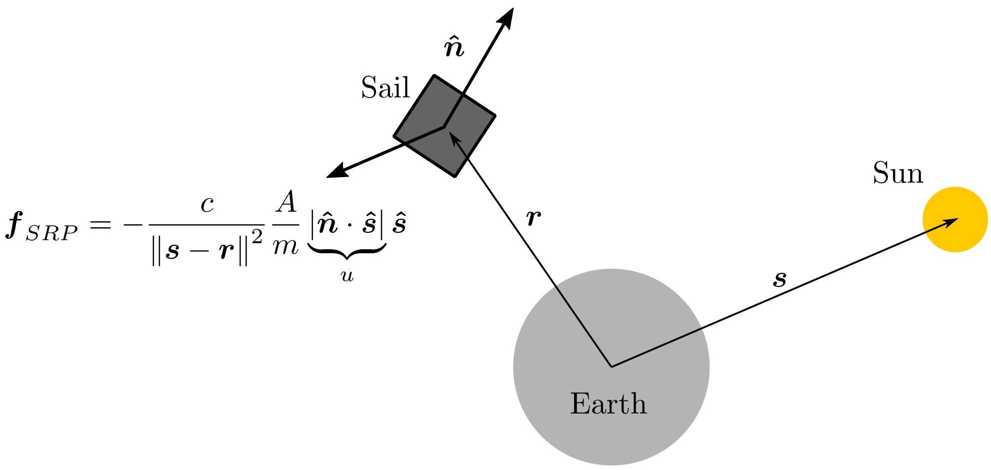

Fig. 1 – Schematic representation of the dynamical problem.

where the constant c is the radiation pressure at 1 AU, $r(I,\varphi_1)$ is the position vector, $\hat{s} = \frac{s}{\|s\|}$ is the unit vector pointing toward the Sun direction, and u is the control variable, which can be physically interpreted as $u = |n \cdot \hat{s}|$, with $\hat{n}$ being the unit vector orthogonal to the sail.

- The only perturbation considered other than SRP is the third-body attraction of the Sun,
$$f_S(I,\varphi) = \mu_S\left(\frac{s-r}{\|s-r\|^3} - \frac{s}{\|s\|^3}\right),$$
where μ_S denotes the gravitational constant of the Sun.

Hence, functions f_0, f_1, g_0 and g_1 are given by
$$f_0 = F(I,\varphi)f_S, \qquad f_1 = F(I,\varphi)f_{SRP}^{max},$$
$$g_0 = G(I,\varphi)f_S \qquad g_1 = G(I,\varphi)f_{SRP}^{max}.$$
Matrix-valued functions $F(I,\varphi)$ and $G(I,\varphi)$ are straightforwardly deduced from the Gauss variational equations (GVE) for equinoctial elements in [8], but they are not detailed herein for the sake of conciseness.

2.2. Necessary conditions for optimality

Denote by p_I and p_φ the adjoints to slow and fast variables, respectively. The application of the infamous Pontryagin maximum principle (PMP) [9] yields the Hamiltonian of the extremal flow assocititated to Eqs (5)–(7)
$$\mathscr{H} = p_\varphi \cdot \omega(I) + \varepsilon K\left(I, p_I, \varphi, p_\varphi\right), \tag{8}$$

where the function K that characterizes the slow component of the Hamiltonian is

$$K = H_0 + \frac{H_1 + |H_1|}{2}, \tag{9}$$

and H_j, for $j = 0, 1$, are defined as

$$H_j = f_j(I,\, \varphi) \cdot p_I + g_j(I,\, \varphi) \cdot p_\varphi.$$

The optimal control law is bang-bang, and it is determined by the sign of H_1, namely

$$u^{opt}\left(I, p_I, \varphi, p_\varphi\right) = \frac{1}{2}\left(1 + \frac{H_1}{|H_1|}\right). \tag{10}$$

Necessary conditions for optimality consist of the flow associated to the Hamiltonian of Eq. (8),

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \varepsilon \frac{\partial K}{\partial p_I}, \\
\frac{\mathrm{d}\varphi}{\mathrm{d}t} &= \varepsilon \frac{\partial K}{\partial p_\varphi} + \omega(I), \\
\frac{\mathrm{d}p_I}{\mathrm{d}t} &= -\varepsilon \frac{\partial K}{\partial I} - p_\varphi \frac{\partial \omega}{\partial I}, \\
\frac{\mathrm{d}p_\varphi}{\mathrm{d}t} &= -\varepsilon \frac{\partial K}{\partial \varphi},
\end{aligned} \tag{11}$$

of the transversality conditions at time t_f

$$\begin{aligned}
& p_{I_2} + \frac{I_1\ p_{I_1}}{1 + \sqrt{I_2^2 + I_3^2}}\ \frac{I_2}{\sqrt{I_2^2 + I_3^2}} = 0 \\
& p_{I_3} + \frac{I_1\ p_{I_1}}{1 + \sqrt{I_2^2 + I_3^2}}\ \frac{I_3}{\sqrt{I_2^2 + I_3^2}} = 0 \\
& p_{I_j} = 0 \qquad \text{for } j = 4, 5, \\
& p_\varphi = 0,
\end{aligned} \tag{12}$$

and of Eqs. (2) and (3).

Solutions of the necessary conditions are obtained by finding triads $\left(t_f,\ p_{I0},\ p_{\varphi_0}\right)$ such that trajectories of System (13) with initial conditions $I(0) = I_0$, $p_I(0) = p_{I0}$, $\varphi(0) = \varphi_0$, and $p_{\varphi_0} = 0$ satisfy the Eq. (12) at time t_f. Finally, because of the homogeneity of the Hamiltonian, the level set $\mathscr{H} = \varepsilon$ is imposed. This normalizing condition is arbitrary and not unique.

3. THE AVERAGED CONTROL SYSTEM

Applying averaging theory to System (13) is questionable because the structure of this vector field differs from the one of classical fast-oscillating system. Specifically, the equation of motion of p_I, includes the term $p_\varphi \dfrac{\partial \omega}{\partial I}$ that may possibly be of order larger than ε. Hence, adjoints to slow variables are not necessary slow themselves.

Recent contributions showed that adjoints to fast variables are systematically ε-small for any extremal trajectory with free phases at boundaries, and, as such, $\frac{\mathrm{d}p_I}{\mathrm{d}t} = \mathscr{O}(\varepsilon)$ when restrained to these trajectories, which justifies the averaging of the extremal flow [4].

Denote by $\overline{K}$ the averaged counterpart of the functional defined in Eq. (9), namely

$$\overline{K} = \frac{1}{4\pi^2}\int_{\mathbb{T}^2} K\left(I, p_I, \varphi, 0\right)\, \mathrm{d}\varphi.$$

Here, $p_\varphi = 0$ because the averaging is carried out by considering the limit of the function as ε approaches zero. Averaging Eq. (13) yields

$$\begin{aligned}
\frac{\mathrm{d}\bar{I}}{\mathrm{d}t} &= \varepsilon\frac{\partial\overline{K}}{\partial\overline{p_I}}, \\
\frac{\mathrm{d}\varphi}{\mathrm{d}t} &= \varepsilon\frac{\partial\overline{K}}{\partial\overline{p_\varphi}} + \omega\left(\bar{I}\right), \\
\frac{\mathrm{d}\overline{p_I}}{\mathrm{d}t} &= -\varepsilon\frac{\partial\overline{K}}{\partial\bar{I}} - \overline{p_\varphi}\,\frac{\partial\,\omega}{\partial\bar{I}}, \\
\frac{\mathrm{d}\overline{p_\varphi}}{\mathrm{d}t} &= 0.
\end{aligned} \tag{13}$$

Adjoints to the fast variables are indeed constant along averaged extremal trajectories.

A near-identity transformation of the initial adjoints to fast variables is mandatory to have trajectories of the original and averaged systems that remain close for long time, as discussed in [4], namely

$$p_\varphi = \overline{p_\varphi} + \nu_{p_\varphi}\left(\bar{I}, \overline{\varphi}, \overline{p_I}\right).$$

This transformation is aimed at obtaining unbiased oscillations of p_φ with respect to $\overline{p_\varphi}$.

Assuming that I_0 is out of any resonant zone of order lower than $N > c_N \log\frac{1}{\varepsilon}$, then ν_{p_φ} is given by

$$\nu_{p_\varphi} = -i\sum_{0<|k|\leq N}\frac{\mathrm{e}^{ik\cdot\overline{\varphi}}}{k\cdot\omega\left(\bar{I}\right)}\,\left[-\frac{\partial K}{\partial\,\varphi}\right]^{(k)}, \tag{14}$$

where $\left[-\frac{\partial K}{\partial\,\varphi}\right]^{(k)}$ denote the coefficients of the Fourier expansion of $-\frac{\partial K}{\partial\,\varphi}$. To obtain exponential convergence of this series, the absolute value in (9) is approximated by

$$|H_1| \approx \sqrt{H_1^2 + \eta} - \sqrt{\eta}, \tag{15}$$

where η is a small parameter. We note that the approximation of Eq. (15) is obtained by adding the penalty function

$$-\sqrt{\eta}\log\left(1-u^2\right)$$

to the cost function of the optimal control problem.

Given the averaged state, Equation (14) establishes a mapping between φ and p_φ.

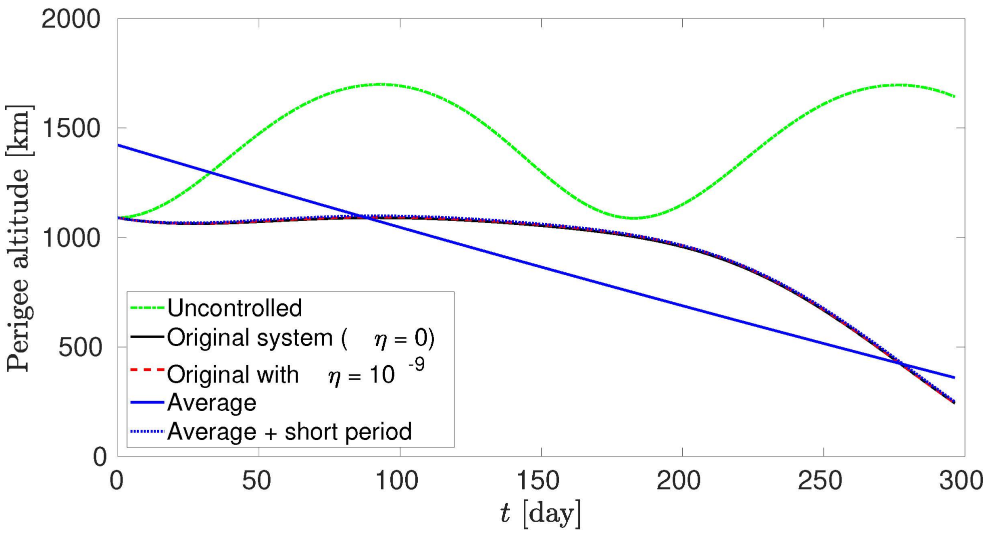

Fig. 2 – Trajectory of the perigee altitude.

Because ν_{p_φ} has zero mean, there exist $\varphi_0 \in \mathbb{T}^2$ such that

$$p_\varphi - \overline{p_\varphi} = \nu_{p_\varphi}\left(\bar{I}, \overline{p_I}, \varphi_0\right) = 0.$$

Restoring fast-variations of slow variables and their adjoints at initial time is less important than the transformation of p_φ. However, the drift between averaged and osculating trajectories can be further reduced by doing so. Direct application of classical perturbation theories to develop this transformation, e.g., Eq. (14) is not sufficient to reconstruct short-period variations of p_I, as discussed in [4]. In fact, the transformation of p_I should be carried out by including ν_{p_φ} in the Fourier expansion, namely

$$\nu_{p_I} = -i \sum_{0<|k|\leq N} \frac{\mathrm{e}^{ik\cdot\overline{\varphi}}}{k\cdot\omega\left(\bar{I}\right)} \left[-\left(\overline{p_\varphi} + \nu_{p_\varphi}\right)\frac{\partial\,\omega}{\partial\,I} - \frac{\partial\,K}{\partial\,I}\right]^{(k)}. \tag{16}$$

This nested transformation is capable of properly reconstructing short-period variations of the adjoints to slow variables.

4. NUMERICAL SIMULATION

The disposal of an object in a highly eccentric orbit is discussed in this section. Table 1 lists the parameters used in this case study.

The optimal control problem is solved via a simple shooting algorithm by approximating the dynamics of the original system with its averaged counterpart. The near identity transformation introduced in Section 3 is used to modify boundary conditions. The numerical solution[1] converges in few iterations regardless the choice of the initial

[1] Matlab's numerical solver *fsolve* with default tolerances was used to achieve the solution.

Table 1
Simulation parameters

Constants	
Area-to-mass ratio, $\frac{A}{m}$	$5\ \frac{\mathrm{m}^2}{\mathrm{kg}}$
Radiation pressure constant, c	$1.0205 \cdot 10^17$ N
Earth's gravitational parameter, μ	$3.986 \cdot 10^5\ \frac{\mathrm{km}^3}{\mathrm{s}^2}$
Earth's equatorial radius, r_E	6378.137 km
Sun' orbit semi-major axis	$1.4960 \cdot 10^8$ km
Sun' orbit eccentricity	0.017
Sun' orbit inclination	23.44 deg
Sun' orbit AoP	282.77 deg
Sun' orbit RAAN	0 deg
Initial conditions	
Sun' orbit semi-major axis	$1.4960 \cdot 10^8$ km
Semi-major axis	26000 km
Eccentricity	0.7
Inclination	65 deg
AoP	0 deg
RAAN	0 deg
Satellite longitude	0 deg
Sun longitude	0 deg
Final conditions	
Perigee radius, r_f	250 km $+ r_E$

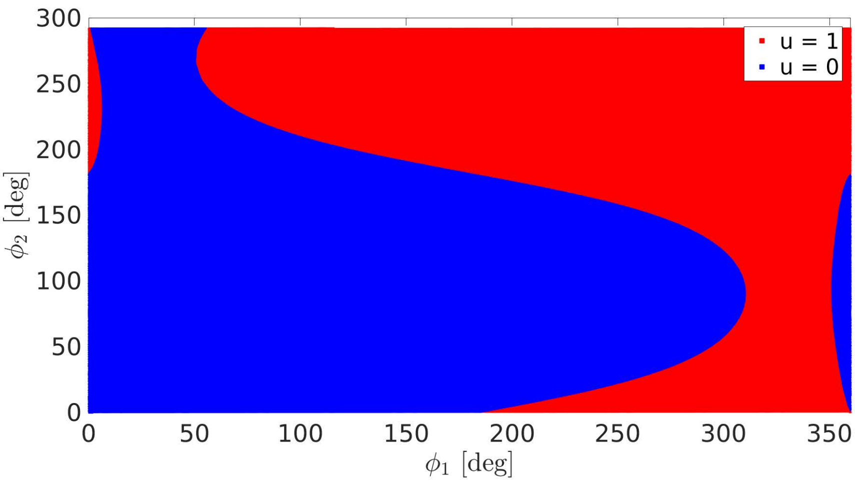

Fig. 3 – Control variable (original system).

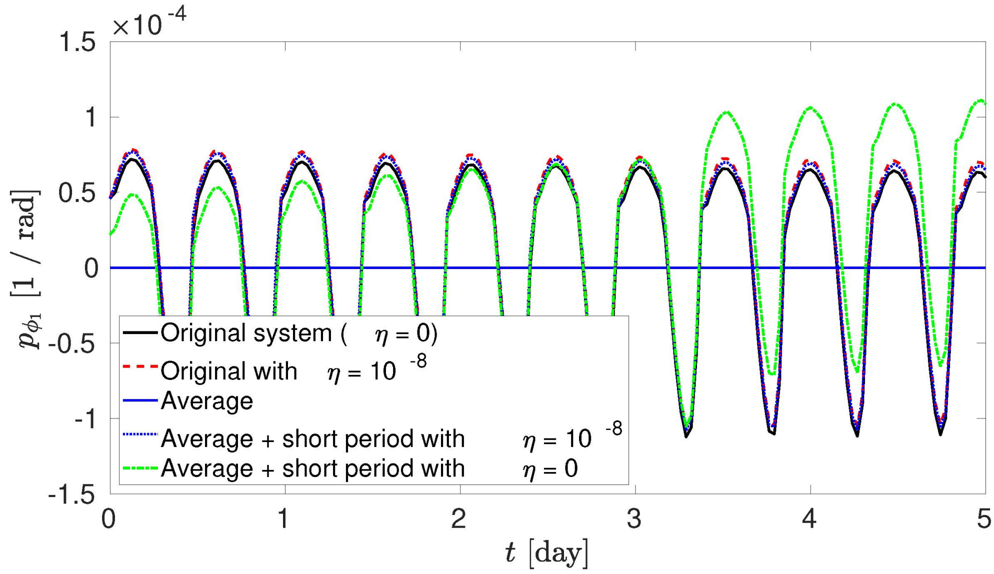

Fig. 4 – Beginning of the trajectory of the adjoint to satellite longitude.

guess thanks to the extremely smoothed and nearly straight trajectories of the averaged system, as shown in Figure 2 (solid blue line). The near identity transformation is finally capable of adequately restoring short-period oscillations. This claim is supported by the good matching between the blue dotted and red dashed trajectories.

Figure 3 depicts the value of the control variable throughout the entire maneuver (we emphasize that φ_2 is proportional to time). Two bang events characterize every orbital revolution. Obtaining a solution with such an involved control structure without leveraging on the simplified dynamics of the averaged system would be extremely difficult even by using direct approaches (e.g., pseudospectral techniques).

Finally, short period oscillations of the adjoint to satellite longitude, p_{φ_1} at the beginning of the maneuver are illustrated in Figure 4. The introduction of the smoothing parameter, η, in Eq. (15) is mandatory to adequately approximate these variations. Setting this parameter to zero would jeopardize the convergence of the Fourier coefficients in Eq. (14) and would result in the inconsistent reconstruction of the short period variations (dash-dotted green curve).

5. CONCLUSION

This paper discussed the realization propellantless time optimal disposal maneuvers of satellites by leveraging on the solar radiation pressure perturbation. The numerical solution of this problem is particularly challenging because of the fast-oscillating nature of orbital dynamics and of the large number of bang events characterizing the optimal trajectory. We showed that the problem can be greatly simplified by averaging the extremal flow with respect to satellite and Sun longitudes. Future work will be aimed at increasing the fidelity of the dynamical system by modeling eclipses and perturbations due to Earth's oblateness.

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